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Algebraic Homology For Real Hyperelliptic and Real Projective Ruled Surfaces

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Abstract. Let X be a reduced nonsingular quasiprojective scheme over $\mathbb{R}$ such that the set of real rational points $X(\mathbb{R})$ is dense in X and compact. Then $X(\mathbb{R})$ is a real algebraic variety. Denote by $H_k^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ the group of homology classes represented by Zariski closed k-dimensional subvarieties of $X(\mathbb{R})$. In this note we show that $H_1^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ is a proper subgroup of $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ for a nonorientable hyperelliptic surface X. We also determine all possible groups $H_1^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ for a real ruled surface X in connection with the previously known description of all possible topological configurations of X.

1 Introduction

Let V be a compact quasiprojective real algebraic variety. Denote by $H_k^{\mathrm{alg}}(V, \mathbb{Z}/2)$ the subgroup of $H_k(V, \mathbb{Z}/2)$ generated by the homology classes represented by Zariski closed k-dimensional subvarieties of V. If V is nonsingular and $d = \dim(V)$, let $H^{d-k}_{\mathrm{alg}}(V, \mathbb{Z}/2)$ be the subgroup of $H^{d-k}(V, \mathbb{Z}/2)$ consisting of all the cohomology classes that are sent via the Poincaré duality isomorphism $H^{d-k}(V, \mathbb{Z}/2) \to H_k(V, \mathbb{Z}/2)$ into $H_k^{\mathrm{alg}}(V, \mathbb{Z}/2)$. For definitions and results of real algebraic geometry the reader is referred to [1]. The important role played by the groups H_k^{alg} and H^k_{alg} in real algebraic geometry is extensively described in the recent survey [2].

We can also adopt a scheme theoretic point of view. Given a reduced quasiprojective scheme X over $\mathbb{R}$, we let $X(\mathbb{R})$ (resp. $X(\mathbb{C})$) denote its set of $\mathbb{R}$-rational (resp. $\mathbb{C}$-rational) points. If $X(\mathbb{R})$ is dense in X, then $\big(X(\mathbb{R}), \mathcal{O}_X \mid X(\mathbb{R})\big)$, where $\mathcal{O}_X$ is the structure sheaf of X, is a real algebraic variety (note that every real algebraic variety is biregularly isomorphic to $X(\mathbb{R})$ for some X as above). Assume now that X is also nonsingular and n-dimensional and $X(\mathbb{R})$ is nonempty and compact. Given a nonnegative integer k, we let $Z^k(X)$ denote the group of algebraic $(n-k)$-cycles on X and $\mathrm{CH}^k(X)$ the Chow group in codimension k of X. There exists a unique group homomorphism,

$$\mathrm{cl}_{\mathbb{R}} \colon \mathrm{CH}^k(X) \to H^k\big(X(\mathbb{R}), \mathbb{Z}/2\big)$$

such that for every closed $(n-k)$-dimensional subvariety V of X, the cohomology class $\mathrm{cl}_{\mathbb{R}}([V])$ is Poincaré dual to the homology class in $H_{n-k}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ determined by $V(\mathbb{R})$ (*cf.* [3]). In particular we have $\mathrm{cl}_{\mathbb{R}}\big(\mathrm{CH}^k(X)\big) = H^k_{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$.

In this paper we are interested in the group $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ when X is a nonsingular algebraic surface over $\mathbb{R}$. We say that a surface X over $\mathbb{R}$ is a real Enriques

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surface, a real ruled surface, *etc.*, if its complexification, $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$, is a complex Enriques surface, resp. a complex ruled surface, *etc.*

This question has been considered before for real rational, real abelian, real $K3$ and real Enriques surfaces by several authors (see [9], [5], [7], [8]).

In Section 2 we show that $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ is a proper subgroup of $H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ for a nonorientable hyperelliptic surface X. In Section 3 we study real ruled surfaces. We use the affine description of some real relatively minimal ruled surfaces as given in [10, V]. There, all possible topological configurations for real ruled surfaces are described. We determine the possible groups $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ in connection with those configurations.

Notation We will denote by $\cong$ a biregular isomorphism between two algebraic varieties and by $\sim$ a homeomorphism between topological spaces.

