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Minimal vector lattice covers

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We show that each archimedean lattice-ordered group is contained in a unique (up to isomorphism) minimal archimedean vector lattice. This improves a result of Paul F. Conrad appearing previously in this Bulletin. Moreover, we show that this relationship between archimedean lattice-ordered groups and archimedean vector lattices is functorial.

The reader is referred to [1] for the basic terminology of lattice-ordered groups (l-groups) and vector lattices. By a vector lattice we always mean a real vector lattice.

THEOREM 1. *Let* G *be an archimedean* l*-group. There exists an archimedean vector lattice* V *satisfying*

(i) G *is an* l*-subgroup of* V *, and*

(ii) if G *is an* l*-subgroup of the vector sublattice* W *of* V *, then* $V = W$.

If V' *is an archimedean vector lattice satisfying (i) and (ii), then there is a unique vector lattice isomorphism of* V *onto* V' *inducing the identity on* G .

Proof. This theorem was proved in [3] under the additional assumption

(iii) each non-zero ideal of V *(respectively,* V' *) has non-zero intersection with* G .

We show that *(iii)* is redundant here.

To this end let V be an archimedean vector lattice satisfying *(i)*

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and *(ii)*. Let $G^d = \{x \in V \mid nx \in G$ for some integer $n\}$. G^d is an l-subgroup of V . To show that *(iii)* holds it is sufficient to show that if $0 < v \in V$ then there exists $g \in G^d$ such that $0 < g \leq v$.

Let $0 < v \in V$. Without loss of generality $v = \bigwedge_J (h_j \vee 0)$ with J finite and $h_j = r_{1j}g_{1j} + \ldots + r_{kj}g_{kj}$, where $r_{ij} \in R$ and $0 < g_{ij} \in G$. Let N denote the natural numbers. For each i, j and for each $n \in N$, there exists a rational number s_{ijn} such that $0 \leq r_{ij} - s_{ijn} \leq 1/n$. Let $f_{jn} = \sum_{i=1}^{k} s_{ijn}g_{ij}$. Then for each $j \in J$ and $n \in N$, we have $0 \leq h_j - f_{jn} = \sum_{i=1}^{k} (r_{ij} - s_{ijn})g_{ij} \leq (1/n) \sum_{i=1}^{k} g_{ij}$. But $\bigwedge_N (1/n) \sum_{i=1}^{k} g_{ij} = 0$ since V is archimedean. Thus $0 = \bigwedge_N (h_j - f_{jn})$, and hence $h_j = \bigvee_N f_{jn}$.

Hence $\bigwedge_J h_j = \bigwedge_J \bigvee_N f_{jn} = \bigvee_{N^J} \bigwedge_J f_{j\gamma(j)}$ (since J is finite), and thus

$$0 < v = \bigwedge_J (h_j \vee 0) = \left(\bigwedge_J h_j\right) \vee 0 = \left(\bigvee_{N^J} \bigwedge_J f_{j\gamma(j)}\right) \vee 0 =$$

$$\bigvee_{N^J} \bigwedge_J (f_{j\gamma(j)} \vee 0)$$

Thus $0 < \bigwedge_J (f_{j\gamma(j)} \vee 0) \leq v$ for some $\gamma \in N^J$. Since $\bigwedge_J (f_{j\gamma(j)} \vee 0) \in G^d$, the proof is complete.

In the terminology of [3] the vector lattice V in Theorem 1 is the *v-hull* of G . We have shown that, in addition to the conclusion of the theorem, V satisfies *(iii)*.

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Let $FVL(S)$ denote the free vector lattice on the set S. It follows from the definition of freedom and the fact that each abelian l-group can be embedded in a vector lattice that [2] the l-subgroup of $FVL(S)$ generated by S is the free abelian l-group $FLG(S)$ on S. With this identification, suppose $\alpha : FLG(S) \to W$ is an l-group homomorphism into a vector lattice W. Then $\alpha|_S$ extends to a vector lattice homomorphism $\bar{\alpha} : FVL(S) \to W$ and $\bar{\alpha}|_{FLG(S)} = \alpha$.

