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Conditional intermediate entropy and Birkhoff average properties of hyperbolic flows

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Abstract. Katok [Lyapunov exponents, entropy and periodic points of diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **51** (1980), 137–173] conjectured that every C^2 diffeomorphism f on a Riemannian manifold has the intermediate entropy property, that is, for any constant $c \in [0, h_{\mathrm{top}}(f))$, there exists an ergodic measure μ of f satisfying $h_\mu(f) = c$. In this paper, we obtain a conditional intermediate metric entropy property and two conditional intermediate Birkhoff average properties for basic sets of flows that characterize the refined roles of ergodic measures in the invariant ones. In this process, we establish a 'multi-horseshoe' entropy-dense property and use it to get the goal combined with conditional variational principles. We also obtain the same result for singular hyperbolic attractors.

Key words: hyperbolic sets, singular hyperbolic attractor, metric entropy, horseshoe, symbolic dynamics
2020 Mathematics Subject Classification: 37A25, 37C10 (Primary); 37D20, 37C70, 37B10 (Secondary)

1. *Introduction*

In [22], Katok showed that every $C^{1+\alpha}$ diffeomorphism f in dimension 2 has horseshoes of large entropies. This implies that the system has the intermediate entropy property, that is, for any constant $c \in [0, h_{\mathrm{top}}(f))$, there exists an ergodic measure μ of f satisfying $h_\mu(f) = c$, where $h_{\mathrm{top}}(f)$ is the topological entropy of f and $h_\mu(f)$ is the metric entropy of μ. Katok believed that this holds in any dimension. In the last decade, a number of partial results on the intermediate entropy property have been obtained, see [10, 17, 21, 23, 24, 26, 27, 32, 35, 37].

Conjecture 1.1. (Katok) Every C^2 diffeomorphism f on a Riemannian manifold has the intermediate entropy property.

In [36], the authors consider the intermediate Birkhoff average. To be precise, they proved that if $f : X \to X$ is a continuous map over a compact metric space X with the periodic gluing orbit property, then there is an ergodic measure $\mu_\alpha \in \mathcal{M}(f, X)$ such that $\int g\, d\mu_\alpha = \alpha$ for any continuous function $g : X \to \mathbb{R}$ and any constant α satisfying

$$\inf_{\mu\in\mathcal{M}(f,X)} \int g\, d\mu < \alpha < \sup_{\mu\in\mathcal{M}(f,X)} \int g\, d\mu,$$

where $\mathcal{M}(f, X)$ is the set of all f-invariant probability measures. (The author obtained the same result for the asymptotically additive functions, readers can refer to [36] for details.)

Imitating the relationships between extremum and conditional extremum, variational principle, and conditional variational principle, we consider the following question.

Question 1.1. Under certain restricted conditions, do the intermediate entropy property and intermediate Birkhoff average property hold?

In fact, a certain restricted condition determines a subset of $\mathcal{M}(f, X)$. So in other words, Question 1.1 means that given a subset F of $\mathcal{M}(f, X)$, does one have

$$\{h_\mu(f) : \mu \in F\} = \{h_\mu(f) : \mu \in F \cap \mathcal{M}_{\mathrm{erg}}(f, X)\} \tag{1.1}$$

and

$$\left\{\int g\, d\mu : \mu \in F\right\} = \left\{\int g\, d\mu : \mu \in F \cap \mathcal{M}_{\mathrm{erg}}(f, X)\right\}? \tag{1.2}$$

Here $\mathcal{M}_{\mathrm{erg}}(f, X)$ is the set of all f-invariant ergodic probability measures.

It is clear that Question 1.1 can only be true under some suitable restricted condition or some suitable F. For example, if $F \cap \mathcal{M}_{\mathrm{erg}}(f, X) = \emptyset$, then Question 1.1 is obviously false. When $F = \mathcal{M}(f, X)$, equation (1.1) is true for every $C^{1+\alpha}$ diffeomorphism f in dimension 2 [22] and equation (1.2) is true for a system satisfying the periodic gluing orbit property [36]. In the two proofs, a basic requirement is that $\mathcal{M}_{\mathrm{erg}}(f, X)$ is dense in $\mathcal{M}(f, X)$. So we believe that a suitable F should satisfy the following properties:

(1) $\mathcal{M}_{\mathrm{erg}}(f, X) \cap F$ is dense in F;

(2) F is convex.

We will see that the following two sets satisfy items (1) and (2) for hyperbolic sets:

$$F_1(c) = \{\mu \in \mathcal{M}(f, X) : h_\mu(f) = c\}, \qquad F_2^g(\alpha) = \left\{\mu \in \mathcal{M}(f, X) : \int g\, d\mu = \alpha\right\},$$

where $c \in [0, h_{\mathrm{top}}(f))$ and $\inf_{\mu\in\mathcal{M}(f,X)} \int g\, d\mu < \alpha < \sup_{\mu\in\mathcal{M}(f,X)} \int g\, d\mu$. Additionally, for two continuous functions g, h, we will consider whether the following three equalities hold: a conditional intermediate entropy property

$$\{h_\mu(f) : \mu \in F_2^g(\alpha)\} = \{h_\mu(f) : \mu \in F_2^g(\alpha) \cap \mathcal{M}_{\mathrm{erg}}(f, X)\}, \tag{1.3}$$

and two conditional intermediate Birkhoff average properties

$$\left\{\int g\,d\mu : \mu \in F_1(c)\right\} = \left\{\int g\,d\mu : \mu \in F_1(c) \cap \mathcal{M}_{\rm erg}(f, X)\right\}, \tag{1.4}$$

$$\left\{\int h\,d\mu : \mu \in F_2^g(\alpha)\right\} = \left\{\int h\,d\mu : \mu \in F_2^g(\alpha) \cap \mathcal{M}_{\rm erg}(f, X)\right\}. \tag{1.5}$$

Note that if the following two sets are equal,

$$\mathrm{Int}\left\{\left(\int g\,d\mu, h_\mu(f)\right) : \mu \in \mathcal{M}(f, X)\right\},$$
$$\mathrm{Int}\left\{\left(\int g\,d\mu, h_\mu(f)\right) : \mu \in \mathcal{M}_{\rm erg}(f, X)\right\},$$

then equations (1.3) and (1.4) hold except extremums for any $c \in [0, h_{\rm top}(f))$ and $\inf_{\mu\in\mathcal{M}(f,X)}\int g\,d\mu < \alpha < \sup_{\mu\in\mathcal{M}(f,X)} \int g\,d\mu$. Additionally, if the following two sets are equal,

$$\mathrm{Int}\left\{\left(\int g\,d\mu, \int h\,d\mu\right) : \mu \in \mathcal{M}(f, X)\right\},$$
$$\mathrm{Int}\left\{\left(\int g\,d\mu, \int h\,d\mu\right) : \mu \in \mathcal{M}_{\rm erg}(f, X)\right\},$$

then equation (1.5) holds except extremums for any $\inf_{\mu\in\mathcal{M}(f,X)} \int g\,d\mu < \alpha < \sup_{\mu\in\mathcal{M}(f,X)} \int g\,d\mu$.

In this paper, we are interested in flows. We first recall some notions of flows. Let $\mathscr{X}^r(M)$, $r \geq 1$, denote the space of C^r-vector fields on a compact Riemannian manifold M endowed with the C^r topology. For $X \in \mathscr{X}^r(M)$, denote by ϕ_t^X or ϕ_t for simplicity the C^r-flow generated by X and denote by $\mathrm{D}\phi_t$ the tangent map of ϕ_t. Given a vector field $X \in \mathcal{X}^1(M)$ and a compact invariant subset Λ of the C^1-flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ generated by X, we denote by $C(\Lambda, \mathbb{R})$ the space of continuous functions on Λ. The set of invariant (respectively ergodic) probability measures of X supported on Λ is denoted by $\mathcal{M}(\Phi, \Lambda)$ (respectively $\mathcal{M}_{\rm erg}(\Phi, \Lambda)$), and it is endowed with the weak*-topology. We denote by $h_\mu(X)$ or $h_\mu(\Phi)$ the metric entropy of the invariant probability measure $\mu \in \mathcal{M}(\Phi, \Lambda)$, defined as the metric entropy of μ with respect to the time-1 map ϕ_1^X of the flow. Let d^* be a translation invariant metric on the space $\mathcal{M}(\Phi, \Lambda)$ compatible with the weak* topology. We focus on the following question for flows in the present paper.

Question 1.2. For every flow $(\phi_t)_{t\in\mathbb{R}}$ generated by a typical vector field X, and two continuous functions g, h on a compact invariant subset Λ of the C^1 flow $(\phi_t)_{t\in\mathbb{R}}$, do the following two equalities hold:

$$\mathrm{Int}\left\{\left(\int g\,d\mu, h_\mu(\Phi)\right) : \mu \in \mathcal{M}(\Phi, \Lambda)\right\} = \mathrm{Int}\left\{\left(\int g\,d\mu, h_\mu(\Phi)\right) : \mu \in \mathcal{M}_{\rm erg}(\Phi, \Lambda)\right\}$$

and

$$\mathrm{Int}\left\{\left(\int g\,d\mu, \int h\,d\mu\right) : \mu\in\mathcal{M}\Big(\Phi, \Lambda\Big)\right\}=\mathrm{Int}\left\{\left(\int g\,d\mu, \int h\,d\mu\right) : \mu \in \mathcal{M}_{\rm erg}\Big(\Phi, \Lambda\Big)\right\}?$$

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We will answer this question partially for hyperbolic flows and singular hyperbolic flows.

Definition 1.3. Given $X \in \mathscr{X}^1(M)$, an invariant compact set Λ is called a *basic set* if it is transitive, hyperbolic, and locally maximal; is not reduced to a single orbit of a hyperbolic critical element; and its intersection with any local cross-section to the flow is totally disconnected.