2 Real Hyperelliptic Surfaces

A complex algebraic surface S is a hyperelliptic surface if $S \cong (E \times F)/G$, where E and F are elliptic curves and G is a finite group of translations of E acting on F such that $F/G \cong \mathbb{P}^1_{\mathbb{C}}$. Let $K_Y \in \mathrm{Pic}(Y)$ denote the canonical class of the algebraic variety Y. The hyperelliptic surfaces are those elliptic surfaces (*families* of elliptic curves) for which $12K = 0$.

Topologically, the real hyperelliptic surfaces are well understood. If X is a real hyperelliptic surface, then $X(\mathbb{R})$ has at most 4 connected components, each of them homeomorphic to a torus or a Klein bottle (*cf.* [10]). In particular $X(\mathbb{R})$ is compact.

We are going to consider the image, by the real class map $\mathrm{cl}_{\mathbb{R}}$, of the classes in $\mathrm{CH}^1(X)$ given by algebraic cycles in $Z^1(X)$ numerically equivalent to zero. Let us recall the definition of numerical equivalence. For simplicity we only consider nonsingular varieties.

Definition 2.1 Let Y be a nonsingular complete n-dimensional variety over a field. A k-cycle α on Y is *numerically equivalent to zero* if $\deg(\alpha \cdot \beta) = 0$ for all $(n-k)$-cycles β on Y. (Where $\alpha \cdot \beta$ denotes the intersection product on Y.)

We denote by $\mathrm{Num}_k(Y)$ the group of k-cycles numerically equivalent to zero. We also use codimensional notation when convenient.

The following lemma shows how the cohomology classes determined by cycles numerically equivalent to zero are *perpendicular* to the algebraic cohomology with respect to $\langle\, , \rangle$, the Kronecker index (pairing) of cohomology and homology classes. We denote by $[X(\mathbb{R})]$ the fundamental class of $X(\mathbb{R})$.

Lemma 2.2 *Let X be a nonsingular, n-dimensional, projective variety over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty and compact. Let $\mathrm{cl}_{\mathbb{R}}\colon \mathrm{CH}^*(X) \to H^*\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ be the real class map. Then for all cohomology classes u in $\mathrm{cl}_{\mathbb{R}}\big(\mathrm{Num}^k(X)\big)$ and v in $H^{n-k}_{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ one has*

$$\langle u \cup v, [X(\mathbb{R})]\rangle = 0.$$

In particular, if $\dim_{\mathbb{Z}/2}\Big(\mathrm{cl}_{\mathbb{R}}\big(\mathrm{Num}^k(X)\big)\Big) \geq d$ *then*

$$\dim_{\mathbb{Z}/2}\Big(H^{n-k}\big(X(\mathbb{R}),\mathbb{Z}/2\big)/H^{n-k}_{\mathrm{alg}}\big(X(\mathbb{R}),\mathbb{Z}/2\big)\Big) \geq d.$$

Proof For each element γ in $\mathrm{CH}^n(X)$ it follows, from the definitions, that

$$\langle \mathrm{cl}_{\mathbb{R}}(\gamma), [X(\mathbb{R})]\rangle = \deg(\gamma) \pmod 2.$$

Let $\alpha \in \mathrm{CH}^k(X)$ be numerically equivalent to zero and such that $\mathrm{cl}_{\mathbb{R}}(\alpha) = u$, and $\beta \in \mathrm{CH}^{n-k}(X)$ such that $\mathrm{cl}_{\mathbb{R}}(\beta) = v$. We have

$$\begin{aligned}\langle u \cup v, [X(\mathbb{R})]\rangle &= \langle \mathrm{cl}_{\mathbb{R}}(\alpha) \cup \mathrm{cl}_{\mathbb{R}}(\beta), [X(\mathbb{R})]\rangle = \langle \mathrm{cl}_{\mathbb{R}}(\alpha\cdot\beta), [X(\mathbb{R})]\rangle \\ &= \deg(\alpha\cdot\beta) \pmod 2,\end{aligned}$$

which vanishes by definition of numerical equivalence.