THEOREM 2. *Let A be an archimedean l-group with v-hull V. Suppose W is an archimedean vector lattice, and $f : A \to W$ is an l-group homomorphism. Then there exists a unique vector lattice homomorphism $\bar{f} : V \to W$ extending f.*

Proof. Choose an l-group epimorphism $\alpha : FLG(S) \to A$. Since $A \subseteq V$ we have a vector lattice homomorphism $\bar{\alpha} : FVL(S) \to V$ extending α, and since A generates V as a vector lattice, $\bar{\alpha}$ is epic. Also, $\ker\bar{\alpha} \cap FLG(S) = \ker\alpha$.

Let

$$C = \{J \mid J \text{ is an ideal of } FVL(S),\ J \supseteq \ker\alpha, \text{ and } FVL(S)/J \text{ is archimedean}\}.$$

Note $\ker\bar{\alpha} \in C$. Let $J^* = \bigcap C$. $FVL(S)/J^*$ is archimedean since $FVL(S)/J$ is for each $J \in C$; moreover, $FVL(S)/J^*$ is generated as a vector lattice by $J^* + FLG(S)/J^*$. Thus $FVL(S)/J^*$ is the v-hull of $J^* + FLG(S)/J^*$.

We have $J^* \subseteq \ker\bar{\alpha}$. Suppose $0 < x \in \ker\bar{\alpha}\backslash J^*$. By *(iii)* the ideal of $FVL(S)/J^*$ generated by $x + J^*$ has non-zero intersection with $J^* + FLG(S)/J^*$. Thus there exists $0 < g \in FLG(S)$ such that $J^* < g+J^* \le nx+J^*$ for some integer n. Hence $g+h \le nx$ for some $h \in J^*$. Thus $h \le g+h \le nx$, and thus $g+h \in \ker\bar{\alpha}$. Hence $g \in \ker\bar{\alpha}$, and thus $g \in \ker\alpha$. But this contradicts $J^* < g+J^*$, since $\ker\alpha \subseteq J^*$. Thus $J^* = \ker\bar{\alpha}$.

There exists a vector lattice homomorphism $\tau : FVL(S) \to W$ such that $f \circ \alpha(x) = \tau(x)$ for all $x \in FLG(S)$. Note that $\ker\tau \supseteq \ker\alpha$ and $\operatorname{Im}\tau$ is archimedean. Thus $\ker\tau \supseteq J^* = \ker\bar{\alpha}$. Hence there is a vector lattice homomorphism $\bar{f} : V \to W$ such that $\bar{f} \circ \bar{\alpha} = \tau$. Now, if $a \in A$ then

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$a = \alpha(x)$ for some $x \in FLG(S)$ and

$$f(a) = f \circ \alpha(x) = \tau(x) = \overline{f} \circ \overline{\alpha}(x) = \overline{f}(a) .$$

Thus $\overline{f}$ extends f .

Since A generates V as a vector lattice, $\overline{f}$ is the unique extension of f to V .

REMARK. If A is an archimedean l-group, let $F(A)$ be its v-hull. If A and B are archimedean l-groups and $f : A \to B$ is an l-group homomorphism, let $F(f)$ be the unique extension of f to a vector lattice homomorphism of $F(A)$ into $F(B)$ given by Theorem 2. Then F is a functor from the category of archimedean l-groups to the category of archimedean vector lattices. F is adjoint to the forgetful functor which "forgets" the scalar multiplication.

We list some corollaries to the main theorems. The last two depend on the fact that free (and projective) abelian l-groups are archimedean.

COROLLARY 1. *If G is an l-subgroup of the archimedean l-group H, then the v-hull of G is a vector sublattice of the v-hull of H.*

COROLLARY 2. (Conrad, [2]). *If V and W are archimedean vector lattices, and $f : V \to W$ is an l-group homomorphism, then f is a vector lattice homomorphism. (This need not hold when W is not archimedean.)*

COROLLARY 3. (Conrad, [2]). *The v-hull of $FLG(S)$ is $FVL(S)$.*

COROLLARY 4. *The v-hull of a projective abelian l-group is a projective vector lattice.*

References

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