On Characterizations of Real Hypersurfaces in a Complex Space Form with η-Parallel Shape Operator

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Abstract. In this paper we study real hypersurfaces in a non-flat complex space form with η-parallel shape operator. Several partial characterizations of these real hypersurfaces are obtained.

1 Introduction

Let $M_n(c)$ be an n-dimensional complete and simply connected non-flat complex space form with constant holomorphic sectional curvature $4c$, *i.e.*, it is either a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$ (according to whether the holomorphic sectional curvature $4c$ is positive or negative). Suppose M is a connected real hypersurface in $M_n(c)$ and N is a unit normal vector field of M. Then the complex structure J of $M_n(c)$ induces an almost contact metric structure $(\phi, \xi, \eta, \langle, \rangle)$ on M, *i.e.*,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = \langle \xi, X \rangle.$$

We denote by $\Gamma(\mathcal{V})$ the module of all differentiable sections on the vector bundle $\mathcal{V}$ over M. Typical examples of real hypersurfaces are the homogeneous real hypersurfaces M. In 1973, Takagi [18] classified these homogeneous real hypersurfaces in $\mathbb{C}P^n$ into six types, so-called real hypersurfaces of type A_1, A_2, B, C, D, and E. A *Hopf hypersurface* M in $M_n(c)$ is characterized by the condition that the structure vector field ξ is principal, *i.e.*, $A\xi = \alpha\xi$, and it can be shown that this principal curvature α is a constant.

By looking at Takagi's classification, one may verify that the homogeneous real hypersurfaces are Hopf and with constant principal curvatures. In 1986, Kimura [7] showed that the converse is also true, *i.e.*, Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ are in fact those real hypersurfaces of type A_1, A_2, etc. Also, Berndt [2] showed a $\mathbb{C}H^n$'s version for Kimura's result, *i.e.*, Hopf hypersurfaces with constant principal curvatures could be divided into four types, nowadays known as type A_0, A_1, A_2, and B. In what follows, by real hypersurfaces of type A, we mean of type A_1, A_2 (resp. of type A_0, A_1, A_2) for $c > 0$ (resp. for $c < 0$). Other than these Hopf hypersurfaces, another example of real hypersurfaces in $M_n(c)$ is the class of ruled

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real hypersurfaces. *Ruled real hypersurfaces* in $M_n(c)$ are characterized by having a one-codimensional foliation whose leaves are complex totally geodesic hyperplanes in $M_n(c)$. The geometry of ruled real hypersurfaces in $M_n(c)$ was studied in [10].

One of the first results in the theory of real hypersurfaces M in $M_n(c)$ is that the shape operator A of M in $M_n(c)$ cannot be parallel, *i.e.*, $\nabla A \neq 0$, where ∇ is the Levi–Civita connection of M. The non-existence of real hypersurfaces in $M_n(c)$ with parallel shape operator motivates the study of the weaker notion of η-parallelism, which was first introduced by Kimura and Maeda [8]. The shape operator A is said to be *η-parallel* if it satisfies the condition

$$\langle (\nabla_X A)Y, Z \rangle = 0$$

for any X, Y and $Z \in \Gamma(\mathcal{D})$, where $\mathcal{D} := \mathrm{span}\{\xi\}^{\perp}$, called the *holomorphic distribution* on M. The complete classification of real hypersurfaces with η-parallel shape operator in $M_n(c)$ remains open up to this point. Nevertheless, many partial characterizations have been obtained either by imposing an additional condition or by considering a condition that is slightly stronger than the η-parallelism (see, for instance,[1, 4, 5, 8, 16, 17]). It is worth noting that real hypersurfaces that appeared in the list of these characterizations are those of type A, B and ruled real hypersurfaces.

In this paper we shall continue the study of real hypersurfaces in $M_n(c)$ with η-parallel shape operator. In particular, several partial characterizations of real hypersurfaces in $M_n(c)$ with η-parallel shape operator are obtained.

This paper is organized as follows. Section 2 recalls some basic formulas and briefly reviews certain known results on real hypersurfaces in $M_n(c)$ with η-parallel shape operator. Some auxiliary lemmas are derived in Section 3. In Section 4 we focus on contact real hypersurfaces in $M_n(c)$ and give a characterization for ruled real hypersurfaces and contact real hypersurfaces. In Section 5 we characterize real hypersurfaces in $M_n(c)$ with η-parallel shape operator under the commutativity assumption on $\phi A\phi$ and $\phi^2 A\phi^2$. In the last section we characterize real hypersurfaces in $M_n(c)$ with prescribed covariant derivative of the shape operator.

2 Preliminaries

Consider a connected real hypersurface M in $M_n(c)$, the induced almost contact metric structure $(\phi, \xi, \eta, \langle, \rangle)$ on M has the following properties

(2.1) $\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$

(2.2) $(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y \rangle\xi, \quad \nabla_X \xi = \phi AX$

for any $X, Y \in \Gamma(TM)$. Let R be the curvature tensor of M. Then the equations of Gauss and Codazzi are given respectively by

$$R(X, Y)Z = c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle \phi Y, Z \rangle \phi X - \langle \phi X, Z \rangle \phi Y - 2\langle \phi X, Y \rangle \phi Z\}$$
$$+ \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,$$
$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y \rangle \xi\}.$$

The second order covariant derivative $\nabla A(X,Y)$ on the shape operator A is defined by

$$(\nabla A(X,Y))Z = (\nabla_X \nabla_Y A - \nabla_{\nabla_X Y} A)Z$$
$$= \nabla_X\{(\nabla_Y A)Z\} - (\nabla_Y A)\nabla_X Z - (\nabla_{\nabla_X Y} A)Z.$$

Next, we state two necessary and sufficient conditions for real hypersurfaces in $M_n(c)$ to be of type A.