Given a continuous function g on a compact invariant subset Λ of the C^1 flow $(\phi_t)_{t\in\mathbb{R}}$, denote $L_g = \{\int g\, d\mu : \mu \in \mathcal{M}(\Phi, \Lambda)\}$. For any $\alpha \in L_g$, denote

$$M_g(\alpha) = \left\{\mu \in \mathcal{M}(\Phi, \Lambda) : \int g\, d\mu = \alpha\right\}, \qquad M_g^{\mathrm{erg}}(\alpha) = M_g(\alpha) \cap \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda)$$

and

$$M_g^{\mathrm{top}}(\alpha) = \{\mu \in M_g(\alpha) : h_\mu(\Phi) = \sup\{h_\nu(\Phi) : \nu \in M_g(\alpha)\}\}.$$

Then $M_g(\alpha)$ is a closed subset of $\mathcal{M}(\Phi, \Lambda)$ and thus

$$\sup\{h_\nu(\Phi) : \nu \in M_g(\alpha)\} = \max\{h_\nu(\Phi) : \nu \in M_g(\alpha)\}$$

for any $\alpha \in L_g$ if the entropy function $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous. For a probability measure $\mu \in \mathcal{M}(\Phi, \Lambda)$, we denote the support of μ by

$$S_\mu := \{x \in \Lambda : \mu(U) > 0 \text{ for any neighborhood } U \text{ of } x\}.$$

Now we state our first main result.

THEOREM A. *Let $X \in \mathscr{X}^1(M)$ and Λ be a basic set of X. If g is a continuous function on Λ, then for any $\alpha \in \mathrm{Int}(L_g)$, any $\mu \in M_g(\alpha) \setminus M_g^{\mathrm{top}}(\alpha)$, any $0 \le c \le h_\mu(\Phi)$, and any $\zeta > 0$, there is $\nu \in M_g^{\mathrm{erg}}(\alpha)$ such that $d^*(\nu, \mu) < \zeta$ and $h_\nu(\Phi) = c$. Moreover, for any $\alpha \in \mathrm{Int}(L_g)$ and $0 \le c < \max\{h_\mu(\Phi) : \mu \in M_g(\alpha)\}$, the set $\{\mu \in M_g^{\mathrm{erg}}(\alpha) : h_\mu(\Phi) = c$, $S_\mu = \Lambda\}$ is residual in $\{\mu \in M_g(\alpha) : h_\mu(\Phi) \ge c\}$.*

Remark 1.4. Let $X \in \mathscr{X}^1(M)$ and Λ be a basic set of X. For a continuous function g, let $g_t = \int_0^t g(\phi_\tau(x))\, d\tau$, then $(g_t)_{t\ge 0}$ is an additive family of continuous functions. So let $\chi \equiv 0$ and $d = 1$ in Theorem 4.1(iv), we have

$$\mathrm{Int}\left\{\left(\int g\, d\mu, h_\mu(\Phi)\right) : \mu \in \mathcal{M}(\Phi, \Lambda)\right\} = \mathrm{Int}\left\{\left(\int g\, d\mu, h_\mu(\Phi)\right) : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda)\right\}. \tag{1.6}$$

We draw the graph of $\left(\int g\, d\mu, h_\mu(\Phi)\right)$ in Figure 1. Then by equation (1.6), every point in the interior of the region can be attained by ergodic measures. From equation (1.6), we obtain one conditional intermediate metric entropy property:

$$\mathrm{Int}\left\{h_\mu(\Phi) : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda) \text{ and } \int g\, d\mu = \alpha\right\}$$
$$= \mathrm{Int}\left\{h_\mu(\Phi) : \mu \in \mathcal{M}(\Phi, \Lambda) \text{ and } \int g\, d\mu = \alpha\right\},$$

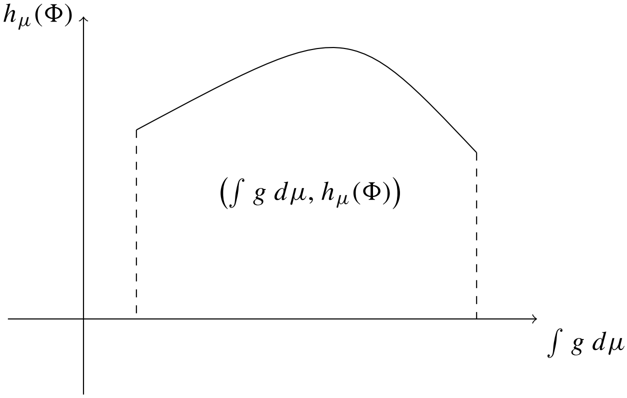

FIGURE 1. Graph of $\big(\int g\,d\mu, h_\mu(\Phi)\big)$.

and one conditional intermediate Birkhoff average property:

$$\begin{aligned}&\mathrm{Int}\bigg\{\int g\,d\mu : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda) \text{ and } h_\mu(\Phi) = c\bigg\}\\ &\quad = \mathrm{Int}\bigg\{\int g\,d\mu : \mu \in \mathcal{M}(\Phi, \Lambda) \text{ and } h_\mu(\Phi) = c\bigg\}\end{aligned}$$

for any $\alpha \in \mathrm{Int}(L_g)$ and any $c \in (0, h_{\mathrm{top}}(\Lambda))$.

Remark 1.5. Given $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, $\mathcal{M}(\Phi, \Lambda)$ is said to have entropy-dense property if for any $\varepsilon > 0$ and any $\mu \in \mathcal{M}(\Phi, \Lambda)$, there exists $\upsilon \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda)$ satisfying

$$d^*(\mu, \upsilon) < \varepsilon \quad \text{and} \quad h_\nu(\Phi) > h_\mu(\Phi) - \varepsilon.$$

Given a continuous function g on Λ, we say $\mathcal{M}(\Phi, \Lambda)$ has entropy-dense property with the same g-level if for any $\alpha \in \mathrm{Int}(L_g)$, any $\mu \in M_g(\alpha)$, and any $\varepsilon > 0$, there is $\nu \in M_g^{\mathrm{erg}}(\alpha)$ such that

$$d^*(\mu, \upsilon) < \varepsilon \quad \text{and} \quad h_\nu(\Phi) > h_\mu(\Phi) - \varepsilon.$$

By Theorem A, if Λ is a basic set of X and g is a continuous function on Λ, then $\mathcal{M}(\Phi, \Lambda)$ has entropy-dense property with the same g-level. Moreover, we can choose $\nu \in M_g^{\mathrm{erg}}(\alpha)$ such that $d^*(\mu, \upsilon) < \varepsilon$ and $h_\nu(\Phi) = h_\mu(\Phi)$ if $\mu \in M_g(\alpha) \setminus M_g^{\mathrm{top}}(\alpha)$.

Given two continuous functions g, h on a compact invariant subset Λ of the C^1 flow $(\phi_t)_{t\in\mathbb{R}}$, we denote

$$L_{g,h} = \bigg\{\bigg(\int g\,d\mu, \int h\,d\mu\bigg) : \mu \in \mathcal{M}(\Phi, \Lambda)\bigg\}.$$

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Then $L_{g,h}$ is a non-empty convex compact subset of $\mathbb{R}^2$. For any $\alpha \in L_{g,h}$, denote

$$M_{g,h}(\alpha) = \bigg\{\mu \in \mathcal{M}(\Phi, \Lambda) : \bigg(\int g\, d\mu, \int h\, d\mu\bigg) = \alpha\bigg\},\ M^{\mathrm{erg}}_{g,h}(\alpha) = M_{g,h}(\alpha) \cap \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda).$$

Now we state our second main result.

THEOREM B. *Let $X \in \mathscr{X}^1(M)$ and Λ be a basic set of X. If g, h are continuous functions on Λ, then for any $\alpha \in \mathrm{Int}(L_{g,h})$, any $\mu \in M_{g,h}(\alpha)$, and any $\zeta > 0$, there is $\nu \in M^{\mathrm{erg}}_{g,h}(\alpha)$ such that $d^*(\nu, \mu) < \zeta$. Moreover, for any $\alpha \in \mathrm{Int}(L_{g,h})$, the set $\{\mu \in M^{\mathrm{erg}}_{g,h}(\alpha) : S_\mu = \Lambda\}$ is residual in $M_{g,h}(\alpha)$.*

Remark 1.6. Let $X \in \mathscr{X}^1(M)$ and Λ be a basic set of X. If g, h are continuous functions on Λ, then by Theorem B, we have

$$\mathrm{Int}\bigg\{\bigg(\int g\, d\mu, \int h\, d\mu\bigg) : \mu \in \mathcal{M}(\Phi, \Lambda)\bigg\} = \mathrm{Int}\bigg\{\bigg(\int g\, d\mu, \int h\, d\mu\bigg) : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda)\bigg\}.$$

Then we obtain a conditional intermediate Birkhoff average property:

$$\mathrm{Int}\bigg\{\int h\, d\mu : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda) \text{ and } \int g\, d\mu = \alpha\bigg\} = \mathrm{Int}\bigg\{\int h\, d\mu : \mu \in \mathcal{M}(\Phi, \Lambda) \text{ and } \int g\, d\mu = \alpha\bigg\}$$

for any $\alpha \in \mathrm{Int}(L_g)$.

In the process of proving Theorems A and B, there are two keypoints: 'multi-horseshoe' entropy-dense property (see Theorem 2.5) and conditional variational principles (see Theorems 3.5 and 3.7) proved by Barreira and Holanda in [**5**, **20**].