The second part of the lemma follows from the first part by considering the dual pairing $H^k\big(X(\mathbb{R}),\mathbb{Z}/2\big) \times H^{n-k}\big(X(\mathbb{R}),\mathbb{Z}/2\big) \to \mathbb{Z}/2$, $(u,v) \mapsto \langle u\cup v, [X(\mathbb{R})]\rangle$. ■

The *same* result for algebraic equivalence of cycles had previously been obtained by Kucharz in [6].

We consider now a real hyperelliptic surface X. By definition, we have that $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ is a complex hyperelliptic surface, and hence $12K_{X_{\mathbb{C}}} = 0$. If we write $K_{X_{\mathbb{C}}}$ for a divisor representing the canonical class $K_{X_{\mathbb{C}}}$ we have, in particular, that a multiple of $K_{X_{\mathbb{C}}}$ is algebraically equivalent to zero. By [4, 19.3] this is equivalent to the fact that $K_{X_{\mathbb{C}}}$ is numerically equivalent to zero. We know that the group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathrm{Pic}(X_{\mathbb{C}})$ and we can identify $\mathrm{Pic}(X)$ with the group $\mathrm{Pic}(X_{\mathbb{C}})^G$ of divisor classes invariant under G (*cf.* [10, I]). Using this identification it is easy to see that K_X is then numerically equivalent to zero in X.

Theorem 2.3 *Let X be a real hyperelliptic surface. If $X(\mathbb{R})$ is nonorientable then*

$$\{0\} \neq H_1^{\mathrm{alg}}\big(X(\mathbb{R}),\mathbb{Z}/2\big) \neq H_1\big(X(\mathbb{R}),\mathbb{Z}/2\big).$$

Proof The nonorientability of $X(\mathbb{R})$ implies that $w_1\big(X(\mathbb{R})\big) \neq 0$, where $w_1\big(X(\mathbb{R})\big)$ denotes the first Stiefel-Whitney class of $X(\mathbb{R})$. The fact that $H^*_{\mathrm{alg}}\big(X(\mathbb{R}),\mathbb{Z}/2\big)$ contains all the Stiefel-Whitney classes of $X(\mathbb{R})$ implies the first inequality.

Consider now the real class map $\mathrm{cl}_{\mathbb{R}}\colon \mathrm{Pic}(X) \to H^1\big(X(\mathbb{R}),\mathbb{Z}/2\big)$. We have that K_X is numerically equivalent to zero. Since $\mathrm{cl}_{\mathbb{R}}(K_X) = w_1\big(X(\mathbb{R})\big) \neq 0$ we get, by Lemma 2.2, that $H^1\big(X(\mathbb{R}),\mathbb{Z}/2\big)/H^1_{\mathrm{alg}}\big(X(\mathbb{R}),\mathbb{Z}/2\big) \neq \{0\}$. We get then

$$\{0\} \neq H^1_{\mathrm{alg}}\big(X(\mathbb{R}),\mathbb{Z}/2\big) \neq H^1\big(X(\mathbb{R}),\mathbb{Z}/2\big),$$

which is equivalent to the claim. ■

3 Real Projective Ruled Surfaces

A complex surface V is ruled if there exists a nonsingular complex curve C together with a projective morphism $\pi\colon V \to C$ such that the fiber of a generic point η is an irreducible curve of genus 0. If a real surface X is such that $X_{\mathbb{C}}$ is ruled over $\mathbb{P}^1_{\mathbb{C}}$, then X is rational, and by [9], $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big)$. For nonrational real ruled surfaces we have the following characterization: X is a nonrational ruled surface over $\mathbb{R}$ if and only if there exists a curve B over $\mathbb{R}$ of genus ≥ 1 and a projective morphism $p\colon X \to B$ such that the fiber $p^{-1}(\eta)$ of a generic point is a smooth curve of genus 0 (*cf.* [10, V]). We will assume that $B(\mathbb{R})$ is irreducible.