Theorem 2.1 ([3, 11–13, 15])   *Let M be a real hypersurface in $M_n(c)$, $n \geq 2$. Then the following are equivalent:*

(i)   *M is locally congruent to a real hypersurface of type A;*
(ii)   *$\phi A = A\phi$;*
(iii)   *$(\nabla_X A)Y = -c\{\langle \phi X, Y \rangle \xi + \eta(Y)\phi X\}$, for any $X, Y \in \Gamma(TM)$.*

The following theorem, proved by Kimura and Maeda, and Suh for $c > 0$ and $c < 0$, respectively, completely classified Hopf hypersurfaces with η-parallel shape operator in $M_n(c)$.

Theorem 2.2 ([8, 13, 17])   *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then M has η-parallel shape operator if and only if M is locally congruent to a real hypersurface of type A or B.*

The above theorem is not true if the condition of M being Hopf is removed.

Theorem 2.3 ([1, 8])   *Let M be a real hypersurface in $M_n(c)$, $n \geq 2$. Suppose M satisfies the following two conditions:*

(i)   *$\phi(\phi A + A\phi)\phi = 0$, i.e., the holomorphic distribution $\mathcal{D}$ is integrable;*
(ii)   *the shape operator A is η-parallel.*

Then M is locally congruent to a ruled real hypersurface.

On the other hand, Ki and Suh studied real hypersurfaces M with η-parallel shape operator without assuming it is Hopf. By restricting condition (ii) and condition (iii) in Theorem 2.1 to the holomorphic distribution $\mathcal{D}$, they obtained the following result.

Theorem 2.4 ([4])   *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies the following two conditions:*

(i)   *$\phi(\phi A - A\phi)\phi = 0$;*
(ii)   *$(\nabla_X A)Y = -c\langle \phi X, Y \rangle \xi$, for any $X, Y \in \Gamma(\mathcal{D})$.*

Then M is locally congruent to a real hypersurface of type A.

Observe that condition (ii) in this theorem is a special form for the shape operator A to be η-parallel. Ahn, Lee, and Suh weaken it to the η-parallelism condition on A and proved the following.

Theorem 2.5 ([1])   *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies the following two conditions:*

(i) $\phi(\phi A - A\phi)\phi = 0$;
(ii) *the shape operator A is η-parallel.*

Then M is locally congruent to a ruled real hypersurface or a real hypersurface of type A.

Before we end this section, we shall state the expression of ∇A on these standard examples of real hypersurfaces with η-parallel shape operator.

Theorem 2.6   *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, and $X, Y \in \Gamma(\mathcal{D})$.*

(i)   *If M is of type A, then $(\nabla_X A)Y = -c\langle \phi X, Y \rangle \xi$.*
(ii) *If M is of type B, then*

$$(\nabla_X A)Y = \left\{ -c\langle \phi X, Y \rangle + \frac{\alpha}{2}\langle (\phi A - A\phi)X, Y \rangle \right\} \xi.$$

(iii) *If M is ruled and $V = \phi A\xi$, then*

$$(\nabla_X A)Y = \{ -c\langle \phi X, Y \rangle + \eta(AY)\langle X, V \rangle + \eta(AX)\langle Y, V \rangle \}\xi.$$

(i) is an immediate consequence of Theorem 2.1(iii), while (iii) was derived in [16]. In order to verify (ii), we need to recall a lemma.

Lemma 2.7 ([6])   *Let M be a real hypersurface in $M_n(c)$. If $A\xi = \alpha\xi$, then α is a constant and $(\nabla_\xi A) = (\alpha/2)(\phi A - A\phi)$.*

Since the shape operator of a real hypersurface of type B is η-parallel, for $X, Y \in \Gamma(\mathcal{D})$, (ii) in the above theorem can be derived as follows:

$$
\begin{aligned}
(\nabla_X A)Y &= \langle (\nabla_X A)Y, \xi \rangle \xi \\
&= \{ -\langle \phi X, Y \rangle + \langle (\nabla_\xi A)X, Y \rangle \}\xi && \text{(by the Codazzi equation)} \\
&= \{ -c\langle \phi X, Y \rangle + \frac{\alpha}{2}\langle (\phi A - A\phi)X, Y \rangle \}\xi && \text{(by Lemma 2.7).}
\end{aligned}
$$

3   Real Hypersurfaces with Non-Principal Structure Vector Field

Hopf hypersurfaces with η-parallel shape operator A have already been completely characterized in Theorem 2.2. In this section, we focus on real hypersurfaces M on which the structure vector field ξ is not principal, or equivalently, with the restriction $\beta := \|\phi A\xi\|\| \neq 0$. Certain auxiliary lemmas that are needed in the following sections are also derived here.