1.1. *Outline of the paper.* In §2, we introduce 'multi-horseshoe' entropy-dense property for flows and prove that it holds for basic sets. In §3, we recall non-additive thermodynamic formalism for flows, give abstract conditions on which the results of Theorems A and B hold in the more general context of asymptotically additive families of continuous functions. In §4, using the 'multi-horseshoe' entropy-dense property, we show that the abstract conditions given in §3 are satisfied for basic sets, and thus we obtain Theorems A and B. In §5, we consider singular hyperbolic attractors and give corresponding results on Question 1.2.

2. *'Multi-horseshoe' entropy-dense property*

In this section, we introduce the 'multi-horseshoe' entropy-dense property and prove it holds for basic sets. We first recall the definition of hyperbolicity.

Definition 2.1. Given a vector field $X \in \mathscr{X}^1(M)$, a compact ϕ_t-invariant set Λ is *hyperbolic* if Λ admits a continuous $D\phi_t$-invariant splitting $T_\Lambda M = E^s \oplus \langle X\rangle \oplus E^u$, where $\langle X\rangle$ denotes the one-dimensional linear space generated by the vector field, and

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E^s (respectively E^u) is uniformly contracted (respectively expanded) by $D\phi_t$, that is to say, there exist constants $C > 0$ and $\eta > 0$ such that for any $x \in \Lambda$ and any $t \geq 0$:

- $\|D\phi_t(v)\| \leq Ce^{-\eta t}\|v\|$ for any $v \in E^s(x)$; and
- $\|D\phi_{-t}(v)\| \leq Ce^{-\eta t}\|v\|$ for any $v \in E^u(x)$,

for any $x \in \Lambda$ and $t \geq 0$. A hyperbolic set Λ is said to be *locally maximal* if there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t(U))$.

Now we give the definition of a horseshoe.

Definition 2.2. Given $X \in \mathscr{X}^1(M)$ and a hyperbolic invariant compact set Λ, we call Λ a *horseshoe* of ϕ_t if there exists a suspension flow $f_t : \Delta_\rho \to \Delta_\rho$ over a transitive subshift of finite type (Δ, σ) with a Lipschitz roof function ρ and a homeomorphism $\pi : \Delta_\rho \to \Lambda$ such that $\pi \circ f_t = \phi_t \circ \pi$.

Remark 2.3. Lipschitz continuity of ρ is used in Lemma 3.1.

For any $m \in \mathbb{N}$ and $\{\nu_i\}_{i=1}^m \subseteq \mathcal{M}(\Phi, \Lambda)$, we denote the convex combination of $\{\nu_i\}_{i=1}^m$ by

$$\mathrm{cov}\{\nu_i\}_{i=1}^m := \left\{ \sum_{i=1}^m \theta_i\nu_i : \theta_i \in [0, 1], 1 \leq i \leq m \text{ and } \sum_{i=1}^m \theta_i = 1\right\}.$$

For any non-empty subsets of $\mathcal{M}(\Phi, \Lambda)$, A and B, we denote the Hausdorff distance between them by

$$d_H(A, B) := \max\left\{\sup_{x\in A}\inf_{y\in B} d^*(x, y), \sup_{y\in B}\inf_{x\in A} d^*(y, x)\right\}.$$

We denote by $h_{\mathrm{top}}(\Lambda)$ the topological entropy of Λ, defined as the topological entropy of Λ with respect to the time-1 map ϕ_1 of the flow.

Definition 2.4. Given $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, let $\mathcal{N} \subset \mathcal{M}(\Phi, \Lambda)$ be a non-empty set. We say Λ satisfies the *'multi-horseshoe' entropy-dense property on $\mathcal{N}$* (abbreviated *'multi-horseshoe' dense property*) if for any $F = \mathrm{cov}\{\mu_i\}_{i=1}^m \subseteq \mathcal{N}$ and any $\eta, \zeta > 0$, there exist compact invariant subsets $\Lambda_i \subseteq \Theta \subsetneq \Lambda$ such that for each $1 \leq i \leq m$:

(1) Λ_i and Θ are horseshoes;

(2) $h_{\mathrm{top}}(\Lambda_i) > h_{\mu_i}(X) - \eta$ and consequently, $h_{\mathrm{top}}(\Theta) > \sup\{h_\kappa(X) : \kappa \in F\} - \eta$;

(3) $d_H(F, \mathcal{M}(\Phi, \Theta)) < \zeta$, $d_H(\mu_i, \mathcal{M}(\Phi, \Lambda_i)) < \zeta$.

For convenience, when $\mathcal{N} = \mathcal{M}(\Phi, \Lambda)$, we say Λ satisfies the 'multi-horseshoe' entropy-dense property.

THEOREM 2.5. *Let $X \in \mathscr{X}^1(M)$ and Λ be a basic set of X. Then Λ satisfies the 'multi-horseshoe' dense property.*

We will prove Theorem 2.5 by three steps. In the process, we give various versions of ther 'multi-horseshoe' dense property on transitive subshifts of finite type, suspension flows over transitive subshifts of finite type, and basic sets.

2.1. *Transitive subshifts of finite type.* For a homeomorphism $f : K \to K$ of a compact metric space (K, d), we denote by $\mathcal{M}(f, K)$ the space of f-invariant probability measures. Additionally, let $h_{\mathrm{top}}(f, \Lambda)$ denote the topological entropy of an f-invariant compact set Λ and $h_\mu(f)$ denote the metric entropy of an f-invariant measure. In [15], a result on the 'multi-horseshoe' dense property of homeomorphisms was obtained.

THEOREM 2.6. [15, Theorem 2.6] *Suppose a homeomorphism $f : K \to K$ of a compact metric space (K, d) is transitive, expansive, and satisfies the shadowing property. Then for any $F = \mathrm{cov}\{\mu_i\}_{i=1}^m \subseteq \mathcal{M}(f, K)$, any $x \in K$, and any $\eta, \zeta, \varepsilon > 0$, there exist compact invariant subsets $\Lambda_i \subseteq \Xi \subsetneq K$ such that for each $1 \le i \le m$:*

(1) *$(\Lambda_i, f|_{\Lambda_i})$ and $(\Xi, f|_\Xi)$ are transitive, expansive, and satisfy the shadowing property;*
(2) *$h_{\mathrm{top}}(f, \Lambda_i) > h_{\mu_i}(f) - \eta$ and, consequently, $h_{\mathrm{top}}(f, \Xi) > \sup\{h_\kappa(f) : \kappa \in F\} - \eta$;*
(3) *$d_H(F, \mathcal{M}(f, \Xi)) < \zeta$, $d_H(\mu_i, \mathcal{M}(f, \Lambda_i)) < \zeta$;*
(4) *there is a positive integer L such that for any $z \in \Xi$, one has $d(f^{j+mL}(z), x) < \varepsilon$ for some $0 \le j \le L - 1$ and any $m \in \mathbb{Z}$.*

Let k be a positive integer. Consider the two-side full symbolic space

$$\Sigma = \prod_{-\infty}^{\infty}\{0, 1, \ldots, k - 1\},$$

and the shift homeomorphism $\sigma : \Sigma \to \Sigma$ defined by

$$(\sigma(w))_n = w_{n+1},$$

where $w = (w_n)_{-\infty}^{\infty}$. A metric on Σ is defined by $d(x, y) = 2^{-m}$ if m is the largest positive integer with $x_n = y_n$ for any $|n| < m$, and $d(x, y) = 1$ if $x_0 \neq y_0$. If Δ is a closed subset of Σ with $\sigma(\Delta) = \Delta$, then $\sigma|_\Delta : \Lambda \to \Delta$ is called a subshift. We usually write this as $\sigma : \Delta \to \Delta$. A subshift $\sigma : \Delta \to \Delta$ is said to be of finite type if there exists some positive integer N and a collection of blocks of length $N + 1$ with the property that $x = (x_n)_{-\infty}^{\infty} \in \Delta$ if and only if each block $(x_i, \ldots, x_{i+N})$ in x of length $N + 1$ is one of the prescribed blocks.

Recall from [38] a subshift satisfies the shadowing property if and only if it is a subshift of finite type. As a subsystem of two-side full shift, it is expansive. So we have the following corollary.

COROLLARY 2.7. *Suppose $\sigma : \Delta \to \Delta$ is a transitive subshift of finite type. Then for any $F = \mathrm{cov}\{\mu_i\}_{i=1}^m \subseteq \mathcal{M}(\sigma, \Delta)$, any $x \in \Delta$, and any $\eta, \zeta, \varepsilon > 0$, there exist compact invariant subsets $\Delta_i \subseteq \Xi \subsetneq \Delta$ such that for each $1 \le i \le m$:*

(1) *$(\Delta_i, \sigma|_{\Delta_i})$ and $(\Xi, \sigma|_\Xi)$ are transitive subshifts of finite type.*
(2) *$h_{\mathrm{top}}(\sigma, \Delta_i) > h_{\mu_i}(\sigma) - \eta$ and consequently, $h_{\mathrm{top}}(\sigma, \Xi) > \sup\{h_\kappa(\sigma) : \kappa \in F\} - \eta$;*
(3) *$d_H(F, \mathcal{M}(\sigma, \Xi)) < \zeta$, $d_H(\mu_i, \mathcal{M}(\sigma, \Delta_i)) < \zeta$;*
(4) *there is a positive integer L such that for any $z \in \Xi$, one has $d(\sigma^{j+mL}(z), x) < \varepsilon$ for some $0 \le j \le L - 1$ and any $m \in \mathbb{Z}$.*