Theorem 3.1 *Let X be a real nonrational projective ruled surface over a curve B.*

(i) If $B(\mathbb{R})$ is connected then

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big).$$

(ii) If $B(\mathbb{R})$ is nonconnected and $X(\mathbb{R})$ has some connected component homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ then

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) \neq H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big).$$

In the proof we use the minimal model program to give a detailed description of all the possible topological types for real projective ruled surfaces $X(\mathbb{R})$ together with all the possible groups $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$. Moreover, due to the constructive nature of the arguments, all cases listed occur. Theorem 3.1 is a corollary of this description.

Proof If X is a smooth, irreducible, real surface with a real ruling $\pi\colon X \to B$ over a smooth real curve B, then X is birationally equivalent to a surface defined in some affine open subset of $\mathbb{A}^2 \times B$ by an equation of the form $x^2 + y^2 = f$, where f is a real rational function in B, regular in $B(\mathbb{R})$ (*cf.* [10, V]). Moreover a classical theorem by Witt establishes the existence of such a function f for any choice of zeros and signs in the different connected components of $B(\mathbb{R})$, provided that the number of zeros in each component is even (*cf.* [11]). We first study the group $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ for a real ruled surface given by an affine equation as above, that is, X is the projective completion of the affine surface defined in $\mathbb{A}^2 \times B$ by the equation $x^2 + y^2 = f$, where $f \in \mathbb{R}(B)^*$ is regular on $B(\mathbb{R})$. We consider two cases.

- $B(\mathbb{R})$ connected: We have $B(\mathbb{R}) \sim \mathbb{S}^1$. If f has some zeros on $B(\mathbb{R})$, then $X(\mathbb{R})$ is homeomorphic a union of 2-spheres and $H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big) = \{0\}$.
  If f is strictly positive on $B(\mathbb{R})$, then $X(\mathbb{R})$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. Let b_0 be a fixed point in $B(\mathbb{R})$, we consider

$$L_2 = \{(x, y, b_0) \in \mathbb{A}^2(\mathbb{R}) \times B(\mathbb{R}) \mid x^2 + y^2 = f(b_0)\} \subset X(\mathbb{R}).$$

  Clearly $[L_2] \in H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$, where $[L_2]$ denotes the homology class represented by L_2 in $X(\mathbb{R})$. That is, the *fiber class* is algebraic. We consider now the *section class*.

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Since f is positive on the curve $B(\mathbb{R})$ we have, by [1, Theorem 6.4.18], that $f = f_1^2 + f_2^2$ with f_1, f_2 regular on $B(\mathbb{R})$. We can then define a regular section $s\colon B(\mathbb{R}) \to X(\mathbb{R})$, $b \mapsto \big(f_1(b), f_2(b), b\big)$, so if we set $L_1 = s\big(B(\mathbb{R})\big)$ we clearly have $[L_1] \in H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ so $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big)$.

- $B(\mathbb{R})$ not connected: We write $B(\mathbb{R}) = B_1 \cup \cdots \cup B_n$, where B_i are the connected components of $B(\mathbb{R})$. Let $n = z + s^+ + s^-$ so

$$B(\mathbb{R}) = \underbrace{B_1 \cup \cdots \cup B_z}_{f \text{ has zeros}} \cup \underbrace{B_{z+1} \cup \cdots \cup B_{z+s^+}}_{f \text{ is positive}} \cup \underbrace{B_{z+s^++1} \cup \cdots \cup B_{z+s^++s^-}}_{f \text{ is negative}}$$

If we write z^* for the number of zeros of f in $B(\mathbb{R})$ we have that

$$X(\mathbb{R}) = \Big(\bigcup_{i=1}^{z^*/2} \mathbb{S}_i^2\Big) \cup \Big(\bigcup_{i=z+1}^{z+s^+} T_1^i\Big), \quad \mathbb{S}_i^2 \sim \mathbb{S}^2,\ T_1^i \sim \mathbb{S}^1 \times \mathbb{S}^1,$$

so $\dim_{\mathbb{Z}/2} H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big) = 2s^+$.