We shall first fix some notations as follows: $V := \nabla_\xi \xi = \phi A\xi$, $\alpha := \eta(A\xi)$, and $F := \nabla_\xi A$. Then it is clear that the shape operator A of a real hypersurface M is η-parallel if and only if $(\nabla_X A)Y = \{ -c\langle \phi X, Y \rangle + \langle FX, Y \rangle \}\xi$, $X, Y \in \Gamma(\mathcal{D})$. The next lemma plays an important role in the rest of the paper.

Lemma 3.1 *Let M be a real hypersurface in $M_n(c)$ with η-parallel shape operator A. Then*

$$
\begin{aligned}
c\big\{ &\langle Y, AZ\rangle\langle X, W\rangle - \langle X, AZ\rangle\langle Y, W\rangle \\
&+ \langle \phi Y, AZ\rangle\langle \phi X, W\rangle - \langle \phi X, AZ\rangle\langle \phi Y, W\rangle - 2\langle \phi X, Y,\rangle\langle \phi AZ, W\rangle \\
&- \langle Y, Z\rangle\langle X, AW\rangle + \langle X, Z\rangle\langle Y, AW\rangle \\
&- \langle \phi Y, Z\rangle\langle \phi X, AW\rangle + \langle \phi X, Z\rangle\langle \phi Y, AW\rangle + 2\langle \phi X, Y,\rangle\langle \phi Z, AW\rangle\big\} \\
&+ \langle AY, AZ\rangle\langle AX, W\rangle - \langle AX, AZ\rangle\langle AY, W\rangle \\
&- \langle AY, Z\rangle\langle AX, AW\rangle + \langle AX, Z\rangle\langle AY, AW\rangle \\
= c\big\{ &\langle Z, \phi AY\rangle\langle \phi X, W\rangle + \langle W, \phi AY\rangle\langle \phi X, Z\rangle \\
&- \langle Z, \phi AX\rangle\langle \phi Y, W\rangle - \langle W, \phi AX\rangle\langle \phi Y, Z\rangle\big\} \\
&+ \langle Y, \phi AX\rangle\langle FZ, W\rangle + \langle Z, \phi AX\rangle\langle FY, W\rangle + \langle W, \phi AX\rangle\langle FZ, Y\rangle \\
&- \langle X, \phi AY\rangle\langle FZ, W\rangle - \langle Z, \phi AY\rangle\langle FX, W\rangle - \langle W, \phi AY\rangle\langle FZ, X\rangle
\end{aligned}
$$

for any $X, Y, Z, W \in \Gamma(\mathcal{D})$.

Proof For any $Y, Z, W \in \Gamma(\mathcal{D})$, by differentiating the following equation covariantly $\langle(\nabla_Y A)Z, W\rangle = 0$ in the direction of $X \in \Gamma(\mathcal{D})$, we obtain

$$
\langle(\nabla A(X, Y))Z + (\nabla_{\nabla_X Y} A)Z + (\nabla_Y A)\nabla_X Z, W\rangle + \langle(\nabla_Y A)Z, \nabla_X W\rangle = 0.
$$

From the η-parallelism condition and (2.2), the above equation reduces to

$$
\begin{aligned}
\langle(\nabla A(X, Y))Z, W\rangle = \langle Y, \phi AX\rangle\langle(\nabla_\xi A)Z, W\rangle &+ \langle Z, \phi AX\rangle\langle(\nabla_Y A)\xi, W\rangle \\
&+ \langle W, \phi AX\rangle\langle(\nabla_Y A)Z, \xi\rangle.
\end{aligned}
$$

Furthermore, by using the Codazzi equation, the above equation becomes

$$
\begin{aligned}
\langle(\nabla A(X, Y))Z, W\rangle = \langle Y, \phi AX\rangle\langle FZ, W\rangle &+ \langle Z, \phi AX\rangle\{\langle FY, W\rangle - c\langle \phi Y, W\rangle\} \\
&+ \langle W, \phi AX\rangle\{\langle FY, Z\rangle - c\langle \phi Y, Z\rangle\}.
\end{aligned}
$$

Finally, by the Ricci identity $(R(X, Y)A)Z = (\nabla A(X, Y))Z - (\nabla A(Y, X))Z$ and the above equation, we obtain the lemma. ∎

Lemma 3.2 *Let M be a real hypersurface in $M_n(c)$ with η-parallel shape operator A. Then*

$$
-\langle A\phi V, Y\rangle\langle \phi V, X\rangle + \langle A\phi V, X\rangle\langle \phi V, Y\rangle = \langle \tfrac{\tau}{2}(\phi A + A\phi)X + (F\phi A + A\phi F)X, Y\rangle
$$

for any $X, Y \in \Gamma(\mathcal{D})$, where $\tau := -\operatorname{trace}\phi F\phi$.

