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A NOTE ON SUBNORMAL DEFECT IN FINITE SOLUBLE GROUPS

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It is shown that for every positive integer n there exists a finite group of derived length n in which all Sylow subgroups are abelian and in which the defect of subnormal subgroups is at most 3.

1.

A subgroup S of a group G is subnormal in G if there exists a chain of subgroups of G,

$$S = S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = G,$$

with S_i normal in $S_{i+1} (0 \leqslant i \leqslant n-1)$. It is of defect n if no shorter chain with this property connects S and G. It is consequence of the Jordan-Hölder Theorem that defect is an invariant.

There has been considerable interest for some time in elucidating properties of groups consequent upon their having bounds on the defects of their subnormal subgroups. Thus in the universe of finite soluble groups we denote by $\mathcal{B}_n$ the class of those groups G whose subnormal subgroups have defect at most n. The class $\mathcal{B}_1$ consists precisely of the so-called T-groups, those groups in which all subnormal subgroups are normal. Well-known investigations of Zacher [7] and Gaschütz [4] yield, among detailed structural information, that $\mathcal{B}_1$-groups are all metabelian. Work of Casolo[1], improving earlier results of McCaughan and Stonehewer [6], shows that $\mathcal{B}_2$-groups have derived length at most 5, and that 5 is best possible. Groups of odd order in $\mathcal{B}_2$ have derived length at most 3 (Cossey [3]).

However, by contrast, Hawkes [5] has shown that every finite soluble group is isomorphic to a subgroup of some $\mathcal{B}_3$-group. There is therefore no bound on the derived length of $\mathcal{B}_3$-groups. Hawkes raises the following question: is there a bound on the derived length of A-groups in $\mathcal{B}_n$? Here A-groups are those whose Sylow subgroups are all abelian. The motivation for the question is that his embedding groups are never A-groups. Recently Casolo [2] has shown that A-groups in $\mathcal{B}_4$ do not have bounded derived length. It is the aim of the present note to show that even in $\mathcal{B}_3$ there are A-groups of arbitrary derived length.

THEOREM. *For each positive integer n there is an A-group in $\mathcal{B}_3$ of derived length n.*

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2.

We begin by observing that if $\sum$ is a finite set of odd primes then there exists a prime p not in $\sum$ and for which $p \not\equiv 1(\mod q)$ for every $q \in \sum$. For, by the well-known theorem of Dirichlet, $p = k\Pi\{q : q \in \sum\} + 2$ is prime for at least one positive value of k. But then $p \equiv 2(\mod q)$ for all $q \in \sum$ so certainly $p \not\equiv 1(\mod q)$ for $q \in \sum$.

What follows is a description of a certain group construction which will enable us to produce in $\mathcal{B}_3$ an A-group of derived length n.

Let H be a finite, soluble group with a unique minimal normal subgroup M. We suppose moreover that M is not cyclic and that if M is a p-group for the prime p then, for every prime q dividing $|H/M|$, $p \not\equiv 0, 1 (\mod q)$. A group with this property we call a †-group.

Now let H be a †-group and, in the notation of the last paragraph, let U be a maximal subgroup of M. Then $M \leqslant \mathrm{N}_H(U)$ and

$$\mathrm{N}_H(U)/U = M/U \times L/U,$$

for some subgroup L normal in $\mathrm{N}_H(U)$: under the †-group hypothesis M/U is central in $\mathrm{N}_H(U)/U$ and of order co-prime to its index.

Next choose a prime q such that $q \not\equiv 0, 1(\mod r)$ for every prime r dividing $|H|$. Let X be an irreducible module for $\mathrm{N}_H(U)$ over $GF(q)$ with kernel precisely L. Then set

$$\mathrm{N} = X^H.$$

I claim that N is irreducible. For, if D (containing 1) is a complete set of right coset representatives of $\mathrm{N}_H(U)$ in H then

$$N = \bigoplus_{d \in D} X^d.$$

Moreover X^d is an M-module whose kernel as M-module is U^d. Since $U^{d_1} \neq U^{d_2}$ whenever $d_1 \neq d_2$ we conclude that distinct X^d are non-isomorphic as M-modules. Hence if N_0 is a non-zero submodule of N it contains some X^d and hence every X^d. That is $N_0 = N$ so N is irreducible.