For $i = z+1, \ldots, z+s^+$ we define

$$L_{i1} = \{(f(g)^{1/2}, 0, b) \in \mathbb{A}^2(\mathbb{R}) \times B(\mathbb{R}) \mid b \in B_i\} \subset T_1^i$$
$$L_{i2} = \{(x, y, b_i) \in \mathbb{A}^2(\mathbb{R}) \times B(\mathbb{R}) \mid x^2 + y^2 = f(b_i)\} \subset T_1^i,$$

where b_i is a fixed point of B_i. Clearly $[L_{i2}] \in H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$. Now, if we consider the regular map $p\colon X(\mathbb{R}) \to B(\mathbb{R})$ induced by the projection $\mathbb{A}^2(\mathbb{R}) \times B(\mathbb{R}) \to B(\mathbb{R})$ we have that $p_*([L_{i1}]) = [B_i] \notin H_1^{\mathrm{alg}}\big(B(\mathbb{R}), \mathbb{Z}/2\big)$, so $[L_{i1}] \notin H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$.

If $u \in H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ we can write $u = \sum_{i=z+1}^{z+s^+} \lambda_i[L_{i1}] + \mu_i[L_{i2}]$, $\quad \lambda_i, \mu_i \in \{0, 1\}$, so

$$p_*(u) = \sum_{i=z+1}^{z+s^+} \mu_i[B_i] \in H_1\big(B(\mathbb{R}), \mathbb{Z}/2\big),$$

where p_* denotes the push-forward homomorphism $H_1(p)$ in homology induced by p.

We have then

$$p_*(u) \in H_1^{\mathrm{alg}}\big(B(\mathbb{R}), \mathbb{Z}/2\big) \quad \text{iff } z = 0,\ s^- = 0,\ \mu_i = 1,\ i = 1, \ldots, s^+,$$

so if $z \neq 0$ or $s^- \neq 0$ we have $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = \bigoplus_{i=z+1}^{z+s^+}([L_{i2}])$.

If $z = s^- = 0$ the function f is strictly positive and, arguing as in the connected case, we get

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = \bigoplus_{i=1}^{n}([L_{i2}]) \bigoplus([L_{11}] + \cdots + [L_{n1}]).$$

Let X now be a general relatively minimal ruled surface X over the curve B. By [10, V.3] we know that there exists a relatively minimal ruled surface X' over B given

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as above and a birational map $\varphi\colon X'_{\mathbb{C}} \to X_{\mathbb{C}}$ that is the product of *real elementary transformations*. A real elementary transformation is a blow up at a real point (or two complex conjugate points) composed with the blow down of the fiber obtained over that point (resp. the fibers over the complex conjugate points considered).

Given a smooth projective surface over $\mathbb{R}$ we understand the *effect* on the real part $X(\mathbb{R})$ of a blow up $\pi\colon Y_{\mathbb{C}} \to X_{\mathbb{C}}$ of the kind mentioned above. If $\pi\colon Y_{\mathbb{C}} \to X_{\mathbb{C}}$ is the blow up at a pair of complex conjugate points then $Y(\mathbb{R}) \cong X(\mathbb{R})$. If π has as center a real point $x \in X(\mathbb{R})$ then $X(\mathbb{R})$ and $Y(\mathbb{R})$ have the same number of connected components and the blow up *only changes* the connected component of $X(\mathbb{R})$ containing the center x. More precisely, if we write T_g for the g-holed torus ($T_0 = \mathbb{S}^2$) and U_h for the compact connected nonorientable surface of Euler characteristic $1-h$ (in particular $U_0 = \mathbb{P}_2(\mathbb{R})$), we can describe the blow up at a real point $x \in X(\mathbb{R})$ as follows: Let $\{X_i\}_{i\in I}$ be the set of connected components of $X(\mathbb{R})$, then $\{Y_i = \pi^{-1}(X_i) \cap Y(\mathbb{R})\}_{i\in I}$ is the set of connected components of $Y(\mathbb{R})$ and $Y_i \sim X_i$ if $x \notin X_i$. For $x \in X_i$ we have that $Y_i \sim U_{2g}$ if $X_i \sim T_g$ and $Y_i \sim U_{h+1}$ if $X_i \sim U_h$. In other words, the blow up *puts* an extra $\mathbb{P}^1(\mathbb{R})$ over the real point $x \in X(\mathbb{R})$. Moreover, the addition of this exceptional divisor is the *only change* in the group $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$, that is, the group $H_1^{\mathrm{alg}}\big(Y(\mathbb{R}), \mathbb{Z}/2\big)$ is, roughly speaking, the group $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big)$ plus the class represented by the extra $\mathbb{P}^1(\mathbb{R})$ (*cf.* [10, II]).