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2.2. *Suspension flows over transitive subshifts of finite type.* Let $f: K \to K$ be a homeomorphism on a compact metric space (K, d) and consider a continuous roof function $\rho: K \to (0, +\infty)$. We define the suspension space to be

$$K_\rho = \{(x, s) \in K \times [0, +\infty): 0 \le s \le \rho(x)\}/\sim,$$

where the equivalence relation $\sim$ identifies $(x, \rho(x))$ with $(f(x), 0)$ for all $x \in K$. Denote π the quotient map from $K \times [0, +\infty)$ to K_ρ. We define the suspension flow over $f: K \to K$ with roof function ρ by

$$f_t(x, s) = \pi(x, s + t).$$

For any function $g: K_\rho \to \mathbb{R}$, we associate the function $\varphi_g: K \to \mathbb{R}$ by

$$\varphi_g(x) = \int_0^{\rho(x)} g(x, t)\, dt.$$

Since the roof function ρ is continuous, φ_g is continuous as long as g is. Moreover, to each invariant probability measure μ, we associate the measure μ_ρ given by

$$\int_{K_\rho} g\, d\mu_\rho = \frac{\int_K \varphi_g\, d\mu}{\int_K \rho\, d\mu} \quad \text{for all } g \in C(K_\rho, \mathbb{R}).$$

Observe that not only the measure μ_ρ is $\mathfrak{F}$-invariant (that is, $\mu_\rho(f_t^{-1}A) = \mu_\rho(A)$ for all $t \ge 0$ and measurable sets A), but also using that ρ is bounded away from zero, the map

$$\mathcal{R}: \mathcal{M}(f, K) \to \mathcal{M}(\mathfrak{F}, K_\rho) \quad \text{given by} \quad \mu \mapsto \mu_\rho$$

is a bijection. Abramov's theorem [1, 31] states that $h_{\mu_\rho}(\mathfrak{F}) = h_\mu(f)/\int \rho\, d\mu$ and hence, the topological entropy $h_{\text{top}}(\mathfrak{F})$ of the flow satisfies

$$h_{\text{top}}(\mathfrak{F}) = \sup\{h_{\mu_\rho}(\mathfrak{F}): \mu_\rho \in \mathcal{M}(\mathfrak{F}, K_\rho)\} = \sup\left\{\frac{h_\mu(f)}{\int \rho\, d\mu}: \mu \in \mathcal{M}(f, K)\right\}.$$

Throughout, we will use the notation $\Phi = (\phi_t)_t$ for a flow on a compact metric space and $\mathfrak{F} = (f_t)_t$ for a suspension flow.

Consider a suspension flow $f_t : \Delta_\rho \to \Delta_\rho$ over a transitive subshift of finite type (Δ, σ) with a continuous roof function ρ. A metric on Δ is defined by $d(x, y) = 2^{-m}$ if m is the largest positive integer with $x_n = y_n$ for any $|n| < m$, and $d(x, y) = 1$ if $x_0 \neq y_0$. There is a natural metric d_{Δ_ρ}, known as the Bowen–Walters metric and we have $d_{\Delta_\rho}((x, 0), (y, 0)) = d(x, y)$ for any $x, y \in \Delta$ by [6]. Now we state a result on the 'multi-horseshoe' dense property of suspension flows.

PROPOSITION 2.8. *Suppose $f_t : \Delta_\rho \to \Delta_\rho$ is a suspension flow over a transitive subshift of finite type (Δ, σ) with a continuous roof function ρ. Then for any $F = \operatorname{cov}\{\mu_\rho^i\}_{i=1}^m \subseteq \mathcal{M}(\mathfrak{F}, \Delta_\rho)$, any $x \in \Delta$, and any $\eta, \zeta, \varepsilon > 0$, there exist compact $\mathfrak{F}$-invariant subsets $\Delta_\rho^i \subseteq \Xi_\rho \subsetneq \Delta_\rho$ such that for each $1 \le i \le m$:*

(1) *$f_t|_{\Delta_\rho^i} : \Delta_\rho^i \to \Delta_\rho^i$ and $f_t|_{\Xi_\rho} : \Xi_\rho \to \Xi_\rho$ are suspension flows over transitive subshifts of finite type $(\Delta_i, \sigma|_{\Delta_i})$ and $(\Xi, \sigma|_\Xi)$ with the roof function ρ;*

(2) *$h_{\text{top}}(\Delta_\rho^i) > h_{\mu_\rho^i}(\mathfrak{F}) - \eta$ and consequently, $h_{\text{top}}(\Xi_\rho) > \sup\{h_\kappa(\mathfrak{F}) : \kappa \in F\} - \eta$;*

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(3) $d_H(F, \mathcal{M}(\mathfrak{F}, \Xi_\rho)) < \zeta$, $d_H(\mu_\rho^i, \mathcal{M}(\mathfrak{F}, \Delta_\rho^i)) < \zeta$;

(4) *for any* $z \in \Xi$, *there is an increasing sequence* $\{t_m\}_{m=-\infty}^{+\infty}$ *such that* $f_{t_m}((z, 0)) \in \Delta \times \{0\}$, $d_{\Delta_\rho}(f_{t_m}((z, 0)), (x, 0)) < \varepsilon$ *for any* $m \in \mathbb{Z}$, *and* $\lim_{m\to+\infty} t_m = \lim_{m\to-\infty} -t_m = +\infty$.

Proof. Each $\mathfrak{F}$-invariant probability measure μ_ρ^i is determined by a σ-invariant probability measure μ_i. Take $\tilde{\zeta}, \tilde{\eta} > 0$ small enough such that if $\mu, \nu \in \mathcal{M}(\sigma, \Delta)$ satisfies $d^*(\mu, \nu) < \tilde{\zeta}$, then one has

$$d^*(\mathcal{R}(\mu), \mathcal{R}(\nu)) < \zeta \quad \text{and} \quad \frac{\int \rho \, d\mu}{\int \rho \, d\nu}\bigg(h_{\mathcal{R}(\mu)}(\mathfrak{F}) - \frac{2\tilde{\eta}}{\int \rho \, d\mu}\bigg) > h_{\mathcal{R}(\mu)}(\mathfrak{F}) - \eta. \tag{2.1}$$

For the $\tilde{F} = \mathrm{cov}\{\mu_i\}_{i=1}^m \subseteq \mathcal{M}(\sigma, \Delta)$, $x \in \Delta$ and $\tilde{\eta}, \tilde{\zeta}, \varepsilon > 0$, by Corollary 2.7, there exist compact invariant subsets $\Delta_i \subseteq \Xi \subsetneq \Delta$ such that for each $1 \le i \le m$:

(a) $(\Delta_i, \sigma|_{\Delta_i})$ and $(\Xi, \sigma|_\Xi)$ are transitive subshifts of finite type;

(b) $h_{\mathrm{top}}(\sigma, \Delta_i) > h_{\mu_i}(\sigma) - \tilde{\eta}$ and consequently, $h_{\mathrm{top}}(\sigma, \Xi) > \sup\{h_\kappa(\sigma) : \kappa \in F\} - \tilde{\eta}$;

(c) $d_H(\tilde{F}, \mathcal{M}(\sigma, \Xi)) < \tilde{\zeta}$, $d_H(\mu_i, \mathcal{M}(\sigma, \Delta_i)) < \tilde{\zeta}$;

(d) there is a positive integer L such that for any $z \in \Xi$, one has $d(\sigma^{j+mL}(z), x) < \varepsilon$ for some $0 \le j \le L - 1$ and any $m \in \mathbb{Z}$.

Then by items (a) and (d), we have items (1) and (4). By equation (2.1) and item (c), we have $d_H(\mu_\rho^i, \mathcal{M}(\mathfrak{F}, \Delta_\rho^i)) < \zeta$ and thus F is contained in the ζ-neighborhood of $\mathcal{M}(\mathfrak{F}, \Xi_\rho)$. Note that for any $\theta_i \in [0, 1]$ with $\sum_{i=1}^m \theta_i = 1$, one has

$$\mathcal{R}\bigg(\sum_{i=1}^m \theta_i \mu_i\bigg) = \sum_{i=1}^m \frac{\theta_i \int \rho \, d\mu_i}{\sum_{j=1}^m \theta_j \int \rho \, d\mu_j} \mathcal{R}(\mu_i).$$

So $\mathcal{M}(\mathfrak{F}, \Xi_\rho)$ is contained in the ζ-neighborhood of F and thus we have item (3).

By the variational principle of the topological entropy, there is $\nu_i \in \mathcal{M}(\sigma, \Delta_i)$ such that $h_{\nu_i}(\sigma) > h_{\mathrm{top}}(\sigma, \Delta_i) - \tilde{\eta} > h_{\mu_i}(\sigma) - 2\tilde{\eta}$. Then

$$h_{\mathrm{top}}(\Delta_\rho^i) \ge h_{\mathcal{R}(\nu_i)}(\mathfrak{F}) = \frac{h_{\nu_i}(\sigma)}{\int \rho \, d\nu_i} > \frac{\int \rho \, d\mu_i}{\int \rho \, d\nu_i} \cdot \bigg(h_{\mu_\rho^i}(\mathfrak{F}) - \frac{2\tilde{\eta}}{\int \rho \, d\mu_i}\bigg) > h_{\mu_\rho^i}(\mathfrak{F}) - \eta$$

by equation (2.1). We obtain item (2). □

2.3. *Basic sets and proof of Theorem 2.5.* Following the classical arguments of Bowen [7, 8] on Axiom A vector fields, every basic set is semi-conjugate to a suspension flow over a transitive subshift of finite type with a continuous roof function. Now we recall some basic results from [7, 8]. Readers can also see [6].

Given $X \in \mathscr{X}^1(M)$ and a basic set Λ, by [8, Theorem 2.5], there is a family of closed sets $R_1, \ldots, R_k$ and a positive number α so that the following properties hold:

(1) $\Lambda = \bigcup_{t\in[-\alpha,0]} \phi_t(\bigcup_{i=1}^k R_i)$;

(2) there exists a transitive subshift of finite type (Δ, σ) and a continuous and onto map $\pi : \Delta \to \bigcup_{i=1}^k R_i$ such that $\pi \circ \sigma = T \circ \pi$ (T is the transfer map whose definition will be recalled later);

(3) π is one-to-one off $\bigcup_{n\in\mathbb{Z}} \sigma^n(\pi^{-1}(\bigcup_{i=1}^k \partial R_i))$.