Proof Let $E_1, E_2, \ldots, E_{2n-2}$ be a local field of orthonormal vectors in $\Gamma(\mathcal{D})$. By putting $Z = W = E_j$, for $j = 1, 2, \ldots, 2n-2$, in Lemma 3.1 and then summing up these equations, we get

$$2\langle \phi A^2 Y, \phi AX \rangle - 2\langle \phi A^2 X, \phi AY \rangle = \sum_{j=1}^{2n-2} \langle FE_j, E_j \rangle \langle \phi AX + A\phi X, Y \rangle$$

$$+ 2\langle FY, \phi AX \rangle - 2\langle FX, \phi AY \rangle.$$

Next, by applying (2.1) in the left-hand side of this equation, we obtain the lemma. ∎

Lemma 3.3 *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel shape operator A. Suppose that β is nowhere zero on M. If there exist two functions ν and $\tilde{\nu}$ such that $AV = \nu V$ and $A\phi V = \tilde{\nu}\phi V - \beta^2 \xi$, then $\phi A\phi$ and $\phi^2 A\phi^2$ can be diagonalized simultaneously.*

Proof Let x be an arbitrary point in M. From the hypothesis, the subspace

$$\mathcal{H} := \text{span}\{V, \phi V, \xi\}$$

and its orthogonal complement $\mathcal{H}^\perp$ in $T_x M$ are both invariant by A and hence by both $\phi A\phi$ and $\phi^2 A\phi^2$ as well. Furthermore, each eigenvector $E \in \mathcal{H}^\perp$ of $\phi^2 A\phi^2$ is a principal vector as well. If ϕE is principal for each principal vector $E \in \mathcal{H}^\perp$, then the statement is clearly true. Hence, we suppose that there is a unit principal vector $E' \in \mathcal{H}^\perp$, but $\phi E'$ is not principal.

First, by letting $X, Y \in \mathcal{H}^\perp$, $Z = V$ and $W = \phi V$ in Lemma 3.1, we obtain

$$-2c\beta^2(\nu - \tilde{\nu})\langle \phi X, Y \rangle = \langle F\phi V, V \rangle \langle (\phi A + A\phi)X, Y \rangle.$$

Since $\phi E'$ is not principal, we can see that $\langle F\phi V, V \rangle = 0 = \nu - \tilde{\nu}$, (for otherwise, putting $X = E'$ in the above equation yields $A\phi E' = \tilde{\lambda}\tilde{\phi}E'$ and a contradiction).

Next, by putting $X = \phi V$, $Y = V$ in Lemma 3.1 and making use of the fact that $\nu = \tilde{\nu}$,

(3.1)
$$2c\beta^2 \langle (\phi A - A\phi)Z, W \rangle - \nu\beta^2 \{\langle V, Z \rangle \langle \phi V, W \rangle + \langle \phi V, Z \rangle \langle V, W \rangle\}$$

$$= -2\nu\beta^2 \langle FZ, W \rangle - \nu \big\{ \langle V, Z \rangle \langle FV, W \rangle + \langle V, W \rangle \langle FV, Z \rangle$$

$$+ \langle \phi V, Z \rangle \langle F\phi V, W \rangle + \langle \phi V, W \rangle \langle F\phi V, Z \rangle \big\}.$$

If we put $Z, W \in \mathcal{H}^\perp$ in (3.1), then

(3.2)
$$c\langle (\phi A - A\phi)Z, W \rangle = -\nu\langle FZ, W \rangle.$$

From the hypothesis $\phi E'$ is not principal, the right-hand side of (3.2) is not identically zero, so we may assume that $\nu \neq 0$. On the other hand, by putting $Z = V$ and $W = \phi V$ in (3.1), and taking account of $\langle FV, \phi V \rangle = \nu - \tilde{\nu} = 0$, we obtain $-\nu\beta^6 = 0$. This contradicts the facts $\nu \neq 0$ and $\beta \neq 0$. ∎

4   Characterizations on Contact Real Hypersurfaces

An almost contact manifold $(M^{2n-1}, \phi, \xi, \eta)$ is said to be a *contact manifold* if

$$\eta \wedge (d\eta)^{n-1} \neq 0$$

on M. If there is a Riemannian metric $\langle \cdot , \cdot \rangle$ which is compatible with this contact structure, then $(\phi, \xi, \eta, \langle \cdot , \cdot \rangle)$ becomes a *contact metric structure* and M is said to be a *contact metric manifold*.

A real hypersurface in a Kaehler manifold is said to be *contact* if its induced almost-contact structure is contact. Okumura proved a necessary and sufficient condition for real hypersurfaces in a Kaehler manifold to be contact.

Theorem 4.1 ([14])   *Let M be a real hypersurface in a Kaehler manifold. Then the induced almost-contact structure (ϕ, ξ, η) is contact if and only if there is a non-vanishing function k on M such that*

$$(4.1) \qquad \phi A + A\phi - k\phi = 0.$$

It can be shown that k is constant. Kon proved the following characterization while the ambient space is $\mathbb{C}P^n$.

Theorem 4.2 ([9])   *Let M be a complete real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. If M satisfies $\phi A + A\phi - \varepsilon\phi = 0$ for some nonzero constant ε, then M is congruent to a real hypersurface of type A_1 or B.*

On the other hand, Vernon gave a characterization of contact real hypersurfaces in $\mathbb{C}H^n$.

Theorem 4.3 ([19])   *Let M be a complete contact real hypersurface in $\mathbb{C}H^n$, $n \geq 3$. Then M is congruent to a real hypersurface of type A_0, A_1 or B.*

Remark   Without the completeness hypothesis in Theorem 4.2 and Theorem 4.3, the conclusion is still true locally, *i.e.*, M is locally congruent to a real hypersurface of the type as stated.