Let Y' and Y be two real ruled surfaces over a real curve B and $\mathrm{elm}\colon Y'_{\mathbb{C}} \to Y_{\mathbb{C}}$ be an elementary transformation. If elm *starts* with the blow up of two conjugate points of $Y'_{\mathbb{C}}$ we have that $Y(\mathbb{R}) \cong Y'(\mathbb{R})$. Consider then that $\mathrm{elm}\colon Y'_{\mathbb{C}} \to Y_{\mathbb{C}}$ starts with a blow up at a real point $y \in Y'(\mathbb{R})$ and assume that $Y'(\mathbb{R})$ is connected. It is clear that if $Y'(\mathbb{R}) \sim \mathbb{S}^2$ then $Y(\mathbb{R}) \sim \mathbb{S}^2$, if $Y'(\mathbb{R}) \sim \mathbb{S}^1 \times \mathbb{S}^1$ then $Y(\mathbb{R}) \sim U_1$ (Klein bottle) and if $Y'(\mathbb{R}) \sim U_1$ then $Y(\mathbb{R}) \sim U_1$.

Again we consider two cases.

- $B(\mathbb{R})$ connected: In this case, $X'(\mathbb{R})$ is, topologically, either a union of 2-spheres or a torus. By the considerations above $X(\mathbb{R})$ is then homeomorphic to a union of 2-spheres, to the torus $\mathbb{S}^1 \times \mathbb{S}^1$ or to the Klein bottle. Moreover, we have that $H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big)$.
- $B(\mathbb{R})$ not connected: We have $X'(\mathbb{R}) = (\bigcup_{i=1}^{z^*/2} \mathbb{S}_i^2) \cup (\bigcup_{i=z+1}^{z+s^+} T_1^i)$.

With a real elementary transformation with a base point x in T_1^i we get U_1^i and since we blow down a *fiber* we get that the transform of the section cycle remains nonalgebraic (same reasons as above, considering the projection onto $B(\mathbb{R})$) and the transform of the fiber cycle remains algebraic. By convenience we keep the same notation for a cycle and its transform under a real elementary transformation.

We have then

$$X(\mathbb{R}) = \Big(\bigcup_{i=1}^{z^*/2} \mathbb{S}_i^2\Big) \cup \Big(\bigcup_{j=1}^{a} T_1^j\Big) \cup \Big(\bigcup_{k=a+1}^{a+b} U_1^k\Big), \quad a + b = s^+.$$

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If $z = 0$ and $a + b = n$ (number of connected components of $B(\mathbb{R})$) then

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = \bigoplus_{i=1}^{n}([L_{i2}]) \bigoplus([L_{11}] + \cdots + [L_{n1}]).$$

If $a + b \neq n$ then

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = \bigoplus_{i=1}^{a+b}([L_{i2}]).$$

We consider now a general nonrational projective ruled surface X over a curve B. We have a relatively minimal real projective ruled surface X' and a finite sequence of birational maps $X_{\mathbb{C}} \to X_{\mathbb{C}}^{(1)} \to X_{\mathbb{C}}^{(2)} \to \cdots \to X'_{\mathbb{C}}$ where $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ are smooth projective surfaces over $\mathbb{R}$, and such that each map is the morphism corresponding to the blow up of a real point or two complex conjugate points (*cf.* [10, II.6]). We can assume that each morphism is the blow up at a real point.

We consider again two cases.

- $B(\mathbb{R})$ connected: We have three possibilities for $X'(\mathbb{R})$.
  - If $X'(\mathbb{R}) \sim \bigcup_{i=1}^{m} \mathbb{S}^2$ we have $X(\mathbb{R}) \sim (\bigcup_{j=1}^{a} \mathbb{S}^2) \cup (\bigcup_{l=a+1}^{a+b} U_{g(l)}^{l})$, $a+b = m$, $g(l) \geq 0$.
  - If $X'(\mathbb{R}) \sim \mathbb{S}^1 \times \mathbb{S}^1$ then either $X(\mathbb{R}) \sim \mathbb{S}^1 \times \mathbb{S}^1$ or $X(\mathbb{R}) \sim U_g$ with $g \geq 2$.
  - If $X'(\mathbb{R}) \sim U_1$ then $X(\mathbb{R}) \sim U_g$ with $g \geq 1$.