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By item (1), we define the transfer function $\tau : \Lambda \to [0, \infty)$ by

$$\tau(x) = \min\left\{t > 0 : \phi_t(x) \in \bigcup_{i=1}^{k} R_i\right\}.$$

There is a positive number β such that $\tau(x) \geq \beta$ for any $x \in \Lambda$. Additionally, there is a metric on Δ such that $\tau \circ \pi$ is Lipschitz. Let $T : \Lambda \to \bigcup_{i=1}^{k} R_i$ be the transfer map given by $T(x) = \phi_{\tau(x)}(x)$. The restriction of T to $\bigcup_{i=1}^{k} R_i$ is invertible and $\bigcup_{i=1}^{k} R_i$ is a T-invariant set. We can obtain a suspension flow $f_t : \Delta_\rho \to \Delta_\rho$ over the transitive subshift of finite type (Δ, σ) with Lipschitz roof function $\rho = \tau \circ \pi$. Then we extend π to a finite-to-one surjection $\pi : \Delta_\rho \to \Lambda$ by $\pi(x, s) = (\phi_s \circ \pi)(x)$ for every $(x, s) \in \Delta_\rho$. We have

$$\pi \circ f_t = \phi_t \circ \pi$$

and π is one-to-one off $\bigcup_{t\in\mathbb{R}} f_t(\pi^{-1}(\bigcup_{i=1}^{k} \partial R_i))$.

The boundary of every R_i consists of two parts $\partial R_i = \partial^s R_i \cup \partial^u R_i$. Denote

$$\Delta^s\Lambda = \phi_{[0,\alpha]}\left(\bigcup_{i=1}^{k} \partial^s R_i\right) \quad \text{and} \quad \Delta^u\Lambda = \phi_{[-\alpha,0]}\left(\bigcup_{i=1}^{k} \partial^u R_i\right).$$

By [8, Proposition 2.6], one has $\phi_t(\Delta^s\Lambda) \subset \Delta^s\Lambda$ and $\phi_{-t}(\Delta^u\Lambda) \subset \Delta^u\Lambda$ for any $t \geq 0$. In fact, in the proof of [8, Proposition 2.6], it is also shown that

$$T\left(\bigcup_{i=1}^{k} \partial^s R_i\right) \subset \bigcup_{i=1}^{k} \partial^s R_i \quad \text{and} \quad T^{-1}\left(\bigcup_{i=1}^{k} \partial^u R_i\right) \subset \bigcup_{i=1}^{k} \partial^u R_i. \tag{2.2}$$

Now we give the proof of Theorem 2.5.

Proof of Theorem 2.5. Let $f_t : \Delta_\rho \to \Delta_\rho$ be the associated suspension flow of the basic set Λ. Since $\pi^{-1}(\bigcup_{i=1}^{k} \partial^s R_i)$ is a proper closed subset of Δ, then there is $\tilde{x} \in \Delta$ and $\tilde{\varepsilon} > 0$ such that

$$B(\tilde{x}, \tilde{\varepsilon}) \cap \pi^{-1}\left(\bigcup_{i=1}^{k} \partial^s R_i\right) = \emptyset,$$

where $B(\tilde{x}, \tilde{\varepsilon}) = \{y \in \Delta : d(\tilde{x}, y) < \tilde{\varepsilon}\}$. □

CLAIM 2.9. *If there is a point $z \in \Delta$ and an increasing sequence $\{t_m\}_{m=-\infty}^{+\infty}$ so that $f_{t_m}((z, 0)) \in \Delta \times \{0\}$, $d_{\Delta_\rho}(f_{t_m}((z, 0)), (\tilde{x}, 0)) < \tilde{\varepsilon}$ for any $m \in \mathbb{Z}$, and $\lim_{m\to+\infty} t_m = \lim_{m\to-\infty} -t_m = +\infty$, then $z \notin \bigcup_{n\in\mathbb{Z}} \sigma^n(\pi^{-1}(\bigcup_{i=1}^{k} \partial R_i))$.*

Proof. Without loss of generality, we assume $\sigma^l(z) \in \pi^{-1}(\bigcup_{i=1}^{k} \partial^s R_i)$ for some integer $l \in \mathbb{Z}$. Then by equation (2.2), we have $\sigma^{l+n}(z) \in \pi^{-1}(\bigcup_{i=1}^{k} \partial^s R_i)$ for any $n \geq 0$. Note that $f_{t_m}((z, 0)) \in \Delta \times \{0\}$ implies there is an integer n_m such that $f_{t_m}((z, 0)) = (\sigma^{n_m}(z), 0)$. So by $\lim_{m\to+\infty} t_m = +\infty$, there is an integer $n_m > l$ such that $d(\sigma^{n_m}(z), \tilde{x}) = d_{\Delta_\rho}(f_{t_m}((z, 0)), (\tilde{x}, 0)) < \tilde{\varepsilon}$. This contradicts $B(\tilde{x}, \tilde{\varepsilon}) \cap \pi^{-1}(\bigcup_{i=1}^{k} \partial^s R_i) = \emptyset$. □

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Fix $F=\mathrm{cov}\{\mu_i\}_{i=1}^m\subseteq\mathcal{M}(\Phi,\Lambda)$ and $\eta,\zeta>0$. Then there is $\tilde{\mu}_i\in\mathcal{M}(\mathfrak{F},\Delta_\rho)$ such that $\mu_i=\tilde{\mu}_i\circ\pi^{-1}$. Since π is continuous, there is $\tilde{\zeta}>0$ such that $d^*(\tilde{\mu}\circ\pi^{-1},\tilde{\nu}\circ\pi^{-1})<\zeta$ for any $\tilde{\mu},\tilde{\nu}\in\mathcal{M}(\mathfrak{F},\Delta_\rho)$ with $d^*(\tilde{\mu},\tilde{\nu})<\tilde{\zeta}$. For $\tilde{F}=\mathrm{cov}\{\tilde{\mu}_i\}_{i=1}^m\subseteq\mathcal{M}(\mathfrak{F},\Delta_\rho)$, $\tilde{x}$ and $\eta,\tilde{\zeta},\tilde{\varepsilon}>0$, by Proposition 2.8, there exist compact $\mathfrak{F}$-invariant subsets $\Delta_\rho^i\subseteq\Xi_\rho\subsetneq\Delta_\rho$ such that for each $1\le i\le m$:

(a) $f_t|_{\Delta_\rho^i}:\Delta_\rho^i\to\Delta_\rho^i$ and $f_t|_{\Xi_\rho}:\Xi_\rho\to\Xi_\rho$ are suspension flows over transitive subshifts of finite type $(\Delta_i,\sigma|_{\Delta_i})$ and $(\Xi,\sigma|_\Xi)$ with the roof function ρ;

(b) $h_{\mathrm{top}}(\Delta_\rho^i)>h_{\tilde{\mu}_i}(\mathfrak{F})-\eta$ and consequently, $h_{\mathrm{top}}(\Xi_\rho)>\sup\{h_\kappa(\mathfrak{F}):\kappa\in F\}-\eta$;

(c) $d_H(\tilde{F},\mathcal{M}(\mathfrak{F},\Xi_\rho))<\tilde{\zeta}$, $d_H(\tilde{\mu}_i,\mathcal{M}(\mathfrak{F},\Delta_\rho^i))<\tilde{\zeta}$;

(d) for any $z\in\Xi$, there is an increasing sequence $\{t_m\}_{m=-\infty}^{+\infty}$ such that $f_{t_m}((z,0))\in\Delta\times\{0\}$, $d_{\Delta_\rho}(f_{t_m}((z,0)),(\tilde{x},0))<\tilde{\varepsilon}$ for any $m\in\mathbb{Z}$, and $\lim_{m\to+\infty}t_m=\lim_{m\to-\infty}-t_m=+\infty$.

By Claim 2.9 and item (d), we have

$$\Delta_i\cap\bigcup_{n\in\mathbb{Z}}\sigma^n\bigg(\pi^{-1}\bigg(\bigcup_{i=1}^k\partial R_i\bigg)\bigg)=\emptyset\quad\text{and}\quad\Xi\cap\bigcup_{n\in\mathbb{Z}}\sigma^n\bigg(\pi^{-1}\bigg(\bigcup_{i=1}^k\partial R_i\bigg)\bigg)=\emptyset.$$

Then

$$\Delta_\rho^i\cap\bigcup_{t\in\mathbb{R}}f_t\bigg(\pi^{-1}\bigg(\bigcup_{i=1}^k\partial R_i\bigg)\bigg)=\emptyset\quad\text{and}\quad\Xi_\rho\cap\bigcup_{t\in\mathbb{R}}f_t\bigg(\pi^{-1}\bigg(\bigcup_{i=1}^k\partial R_i\bigg)\bigg)=\emptyset.$$

This implies π is one-to-one on Δ_ρ^i and Ξ_ρ. So $\Lambda_i=\pi(\Delta_\rho^i)$ and $\Theta=\pi(\Xi_\rho)$ are the horseshoes we want. □

3. *Asymptotically additive families and almost additive families*

In this section, we consider Theorems A and B in the more general context of asymptotically additive families and almost additive families of continuous functions and give abstract conditions on which the results of Theorems A and B hold.