In this section, we study real hypersurfaces in $M_n(c)$ under a weaker version of (4.1), *i.e.*,

$$(4.2) \qquad \phi(\phi A + A\phi - k\phi)\phi = 0,$$

for some function k on M. We shall first derive some identities from the condition (4.2). Note that (4.2) is equivalent to $\langle (\phi A + A\phi - k\phi)Y, Z \rangle = 0$, $Y, Z \in \Gamma(\mathcal{D})$. Differentiating this equation covariantly in the direction of $X \in \Gamma(\mathcal{D})$ we get

$$\langle \phi AY, \nabla_X Z \rangle + \langle (\nabla_X \phi)AY + \phi(\nabla_X A)Y + \phi A \nabla_X Y, Z \rangle$$
$$+ \langle A\phi Y, \nabla_X Z \rangle + \langle (\nabla_X A)\phi Y + A(\nabla_X \phi)Y + A\phi \nabla_X Y, Z \rangle$$
$$- (Xk)\langle \phi Y, Z \rangle - k\langle \phi Y, \nabla_X Z \rangle - k\langle (\nabla_X \phi)Y + \phi \nabla_X Y, Z \rangle = 0.$$

By using (2.2) and (4.2), this equation can be reformed as

$$- \langle Z, V \rangle \langle \phi AX, Y \rangle + \langle Y, V \rangle \langle \phi AX, Z \rangle - \langle (\nabla_X A)Y, \phi Z \rangle + \langle (\nabla_X A)Z, \phi Y \rangle$$
$$+ \eta(AY)\langle AX, Z \rangle - \eta(AZ)\langle AX, Y \rangle - (Xk)\langle \phi Y, Z \rangle = 0.$$

Now by replacing X, Y, and Z cyclically in the above equation and then summing these equations, with the help of the Codazzi equation and (4.2), we obtain

$$\mathfrak{S}(k\langle X, V \rangle + Xk)\langle \phi Y, Z \rangle = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum over X, Y, and Z. Let X be an arbitrary vector field in $\Gamma(\mathcal{D})$. If we choose $Y \perp X, \phi X$, and $Z = \phi Y$ in the above equation, then $k\langle X, V \rangle + Xk = 0$.

We summarize the above observations in the following lemma.

Lemma 4.4 *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies*

$$\phi(\phi A + A\phi - k\phi)\phi = 0$$

for some function k on M. Then for any X, Y, and $Z \in \Gamma(\mathcal{D})$,

$$(4.3) - \langle Z, V \rangle \langle \phi AX, Y \rangle + \langle Y, V \rangle \langle \phi AX, Z \rangle - \langle (\nabla_X A)Y, \phi Z \rangle + \langle (\nabla_X A)Z, \phi Y \rangle$$
$$+ \eta(AY)\langle AX, Z \rangle - \eta(AZ)\langle AX, Y \rangle - (Xk)\langle \phi Y, Z \rangle = 0,$$

$$(4.4) \qquad\qquad k\langle X, V \rangle + Xk = 0.$$

We first look at the case where k is a nonzero constant. In this case, the equation (4.4) implies that $V = 0$, *i.e.*, ξ is principal and so $(\phi A + A\phi - k\phi)\xi = 0$. Consequently, we have $\phi A + A\phi - k\phi = 0$, for some nonzero constant k, and hence it follows from Theorem 4.2 and Theorem 4.3 that we obtain the following.

Theorem 4.5 *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. If M satisfies*

$$\phi(\phi A + A\phi - \varepsilon\phi)\phi = 0$$

for some constant $\varepsilon \neq 0$, then M is locally congruent to a real hypersurface of type A_0, A_1 or B.

On the other hand, by adding the η-parallelism condition on the shape operator, we have the following characterization.

Theorem 4.6 *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies the following two conditions:*
(i)   *$\phi(\phi A + A\phi - k\phi)\phi = 0$, for some function k on M;*
(ii)   *the shape operator A is η-parallel.*
Then M is locally congruent to a ruled real hypersurface or a real hypersurface of type A_0, A_1 or B.

Proof In this case, the equation (4.3) can be reduced as

$$- \langle Z, V \rangle \langle \phi AX, Y \rangle + \langle Y, V \rangle \langle \phi AX, Z \rangle$$
$$+ \eta(AY) \langle AX, Z \rangle - \eta(AZ) \langle AX, Y \rangle - (Xk) \langle \phi Y, Z \rangle = 0.$$

If we choose $Y \perp V, \phi V$ and $Z = \phi Y$, then $Xk = 0$ for all $X \in \Gamma(\mathcal{D})$ and together with (4.4), we obtain $k \langle V, V \rangle = 0$ on M.

If k is identically zero on M, then M is ruled by Theorem 2.3.

Now suppose that $M_0 := \{x \in M : k(x) \neq 0\}$ is nonempty. Then on such open set M_0, ξ is principal and so $(\phi A + A\phi - k\phi)\xi = 0$. Consequently, we have $\phi A + A\phi - k\phi = 0$, and hence by Theorem 4.2 and Theorem 4.3, M_0 is locally congruent to a real hypersurface of type A_0, A_1 or B. It follows that k is a nonzero constant on M_0. By the connectedness of M, $M_0 = M$ and so it is locally congruent to a real hypersurface of type A_0, A_1, or B. ∎

5 Real Hypersurfaces with a Commutative Condition

Observe that the condition (i) in Theorem 2.3, Theorem 2.5, and Theorem 4.6 imply that $\phi^2 A \phi^2$ and $\phi A \phi$ are commutative. Hence, it is natural to ask if the condition (i) in these theorems is replaceable by this condition. The main purpose of this section is to give an affirmative answer to this question. We first prove the following lemma.