Observe that the group which is the semi-direct product of N by H has the †-property. We will denote this group by $H(U, q)$. (It is in fact uniquely determined by H, U, q since the irreducible modules for $\mathrm{N}_H(U)$ with kernel L over $GF(q)$ form a single linear isomorphism class.) For convenience write $K = H(U, q)$.

Now choose a prime r satisfying $r \not\equiv 0, 1 (\mod s)$ for every prime s dividing $|K|$. Select a maximal subgroup V of N with the property that no $X^d (d \in D)$ is contained in it. For example, if

$$V_d = \{x^d - x : x \in X\}, \quad 1 \neq d \in D,$$

let V be a maximal subgroup of N containing all V_d. Then we can prove the following result.

LEMMA. *$G = K(V, r)$ has the property that for every one of its subnormal subgroups S either $S \leqslant F_2(G)$ or $F_1(G) \leqslant S$. (Here F_1, F_2 denote respectively the first and second terms of the ascending nilpotent series of G).*

PROOF: Let P be the unique minimal normal subgroup of G. Then P as $GF(r)K$-module is Y^K where Y is irreducible for $\mathrm{N}_K(V)$; and note that for all $k \in K$, $\ker(Y_N)^k = V^k$.

Suppose that S is a subnormal subgroup of G not in $F_2(G) = PN$. It follows that $1 \neq H \cap PNS$ is subnormal in H and therefore that $1 \neq M \cap PNS$. (The Fitting subgroup of a subnormal subgroup is in the Fitting subgroup of the whole group). Since $p \nmid |PN|$ it follows that a Sylow p-subgroup of S is a Sylow subgroup of PNS. Consequently some conjugate of S contains $M \cap PNS = T$ say. Now the intersection of all $U^d (d \in D)$ is trivial so T is not in some U^d. Again, up to conjugacy, $T \nleq U$. It follows that

$$\begin{aligned} X &= [X, M] \\ &= [X, T] \\ &= [X, \underbrace{T, T, \ldots, T}_{n}] \\ &\leqslant S \end{aligned}$$

for sufficiently large n, since S is subnormal. Finally note that $X \nleq V^k (k \in K)$ since V contains no $X^d (d \in D)$. It follows that for all $k \in K$,

$$\begin{aligned} Y^k &= [Y^k, X] \\ &= [Y^k, \underbrace{X, \ldots, X}_{l}] \\ &\leqslant S \end{aligned}$$

for large enough l. Hence P is contained in S and therefore in all its conjugates, as required. ∎

3. Proof of the Theorem

Choose an infinite sequence of distinct odd primes $p_0, p_1, \dots, p_n, \dots$ with the property that for all n, $p_n \not\equiv 1 \pmod{p_i}$, $0 \leqslant i \leqslant n-1$. Construct a sequence G_n of groups as follows. G_0 is cyclic of order p_0. $G_1 = MG_0$ where M is a faithful and irreducible module for G_0 over $GF(p_1)$. Note that G_1 is a †-group. For $n \geqslant 2$ define G_n inductively by

$$G_n = G_{n-1}(V_{n-1}, p_n),$$

where V_1 is an arbitrary maximal subgroup of M and where, for $n \geqslant 3$, V_{n-1} is a suitable maximal subgroup of the monolith of G_{n-1}, suitable in the sense of the construction of V above, immediately before the Lemma.

Now every subnormal subgroup of a metabelian A-group has defect at most 2. This is true of G_1. Let S be a subnormal subgroup of G_n. Either $S \leqslant F_2(G_n)$ which is a metabelian A-group and so S has defect at most 3 in G_n; or $F_1(G_n) \leqslant S$ in which case the defect of S is that of $S/F_1(G_n)$ in $G_n/F_1(G_n) \cong G_{n-1}$. By induction therefore S has defect at most 3 in G_n. Since G_n are all A-groups in $\mathcal{B}_3$ and G_n has derived length n we are done. ∎

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