3.1. *Non-additive thermodynamic formalism for flows.*

In this subsection, we recall a few basic notions and results on the non-additive thermodynamic formalism for flows. Consider a continuous flow $\Phi=(\phi_t)_{t\in\mathbb{R}}$ on a compact metric space (M,d). Let $\Lambda\subset M$ be a compact ϕ_t-invariant set. Given $x\in\Lambda$ and $t,\varepsilon>0$, we consider the set

$$B_t(x,\varepsilon)=\{y\in\Lambda:d(\phi_s(y),\phi_s(x))<\varepsilon\text{ for any }s\in[0,t]\}.$$

Moreover, let $a=(a_t)_{t\ge0}$ be a family of continuous functions $a_t:\Lambda\to\mathbb{R}$ with *tempered variation*, that is, such that

$$\lim_{\varepsilon\to0}\limsup_{t\to\infty}\frac{\gamma_t(a,\varepsilon)}{t}=0,$$

where

$$\gamma_t(a,\varepsilon)=\sup\{|a_t(y)-a_t(x)|:y\in B_t(x,\varepsilon),\ x\in\Lambda\}.$$

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Given $\varepsilon > 0$, we say that a set $\Gamma \subset \Lambda \times \mathbb{R}_0^+$ covers Λ if

$$\bigcup_{(x,t)\in\Gamma} B_t(x,\varepsilon) \supset \Lambda,$$

and we write

$$a(x,t,\varepsilon) = \sup\{a_t(y) : y \in B_t(x,\varepsilon)\} \quad \text{for } (x,t) \in \Gamma.$$

For each $\alpha \in \mathbb{R}$, let

$$M(\Lambda, a, \alpha, \varepsilon) = \lim_{T\to+\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t) \tag{3.1}$$

with the infimum taken over all countable sets $\Gamma \subset \Lambda \times [T, +\infty)$ covering Λ. When α goes from $-\infty$ to $+\infty$, the quantity in equation (3.1) jumps from $+\infty$ to 0 at a unique value and so one can define

$$P(a, \Lambda, \varepsilon) = \inf\{\alpha \in \mathbb{R} : M(\Lambda, a, \alpha, \varepsilon) = 0\}.$$

Moreover, the limit

$$P(a, \Lambda) = \lim_{\varepsilon\to 0} P(a, \Lambda, \varepsilon)$$

exists and is called the non-additive topological pressure of the family a on the set Λ.

We recall that a family of functions $a = (a_t)_{t\geq 0}$ on Λ is said to be *almost additive* with respect to the flow $(\phi_t)_{t\in\mathbb{R}}$ if there is a constant $C > 0$ such that

$$-C + a_t + a_s \circ \phi_t \leqslant a_{t+s} \leqslant a_t + a_s \circ \phi_t + C$$

for every $t, s \geq 0$. Let $A(\Phi, \Lambda)$ be the set of all almost additive families of continuous functions $a = (a_t)_{t\geqslant 0}$ on Λ with tempered variation such that

$$\sup_{t\in[0,s]} \|a_t\|_\infty < \infty \quad \text{for some } s > 0.$$

From [5, Proposition 4], the limit $\lim_{t\to\infty} 1/t \int_\Lambda a_t \, d\mu$ exists for any $a = (a_t)_{t\geq 0} \in A(\Phi, \Lambda)$ and any $\mu \in \mathcal{M}(\Phi, \Lambda)$, and the function

$$\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\mu \tag{3.2}$$

is continuous with the weak* topology. Let $a = (a_t)_{t\geq 0} \in A(\Phi, \Lambda)$. Then by [4, Theorem 2.1 and Lemma 2.4], we have

$$\begin{aligned} P(a, \Lambda) &= \sup_{\mu\in\mathcal{M}(\Phi,\Lambda)} \bigg(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_M a_t \, d\mu \bigg) \\ &= \sup_{\mu\in\mathcal{M}_{\mathrm{erg}}(\Phi,\Lambda)} \bigg(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_M a_t \, d\mu \bigg). \end{aligned} \tag{3.3}$$

A measure $\mu \in \mathcal{M}(\Phi, \Lambda)$ is said to be an *equilibrium measure* for the almost additive family a if

$$P(a, \Lambda) = h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_M a_t \, d\mu.$$

We say that a has *bounded variation* if for every $\kappa > 0$, there exists $\varepsilon > 0$ such that

$$|a_t(x) - a_t(y)| < \kappa \qquad \text{whenever } y \in B_t(x, \varepsilon).$$

Following the arguments of [4, §3.2], we have the following result on the uniqueness of equilibrium measure for suspension flows.

LEMMA 3.1. *Suppose $f_t : \Delta_\rho \to \Delta_\rho$ is a suspension flow over a transitive subshift of finite type (Δ, σ) with a Lipschitz roof function ρ. Let $a = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions on Δ_ρ with bounded variation such that $P(a, \Delta_\rho) = 0$ and $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$. Then there is a unique equilibrium measure μ_a for a.*

Proof. Define a sequence of functions $c_n : \Delta \to \mathbb{R}$ by

$$c_n(x) = a_{\rho_n(x)}(x),$$

where $\rho_n(x) = \sum_{i=0}^{n-1} \rho(\sigma^i(x))$. Following the arguments of [4, §3.2], we have:

(1) $c = (c_n)_{n\in\mathbb{N}}$ is an almost additive sequence of continuous functions, that is, there is a constant $\tilde{C} > 0$ such that for every $n, m \in \mathbb{N}$, we have

$$-\tilde{C} + c_n + c_m \circ \sigma^n \leqslant c_{n+m} \leqslant \tilde{C} + c_n + c_m \circ \sigma^n;$$

(2) $c = (c_n)_{n\in\mathbb{N}}$ has bounded variation, that is, there exists $\varepsilon > 0$ for which $\sup_{n\in\mathbb{N}} \gamma_n(c, \varepsilon) < \infty$ with

$$\gamma_n(c, \varepsilon) = \sup\{|c_n(x) - c_n(y)| : x, y \in \Delta,\ d(\sigma^k(x), \sigma^k(y)) < \varepsilon \text{ for } k = 0, \ldots, n\};$$

(3) for any ergodic measure $\nu \in \mathcal{M}(\sigma, \Delta)$ and $\mu = \mathcal{R}(\nu)$ (recall §2.2),

$$h_\mu(\mathfrak{F}) + \lim_{t\to\infty} \frac{1}{t} \int_{\Delta_\rho} a_t \, d\mu = \Big(h_\nu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int_\Delta c_n \, d\nu\Big) \Big/ \int_\Delta \rho \, d\nu.$$

By item (3), we have $h_\mu(\mathfrak{F}) + \lim_{t\to\infty}(1/t) \int_{\Delta_\rho} a_t \, d\mu = 0$ if and only if $h_\nu(\sigma) + \lim_{n\to\infty}(1/n) \int_\Delta c_n \, d\nu = 0$. Since $P(a, \Delta_\rho) = 0$, then μ is an equilibrium measure for a if and only if ν is an equilibrium measure for c by equation (3.3).

It is proved in [15, Theorem 5.3] that for every homeomorphism which is transitive, expansive, and has the shadowing property, and for every almost additive sequence of continuous functions with bounded variation, there is a unique equilibrium measure. Recall from [38] a subshift satisfies the shadowing property if and only if it is a subshift of finite type. As a subsystem of a two-side full shift, it is expansive. Then there is a unique equilibrium measure ν_c for c, and $\mathcal{R}(\mu_c)$ is the unique equilibrium measure for a. □

COROLLARY 3.2. *Suppose $f_t : \Delta_\rho \to \Delta_\rho$ is a suspension flow over a transitive subshift of finite type (Δ, σ) with a Lipschitz roof function ρ. Let $a = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions on Δ_ρ with bounded variation such that $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$. Then there is a unique equilibrium measure μ_a for a.*

Proof. For any $t \geq 0$, define

$$b_t = a_t - P(a)t.$$

Then b is an almost additive family with bounded variation and satisfies $\sup_{t\in[0,s]} \|b_t\|_\infty < \infty$ and $P(b) = 0$. For each $\mathfrak{F}$-invariant probability measure μ on Δ_ρ, we have

$$\frac{1}{t}\int_X a_t \, d\mu = \frac{1}{t}\int_X b_t \, d\mu - P(a).$$

This implies that a and b have the same equilibrium measures. Then by Lemma 3.1, we complete the proof. □

LEMMA 3.3. *Given $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, assume that there exists a suspension flow $f_t : \Delta_\rho \to \Delta_\rho$ over a transitive subshift of finite type (Δ, σ) with a Lipschitz roof function ρ, and a finite-to-one continuous surjection $\pi : \Delta_\rho \to \Lambda$, such that $\pi \circ f_t = \phi_t \circ \pi$. Then for every almost additive family of continuous functions $a = (a_t)_{t\geq 0}$ on Λ with bounded variation such that $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$, there is a unique equilibrium measure μ_a for a.*

Proof. Define $\tilde{a}_t = a_t \circ \pi$ for any $t \geq 0$. Since π is continuous, $\tilde{a} = (\tilde{a}_t)_{t\geq 0}$ is an almost additive family of continuous functions on Δ_ρ with bounded variation and $\sup_{t\in[0,s]} \|\tilde{a}_t\|_\infty < \infty$. By Corollary 3.2, there is a unique equilibrium measure $\mu_{\tilde{a}}$ for $\tilde{a}$. Since π is finite-to-one, we have $h_{\mu_{\tilde{a}}\circ\pi^{-1}}(\Phi) = h_{\mu_{\tilde{a}}}(\mathfrak{F})$ and thus

$$h_{\mu_{\tilde{a}}\circ\pi^{-1}}(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int a_t \, d\mu_{\tilde{a}}\circ\pi^{-1} = h_{\mu_{\tilde{a}}}(\mathfrak{F}) + \lim_{t\to\infty}\frac{1}{t}\int \tilde{a}_t \, d\mu_{\tilde{a}} = P(\tilde{a}) \geq P(a).$$

Then by equation (3.3), $\mu_{\tilde{a}} \circ \pi^{-1}$ is an equilibrium measure for a. To show that the measure is unique, suppose ν is an equilibrium measure for a, and choose $\mu \in \mathcal{M}(\mathfrak{F}, \Delta_\rho)$ with $\mu \circ \pi^{-1} = \nu$. Then

$$h_\mu(\mathfrak{F}) + \lim_{t\to\infty}\frac{1}{t}\int \tilde{a}_t \, d\mu = h_\nu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int a_t \, d\nu = P(a) = P(\tilde{a}).$$

Thus, $\mu = \mu_{\tilde{a}}$ by the uniqueness of $\mu_{\tilde{a}}$. So $\nu = \mu \circ \pi^{-1} = \mu_{\tilde{a}} \circ \pi^{-1}$ is the unique equilibrium measure for a. □

Then we have the following theorem, combining §2.3 and Definition 2.2.