Lemma 5.1 *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel shape operator A. If $\phi A \phi$ and $\phi^2 A \phi^2$ commute at a point $x \in M$, then at $x \in M$, either*

(i) $\phi(\phi A - A\phi)\phi = 0$, *or*

(ii) $\phi(\phi A + A\phi - k\phi)\phi = 0$, *where $k = (\operatorname{trace} A - \alpha)/(n-1)$.*

Proof As $\phi A \phi$ and $\phi^2 A \phi^2$ are commutative at $x \in M$, they can be diagonalized simultaneously and hence there are orthonormal vectors $E_j, \phi E_j$ $(1 \leq j \leq n-1)$ in $\mathcal{D}_x$ such that

$$AE_j = e_j \xi + \lambda_j E_j \quad A\phi E_j = \tilde{e}_j \xi + \tilde{\lambda}_j \phi E_j.$$

By making the following substitutions for the vectors X, Y, Z, and W in Lemma 3.1:

- $Y = Z = E_i, W = X = \phi E_j, (i \neq j)$,
- $Y = Z = E_i, W = X = E_j, (i \neq j)$,
- $Y = Z = E_i, W = X = \phi E_i$,
- $X = E_j, Y = \phi E_j, Z = \phi E_i, W = E_i, (i \neq j)$,

we obtain the following equations

(5.1) $$\tilde{\lambda}_j \lambda_i^2 - (\tilde{\lambda}_j^2 - c + \tilde{e}_j^2)\lambda_i + (e_i^2 - c)\tilde{\lambda}_j = 0,$$

(5.2) $$\lambda_j \lambda_i^2 - (\lambda_j^2 - c + e_j^2)\lambda_i + (e_i^2 - c)\lambda_j = 0,$$

(5.3) $$(\lambda_i - \tilde{\lambda}_i)(\lambda_i \tilde{\lambda}_i + 5c) + \tilde{\lambda}_i e_i^2 - \lambda_i \tilde{e}_i^2 + 2\langle FE_i, \phi E_i \rangle(\lambda_i + \tilde{\lambda}_i) = 0,$$

(5.4) $$2c(\lambda_i - \tilde{\lambda}_i) + (\lambda_j + \tilde{\lambda}_j)\langle FE_i, \phi E_i \rangle = 0.$$

If $\lambda_i = \tilde{\lambda}_i$ for all i, then $\phi(\phi A - A\phi)\phi = 0$ and we obtain statement (i). Hence, we suppose $\lambda_i \neq \tilde{\lambda}_i$ for some i, say $\lambda_1 \neq \tilde{\lambda}_1$. From (5.4), we obtain $\langle FE_1, \phi E_1 \rangle \neq 0$ and

$$(5.5) \qquad \lambda_i + \tilde{\lambda}_i = 2c\frac{\tilde{\lambda}_1 - \lambda_1}{\langle FE_1, \phi E_1 \rangle}, \quad i \neq 1.$$

Since the right-hand side of the above equation is nonzero, for each $i \neq 1$, at least one of λ_i and $\tilde{\lambda}_i$ is nonzero, say, $\lambda_i \neq 0$.

Now taking $j = 1$ and $i \neq 1$, and then by taking the operation $\lambda_j \times$ (5.1) $- \tilde{\lambda}_j \times$ (5.2), yields $(\lambda_1 - \tilde{\lambda}_1)(\lambda_1\tilde{\lambda}_1 + c) + \tilde{\lambda}_1 e_1^2 - \lambda_1 \tilde{e}_1^2 = 0$. From this equation and (5.3), we can see

$$\lambda_1 + \tilde{\lambda}_1 = 2c\frac{\tilde{\lambda}_1 - \lambda_1}{\langle FE_1, \phi E_1 \rangle}.$$

By observing (5.5) and the above equation, we obtain

$$\lambda_i + \tilde{\lambda}_i = 2c\frac{\tilde{\lambda}_1 - \lambda_1}{\langle FE_1, \phi E_1 \rangle}, \quad \text{for all } i.$$

Therefore, we obtain statement (ii) with $k = 2c(\tilde{\lambda}_1 - \lambda_1)\langle FE_1, \phi E_1 \rangle^{-1}$. Furthermore, we have

$$k = \sum_{i=1}^{n-1} \frac{\lambda_i + \tilde{\lambda}_i}{n-1} = \frac{\operatorname{trace} A - \alpha}{n-1}. \qquad \blacksquare$$

Theorem 5.2 *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel shape operator A. If $\phi A\phi$ and $\phi^2 A\phi^2$ commute, then M is locally congruent to a ruled real hypersurface, or a real hypersurface of type A or B.*