  In all three cases it follows, from the results above, that

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}), \mathbb{Z}/2\big) = H_1\big(X(\mathbb{R}), \mathbb{Z}/2\big).$$

- $B(\mathbb{R})$ not connected: In this case we have $X'(\mathbb{R}) = (\bigcup_{i=1}^{z^*/2} \mathbb{S}_i^2) \cup (\bigcup_{j=1}^{a} T_1^j) \cup (\bigcup_{k=a+1}^{a+b} U_1^k)$, where we follow the notations introduced above.
  If $a + b = n$ (number of connected components of $B(\mathbb{R})$) we have

$$X(\mathbb{R}) = \Big(\bigcup_{j=1}^{a'} T_1^j\Big) \cup \Big(\bigcup_{k=a'+1}^{a'+b'} U_1^k\Big) \cup \Big(\bigcup_{p=a'+b'+1}^{a'+b'+c'} U_{g(p)}^p\Big),\ a' + b' + c' = n,\ g(p) \geq 2,$$

and since

$$H_1^{\mathrm{alg}}(X'(\mathbb{R}), \mathbb{Z}/2) = \bigoplus_{i=1}^{n}([L_{i2}]) \bigoplus([L_{11}] + \cdots + [L_{n1}]),$$

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we have

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}),\mathbb{Z}/2\big) = \bigoplus_{i=1}^{a'}([L_{i2}]) \bigoplus_{k=a'+1}^{a'+b'}([L_{k2}]) \bigoplus_{p=a'+b'+1}^{a'+b'+c'}\{([L_{p2}])\oplus\cdots\oplus([L_{pg(p)}])\}$$
$$\bigoplus([L_{11}]+\cdots+[L_{n1}]),$$

where $[L_{p1}], [L_{p2}], \ldots, [L_{pg(p)}]$ denote the generators of $H_1(U^p_{g(p)}, \mathbb{Z}/2)$.
If $a + b \neq n$ we have

$$X(\mathbb{R}) = \Big(\bigcup_{i=1}^{a''}\mathbb{S}_i^2\Big) \cup \Big(\bigcup_{j=1}^{b''}T_1^j\Big) \cup \Big(\bigcup_{k=b''+1}^{b''+c''}U_1^k\Big) \cup \Big(\bigcup_{p=b''+c''+1}^{b''+c''+d''}U^p_{g(p)}\Big)$$
$$\cup \Big(\bigcup_{q=b''+c''+d''+1}^{b''+c''+d''+e''}U^q_{g(q)}\Big),$$

where $a'' + d'' = z^*/2$, $b'' + c'' + e'' = a + b$, $g(p) \geq 0$ and $g(q) \geq 2$.

Since $H_1^{\mathrm{alg}}\big(X'(\mathbb{R}),\mathbb{Z}/2\big) = \bigoplus_{i=1}^{a+b}([L_{i2}])$, we have

$$H_1^{\mathrm{alg}}\big(X(\mathbb{R}),\mathbb{Z}/2\big) = \bigoplus_{j=1}^{b''}([L_{j2}]) \bigoplus_{k=b''+1}^{b''+c''}([L_{k2}]) \bigoplus_{p=b''+c''+1}^{b''+c''+d''}\{([L_{p1}])\oplus\cdots\oplus([L_{pg(p)}])\}$$
$$\bigoplus_{q=b''+c''+d''+1}^{b''+c''+d''+e''}\{([L_{q2}])\oplus\cdots\oplus([L_{qg(q)}])\}.$$

That is, in this case the U's obtained from spheres have all 1-cycles algebraic, but the U's obtained from the tori have algebraic all 1-cycles but the one obtained from the *section cycle* in T_1^j. ∎

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