THEOREM 3.4. *Given $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, if Λ is a basic set or a horseshoe of ϕ_t, then for every almost additive family of continuous functions $a = (a_t)_{t\geq 0}$ on Λ with bounded variation such that $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$, there is a unique equilibrium measure μ_a for a.*

3.2. *Conditional variational principles*

3.2.1. *Conditional variational principle of almost additive families.* Consider a continuous flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on a compact metric space (M, d). Let $\Lambda \subset M$ be a compact

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ϕ_t-invariant set. Let $d \in \mathbb{N}$ and $(A, B) \in A(\Phi, \Lambda)^d \times A(\Phi, \Lambda)^d$, where

$$A = (a^1, \ldots, a^d) \quad \text{and} \quad B = (b^1, \ldots, b^d),$$

and $a^i = (a_t^i)_{t\in\mathbb{R}}$ and $b^i = (b_t^i)_{t\in\mathbb{R}}$. Assume that

$$\liminf_{t\to\infty} \frac{b_t^i(x)}{t} > 0 \quad \text{and} \quad b_t^i(x) > 0 \tag{3.4}$$

for every $i = 1, \ldots, d$, $x \in \Lambda$, and $t \in \mathbb{R}$. Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, we define

$$R_{A,B}(\alpha) = \bigcap_{i=1}^{d} \bigg\{ x \in \Lambda : \lim_{t\to\infty} \frac{a_t^i(x)}{b_t^i(x)} = \alpha_i \bigg\}.$$

We also define the map $\mathcal{P}_{A,B} : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ by

$$\mathcal{P}_{A,B}(\mu) = \lim_{t\to\infty} \bigg(\frac{\int a_t^1 \, d\mu}{\int b_t^1 \, d\mu}, \ldots, \frac{\int a_t^d \, d\mu}{\int b_t^d \, d\mu} \bigg). \tag{3.5}$$

Equation (3.2) ensures that the function $\mathcal{P}_{A,B}$ is continuous. Denote

$$L_{A,B} = \{\mathcal{P}_{A,B}(\mu) : \mu \in \mathcal{M}(\Phi, \Lambda)\}.$$

Let $E(\Phi, \Lambda) \subset A(\Phi, \Lambda)$ be the set of families with a unique equilibrium measure. Define the sequence of constant functions $u = (u_t)_{t\geq 0}$ with $u_t \equiv t$ for any $t \geq 0$. In [**5**], L. Barreira and C. Holanda give the conditional variational principle as the following theorem.

THEOREM 3.5. [**5**, Theorem 9] *Suppose that $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is a continuous flow on a compact metric space (M, d) and $\Lambda \subset M$ is a compact invariant set such that the entropy function $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous. Let $d \in \mathbb{N}$ and $(A, B) \in A(\Phi, \Lambda)^d \times A(\Phi, \Lambda)^d$ such that B satisfies equation (3.4) and span$\{a^1, b^1, \ldots, a^d, b^d, u\} \subseteq E(\Phi, \Lambda)$. If $\alpha \in \operatorname{Int}(L_{A,B})$, then $R_{A,B}(\alpha) \neq \emptyset$, and the following properties hold.*

(1) *$h_{\text{top}}(R_{A,B}(\alpha))$ satisfies the variational principle:*

$$h_{\text{top}}(R_{A,B}(\alpha)) = \max\{h_\mu(\Phi) : \mu \in \mathcal{M}(\Phi, \Lambda) \text{ and } \mathcal{P}_{A,B}(\mu) = \alpha\}.$$

(2) *There is an ergodic measure $\mu_\alpha \in \mathcal{M}(\Phi, \Lambda)$ with $\mathcal{P}_{A,B}(\mu_\alpha) = \alpha$, $\mu_\alpha(R_{A,B}(\alpha)) = 1$, and*

$$h_{\text{top}}(R_{A,B}(\alpha)) = h_{\mu_\alpha}(\Phi).$$

Remark 3.6. In fact, Barreira and Holanda in [**5**, Theorem 9] give the proof of Theorem 3.5 under the assumption $d = 1$. However, Barreira and Doutor in [**3**, Theorem 3] proved the case of discrete time for any $d \geq 1$. Combining the two proofs, one can obtain Theorem 3.5 for any $d \geq 1$.

3.2.2. *Conditional variational principle of asymptotically additive families.* Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space (M, d) and $\Lambda \subset M$ be a compact

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ϕ_t-invariant set. A family of functions $a = (a_t)_{t\geq 0}$ is said to be asymptotically additive with respect to Φ on Λ if for each $\varepsilon > 0$, there exists a function $b_\varepsilon : \Lambda \to \mathbb{R}$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\Lambda} \left| a_t(x) - \int_0^t (b_\varepsilon \circ \phi_s(x))\, ds \right| \leq \varepsilon.$$

Let $AA(\Phi, \Lambda)$ be the set of all asymptotically additive families of continuous functions $a = (a_t)_{t\geqslant 0}$ on Λ with tempered variation such that

$$\sup_{t\in[0,s]} \|a_t\|_\infty < \infty \quad \text{for some } s > 0.$$

Proceeding as in [16], one can see that every almost additive family of functions is asymptotically additive.

Now assume that Φ is expansive on Λ. Holanda in [20, Corollary 6] proved that for any $a = (a_t)_{t\geq 0} \in AA(\Phi, \Lambda)$, there exists a continuous function $b : \Lambda \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\mu = \int_\Lambda b \, d\mu$$

for any $\mu \in \mathcal{M}(\Phi, \Lambda)$. This implies that the function

$$\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\mu \tag{3.6}$$

is continuous with the weak* topology. Let $d \in \mathbb{N}$ and $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$, where

$$A = (a^1, \ldots, a^d) \quad \text{and} \quad B = (b^1, \ldots, b^d),$$

and $a^i = (a^i_t)_{t\in\mathbb{R}}$ and $b^i = (b^i_t)_{t\in\mathbb{R}}$. Assume that

$$\lim_{t\to\infty} \frac{1}{t} \int_\Lambda b^i_t \, d\mu \geq 0 \quad \text{for all } \mu \in \mathcal{M}(\Phi, \Lambda) \tag{3.7}$$

with equality only permitted when

$$\lim_{t\to\infty} \frac{1}{t} \int_\Lambda a^i_t \, d\mu \neq 0 \tag{3.8}$$

for every $i = 1, \ldots, d$. Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, we define

$$R_{A,B}(\alpha) = \bigcap_{i=1}^d \left\{ x \in \Lambda : \lim_{t\to\infty} \frac{a^i_t(x)}{b^i_t(x)} = \alpha_i \right\}.$$

We also define the map $\mathcal{P}_{A,B} : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ by

$$\mathcal{P}_{A,B}(\mu) = \lim_{t\to\infty} \left(\frac{\int a^1_t \, d\mu}{\int b^1_t \, d\mu}, \ldots, \frac{\int a^d_t \, d\mu}{\int b^d_t \, d\mu} \right). \tag{3.9}$$

Equation (3.6) ensures that the function $\mathcal{P}_{A,B}$ is continuous. Denote

$$L_{A,B} = \{\mathcal{P}_{A,B}(\mu) : \mu \in \mathcal{M}(\Phi, \Lambda)\}.$$

Without using the uniqueness of equilibrium assumption, C. Holanda obtains a conditional variational principle for asymptotically additive families of continuous functions.

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THEOREM 3.7. [20, Theorem 11] *Given $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, assume that Λ is a basic set or a horseshoe of ϕ_t. Let $d \in \mathbb{N}$ and $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$ satisfying equation (3.7). If $\alpha \in \mathrm{Int}(L_{A,B})$, then $R_{A,B}(\alpha) \neq \emptyset$, and the following properties hold.*

(1) *$h_{\mathrm{top}}(R_{A,B}(\alpha))$ satisfies the variational principle:*

$$h_{\mathrm{top}}(R_{A,B}(\alpha)) = \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}(\Phi, \Lambda) \text{ and } \mathcal{P}_{A,B}(\mu) = \alpha\}.$$

(2) *For any $\varepsilon > 0$, there exists an ergodic measure $\mu_\alpha \in \mathcal{M}(\Phi, \Lambda)$ such that $\mathcal{P}_{A,B}(\mu_\alpha) = \alpha$, $\mu_\alpha(R_{A,B}(\alpha)) = 1$ and*

$$|h_{\mathrm{top}}(R_{A,B}(\alpha)) - h_{\mu_\alpha}(\Phi)| < \varepsilon.$$

Remark 3.8. Using the work of Climenhaga [11, Theorem 3.3] and Cuneo [13, Theorem 1.2], and the conclusion that every Hölder continuous function has a unique equilibrium measure with respect to the map T (recall that T is the transfer map, see §2.3), Holanda in [20, Theorem 11] gives the proof of Theorem 3.7 under the assumption $d = 1$ and Λ is a mixing basic set. However, [11, Theorem 3.3] is stated for any $d \geq 1$ and by [9, Example 2], every Hölder continuous function has a unique equilibrium measure with respect to the map T when Λ is just a transitive basic set. So one can obtain Theorem 3.7 for any $d \geq 1$ and any basic set Λ. When Λ is a horseshoe of ϕ_t, we can verify the results also hold following the argument of [20, Theorem 11].