Proof Let $\rho := \|\phi(\phi A - A\phi)\phi\|$ and $M_0 := \{x \in M : \rho(x) \neq 0\}$. Then by Lemma 5.1 we have $\phi(\phi A + A\phi - k\phi)\phi = 0$ on M_0 and it follows from Theorem 4.6 that the open submanifold M_0 is locally congruent to a real hypersurface of type B. By a direct calculation, we see that

$$\rho^2 = 4(\|A\|^2 - \alpha^2) - \frac{2(\operatorname{trace} A - \alpha)^2}{n-1}$$

on M_0. As a real hypersurface of type B is of constant principal curvatures, ρ is a positive constant on M_0. By the connectedness of M, either M_0 is empty or M_0 is the whole of M, *i.e.*, either $\phi(\phi A - A\phi)\phi = 0$ or $\phi(\phi A + A\phi - k\phi)\phi = 0$ on M. By virtue of Theorem 2.5 and Theorem 4.6, M is locally congruent to a ruled real hypersurface or a real hypersurface of type A or B. $\qquad \blacksquare$

6 Real Hypersurfaces with Prescribed Covariant Derivative of the Shape Operator

In the previous sections, we characterized real hypersurfaces M with η-parallel shape operator A under certain additional conditions on M. In this section we study these

real hypersurfaces from another aspect, *i.e.*, by looking at a condition that is slightly stronger than the η-parallelism on A.

In Theorem 2.6 we see that these "standard examples" of real hypersurfaces with η-parallel shape operator have a nice form for the covariant derivative of the shape operator on the holomorphic distribution $\mathcal{D}$. Motivated by these identities, it is natural to ask if the converses of the identities in Theorem 2.1 are true. In 1995, Suh proved the following.

Theorem 6.1 ([16])   *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. If M satisfies*

$$(\nabla_X A)Y = \{-c\langle\phi X, Y\rangle + \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle\}\xi$$

for any $X, Y \in \Gamma(\mathcal{D})$, then M is locally congruent to a ruled real hypersurface or a real hypersurface of type A.

It follows from the above theorem that, since $V = 0$ is necessary and sufficient for ξ to be principal, we can easily obtain the following characterization for real hypersurfaces of type A.

Corollary 6.2   *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies*

$$(\nabla_X A)Y = -c\langle\phi X, Y\rangle\xi$$

for any $X, Y \in \Gamma(\mathcal{D})$. Then M is locally congruent to a real hypersurface of type A.

The condition in Theorem 6.1 is too strong to be used to characterize all the standard examples of real hypersurfaces with η-parallel shape operator. It shall be replaced by a weaker condition in order to broaden the list of characterization. In this sense, we have the following.

Theorem 6.3   *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies*

$$(6.1) \quad (\nabla_X A)Y = \big\{ -c\langle\phi X, Y\rangle + \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle$$
$$+ \varepsilon\langle(\phi A - A\phi)X, Y\rangle\big\}\xi$$

for any $X, Y \in \Gamma(\mathcal{D})$, where ε is a constant. Then M is locally congruent to a ruled real hypersurface or a real hypersurface of type A or B.

Proof   The condition (6.1) implies that A is η-parallel. Let

$$M_0 := \{x \in M : \beta(x) \neq 0\}.$$

Then by virtue of Theorem 2.2, we conclude that the interior of $M - M_0$, $\text{int}(M \setminus M_0)$ is of type A or B. Each of its principal curvatures is constant and so $\|\phi A\phi\|$ is a positive constant on $\text{int}(M \setminus M_0)$.

Next consider the open submanifold M_0. From the condition (6.1), the tensor field F takes the form

$$(6.2) \qquad \langle FX, Y\rangle = \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle + \varepsilon\langle(\phi A - A\phi)X, Y\rangle$$

for any $X, Y \in \Gamma(\mathcal{D})$. It follows from this equation that $\tau = -\operatorname{trace} \phi F \phi = 0$. Moreover, the identity in Lemma 3.2 can be reduced to

$$(6.3) \qquad -\langle AX, V \rangle \langle Y, V \rangle + \langle AY, V \rangle \langle X, V \rangle = \varepsilon \langle (A\phi A\phi - \phi A\phi A)X, Y \rangle.$$

First, by putting $X = V$ and $Y = \phi V$ in (6.3), we obtain $\langle AV, \phi V \rangle = 0$. Next, if we put $Y = \phi V$ in (6.3), then

$$(6.4) \qquad\qquad\qquad \varepsilon \langle (A\phi A + \phi A\phi A\phi)V, X \rangle = 0$$

for any $X \in \Gamma(\mathcal{D})$. Finally, when we put $Y = V$ in (6.3), we get

$$\beta^2 \langle AX, V \rangle - \langle AV, V \rangle \langle X, V \rangle = \varepsilon \langle \phi X, (A\phi A + \phi A\phi A\phi)V \rangle$$
$$= 0 \qquad\qquad\qquad \text{(from (6.4))}.$$

This equation tells us that $AV = \nu V$. Next, we wish to prove that $A\phi V = \bar{\nu}\phi V - \beta^2 \xi$. For this purpose, we put $Y = \phi V$ and $Z = W = V$ in Lemma 3.1. Then

$$0 = c\{\beta^2 \langle A\phi V, X \rangle - \langle A\phi V, \phi V \rangle \langle \phi V, X \rangle\} + \frac{\langle FV, V \rangle}{2} \langle \phi A\phi V - \nu V, X \rangle$$
$$- \langle A\phi V, \phi V \rangle \langle FV, X \rangle + \langle FV, \phi V \rangle \langle A\phi V, X \rangle.$$

On the other hand, by putting $Y = V$ and $Z = W = \phi V$ in Lemma 3.1, we get

$$c\{\beta^2 \langle A\phi V, \phi X \rangle - \langle A\phi V, \phi V \rangle \langle V, X \rangle\} = \frac{\langle F\phi V, \phi V \rangle}{2} \langle \nu \phi V + A\phi V, X \rangle$$
$$+ \nu\{\beta^2 \langle F\phi V, X \rangle - \langle FV, \phi V \rangle \langle V, X \rangle\}.$$

By using (6.2), the above two equations become

$$(\beta^2 - \varepsilon\nu - c)\{\beta^2 \langle A\phi V, X \rangle - \langle A\phi V, \phi V \rangle \langle \phi V, X \rangle\} = 0$$
$$(\varepsilon\nu + c)\{\beta^2 \langle A\phi V, \phi X \rangle - \langle A\phi V, \phi V \rangle \langle \phi V, \phi X \rangle\} = 0$$

for any $X \in \Gamma(\mathcal{D})$. From these two equations and the fact that $\beta \neq 0$,

$$\langle A\phi V, X \rangle = \beta^{-2} \langle A\phi V, \phi V \rangle \langle \phi V, X \rangle, \quad X \in \Gamma(\mathcal{D})$$

and hence we have $A\phi V = \bar{\nu}\phi V - \beta^2 \xi$, where $\bar{\nu} = \beta^{-2} \langle A\phi V, \phi V \rangle$. According to Lemma 3.3 and Theorem 5.2, we conclude that M_0 is ruled. It follows that $\|\phi A\phi\| = 0$ on M_0. By the continuity of $\|\phi A\phi\|$ and the connectedness of M, we have either M_0 is empty or M_0 is the whole of M. Consequently, M is either of type A or B in the former case or ruled in the latter case. ∎

References

[1] S.-S. Ahn, S. B. Lee, and Y. J. Suh, *On ruled real hypersurfaces in a complex space form*. Tsukuba J. Math. **17**(1993), no. 2, 311–322.

[2] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*. J. Reine Angew Math. **395**(1989), 132–141.

[3] Y.-W. Choe, *Characterization of certain real hypersurfaces of a complex space form*. Nihonkai Math. J. **6**(1995), no. 1, 97–114.

[4] U.-H. Ki and Y. J. Suh, *On a characterization of real hypersurfaces of type A in a complex space form*. Canad. Math. Bull. **37**(1994), no. 2, 238–244.   doi:10.4153/CMB-1994-035-8

[5] I.-B. Kim, K. H. Kim and W. H. Sohn, *Characterizations of real hypersurfaces in a complex space form*. Canad. Math. Bull. **50**(2007), no. 1, 97–104.   doi:10.4153/CMB-2007-009-5

[6] H. S. Kim and Y.-S. Pyo, *On real hypersurfaces of type A in a complex space form. III*. Balkan J. Geom. Appl. **3**(1998), no. 2, 101–110.

[7] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*. Trans. Amer. Math. Soc. **296**(1986), no. 1, 137–149.   doi:10.1090/S0002-9947-1986-0837803-2

[8] M. Kimura and S. Maeda, *On real hypersurfaces of a complex projective space*. Math. Z. **202**(1989), no. 3, 299–311.   doi:10.1007/BF01159962

[9] M. Kon, *Pseudo-Einstein real hypersurfaces in complex space forms*. J. Diff. Geom. **14**(1979), no. 3, 339–354.

[10] M. Lohnherr and H. Reckziegel, *On ruled real hypersurfaces in complex space forms*. Geom. Dedicata. **74**(1999), no. 3, 267–286.   doi:10.1023/A:1005000122427

[11] Y. Maeda, *On real hypersurfaces of a complex projective space*. J. Math. Soc. Japan **28**(1976), no. 3, 529–540.   doi:10.2969/jmsj/02830529

[12] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*. Geom. Dedicata. **20**(1986), no. 2, 245–261.

[13] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms*. In : Tight and Taut Submanifolds, Math. Sci., Res. Inst. Publ. 32, Cambrigde University Press, Cambridge, 1997, pp. 233–305.

[14] M. Okumura, *Contact hypersurfaces in certain Kaehlerian manifolds*. Tohoku Math. J. **18**(1966), 74–102.   doi:10.2748/tmj/1178243483

[15] _____, On some real hypersurfaces of a complex projective space. Trans. Amer. Math. Soc. **212**(1975), 355–364.   doi:10.1090/S0002-9947-1975-0377787-X

[16] Y. J. Suh, *Characterizations of real hypersurfaces in complex space forms in terms of Weingarten map*. Nihonkai Math. J. **6**(1995), no. 1, 63–79.

[17] _____, *On real hypersurfaces of a complex space form with η-parallel Ricci tensor*. Tsukuba J. Math. **14**(1990), no. 1, 27–37.

[18] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*. Osaka J. Math. **10**(1973), 495–506.

[19] M. H. Vernon, *Contact hypersurfaces of a complex hyperbolic space*. Tohoku Math. J. **39**(1987), no. 2, 215–222.   doi:10.2748/tmj/1178228324

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