3.3. *Abstract conditions on which Theorems A and B hold.* Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space (M, d), and $\Lambda \subset M$ be a compact ϕ_t-invariant set. Assume that Φ is expansive on Λ. Let $d \in \mathbb{N}$ and $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$. For any $\alpha \in L_{A,B}$, denote

$$M_{A,B}(\alpha) = \{\mu \in \mathcal{M}(\Phi, \Lambda) : \mathcal{P}_{A,B}(\mu) = \alpha\},$$
$$M^{\mathrm{erg}}_{A,B}(\alpha) = \{\mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda) : \mathcal{P}_{A,B}(\mu) = \alpha\}.$$

Then $M_{A,B}(\alpha)$ is closed in $\mathcal{M}(\Phi, \Lambda)$ since the map $\mathcal{P}_{A,B}$ is continuous.

Let $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ be a continuous function. We define the pressure of χ with respect to μ by $P(\Phi, \chi, \mu) = h_\mu(\Phi) + \chi(\mu)$. Now we give a result in the context of asymptotically additive families.

THEOREM 3.9. *Suppose that $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is a continuous flow on a compact metric space (M, d), and $\Lambda \subset M$ is a compact invariant set such that Φ is expansive on Λ. Let $d \in \mathbb{N}$ and $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$ satisfying equation (3.7). Let $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ be a continuous function. Assume that the following holds: for any $F = \mathrm{cov}\{\mu_i\}_{i=1}^m \subseteq \mathcal{M}(\Phi, \Lambda)$ and any $\eta, \zeta > 0$, there are compact invariant subsets $\Lambda_i \subseteq \Theta \subsetneq \Lambda$ such that for each $i \in \{1, 2, \ldots, m\}$:*

(1) *for any $a \in \mathrm{Int}(\mathcal{P}_{A,B}(\mathcal{M}(\Phi, \Theta)))$ and any $\varepsilon > 0$, there exists an ergodic measure μ_a supported on Θ with $\mathcal{P}_{A,B}(\mu_a) = a$ such that $|h_{\mu_a}(\Phi) - H(\Phi, a, \Theta)| < \varepsilon$, where $H(\Phi, a, \Theta) = \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}(\Phi, \Theta) \text{ and } \mathcal{P}_{A,B}(\mu) = a\}$;*

(2) *$h_{\mathrm{top}}(\Lambda_i) > h_{\mu_i}(\Phi) - \eta$;*

(3) *$d_H(K, \mathcal{M}(\Phi, \Theta)) < \zeta$, $d_H(\mu_i, \mathcal{M}(\Phi, \Lambda_i)) < \zeta$.*

Then for any $\alpha \in \mathrm{Int}(L_{A,B})$*, any* $\mu_0 \in M_{A,B}(\alpha)$*, and any* $\eta, \zeta > 0$*, there is* $\nu \in M^{\mathrm{erg}}_{A,B}(\alpha)$ *such that* $d^*(\nu, \mu_0) < \zeta$ *and* $|P(\Phi, \chi, \nu) - P(\Phi, \chi, \mu_0)| < \eta$*.*

For any $\alpha \in L_{A,B}$, denote

$$H_{A,B}(\Phi, \chi, \alpha) = \sup\{P(\Phi, \chi, \mu) : \mu \in M_{A,B}(\alpha)\}.$$

In particular, when $\chi \equiv 0$, we write

$$H_{A,B}(\Phi, \alpha) = H_{A,B}(\Phi, 0, \alpha) = \sup\{h_\mu(\Phi) : \mu \in M_{A,B}(\alpha)\}.$$

We list two conditions for χ :

(A.1) for any $\mu_1, \mu_2 \in \mathcal{M}(\Phi, \Lambda)$ with $P(\Phi, \chi, \mu_1) \neq P(\Phi, \chi, \mu_2)$,

$$\beta(\theta) := P(\Phi, \chi, \theta\mu_1 + (1-\theta)\mu_2) \text{ is strictly monotonic on } [0, 1]; \tag{3.10}$$

(A.2) for any $\mu_1, \mu_2 \in \mathcal{M}(\Phi, \Lambda)$ with $P(\Phi, \chi, \mu_1) = P(\Phi, \chi, \mu_2)$,

$$\beta(\theta) := P(\Phi, \chi, \theta\mu_1 + (1-\theta)\mu_2) \text{ is constant on } [0, 1]. \tag{3.11}$$

Now we give abstract conditions on which Theorems A and B hold in the context of asymptotically additive families.

THEOREM 3.10. *Suppose that* $\Phi = (\phi_t)_{t\in\mathbb{R}}$ *is a continuous flow on a compact metric space* (M, d)*, and* $\Lambda \subset M$ *is a compact invariant set such that* Φ *is expansive on* Λ*. Let* $d \in \mathbb{N}$ *and* $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$ *satisfying equation (3.7). Let* $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ *be a continuous function satisfying equations (3.10) and (3.11). Assume that for any* $\alpha \in \mathrm{Int}(L_{A,B})$*, any* $\mu_0 \in M_{A,B}(\alpha)$*, and any* $\eta, \zeta > 0$*, there is* $\nu \in M^{\mathrm{erg}}_{A,B}(\alpha)$ *such that* $d^*(\nu, \mu_0) < \zeta$ *and* $|P(\Phi, \chi, \nu) - P(\Phi, \chi, \mu_0)| < \eta$*. If* $\{\mu \in \mathcal{M}(\Phi, \Lambda) : h_\mu(\Phi) = 0\}$ *is dense in* $\mathcal{M}(\Phi, \Lambda)$*, then we have the following.*

(1) *For any* $\alpha \in \mathrm{Int}(L_{A,B})$*, any* $\mu_0 \in M_{A,B}(\alpha)$*, any* $\max_{\mu\in M_{A,B}(\alpha)} \chi(\mu) \le c \le P(\phi, \chi, \mu_0)$*, and any* $\eta, \zeta > 0$*, there is* $\nu \in M^{\mathrm{erg}}_{A,B}(\alpha)$ *such that* $d^*(\nu, \mu_0) < \zeta$ *and* $|P(\Phi, \chi, \nu) - c| < \eta$*.*

(2) *For any* $\alpha \in \mathrm{Int}(L_{A,B})$ *and* $\max_{\mu\in M_{A,B}(\alpha)} \chi(\mu) \le c < H_{A,B}(\Phi, \chi, \alpha)$*, the set* $\{\mu \in M^{\mathrm{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c\}$ *is residual in* $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \ge c\}$*. If further there is an invariant measure* $\tilde{\mu}$ *with* $S_{\tilde{\mu}} = \Lambda$*, then for any* $\alpha \in \mathrm{Int}(L_{A,B})$ *and* $\max_{\mu\in M_{A,B}(\alpha)} \chi(\mu) \le c < H_{A,B}(\Phi, \chi, \alpha)$*, the set* $\{\mu \in M^{\mathrm{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c,\ S_\mu = \Lambda\}$ *is residual in* $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \ge c\}$*.*

(3) *The set* $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}(\Phi, \Lambda),\quad \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}), \max_{\mu\in M_{A,B}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H_{A,B}(\Phi, \chi, \alpha)\}$ *coincides with* $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda),\ \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}), \max_{\mu\in M_{A,B}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H_{A,B}(\Phi, \chi, \alpha)\}$*.*

Example 3.11. The function $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ can be defined as follows.

(1) $\chi \equiv 0$. Then $P(\Phi, \chi, \mu) = h_\mu(\Phi)$ is the metric entropy of μ.

(2) $\chi(\mu) = \int g\, d\mu$ with a continuous function g. Then from the weak*-topology on $\mathcal{M}(\Phi, \Lambda)$, $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ is a continuous function. Here, $P(\Phi, \chi, \mu) = h_\mu(\Phi) + \chi(\mu)$ is the pressure of g with respect to μ.

(3) $\chi(\mu) = \lim_{t\to\infty}(1/t)\int a_t\, d\mu$ with an asymptotically additive family of continuous functions $a = (a_t)_{t\in\mathbb{R}}$ on Λ. Then $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ is a continuous function from equation (3.6). Here, $P(\Phi, \chi, \mu) = h_\mu(\Phi) + \lim_{t\to\infty}(1/t)\int a_t\, d\mu$ is the pressure of a with respect to μ.

Furthermore, if χ is defined as above, then equations (3.10) and (3.11) hold for χ since it is affine.

Remark 3.12. In Theorems 3.9 and 3.10, the expansivity of Φ is used to guarantee the function $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto \lim_{t\to\infty}(1/t)\int_\Lambda a_t\, d\mu$ is continuous and the metric entropy $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous. When we consider almost additive families, by equation (3.2), we only need to assume that the metric entropy $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous.

3.4. *Proof of Theorem 3.9*

3.4.1. *Some lemmas.* We establish several auxiliary results. For any $r \in \mathbb{R}$, denote $r^+ = \{s \in \mathbb{R} : s > r\}$ and $r^- = \{s \in \mathbb{R} : s < r\}$. For any $d \in \mathbb{N}$, $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$, and $\xi = (\xi_1, \ldots, \xi_d) \in \{+, -\}^d$, we define

$$r^\xi = \{s = (s_1, \ldots, s_d) \in \mathbb{R}^d : s_i \in r_i^{\xi_i} \text{ for } i = 1, 2, \ldots, d\}.$$

We denote $F^d = \{(p_1/q_1, \ldots, p_d/q_d) : p_i, q_i \in \mathbb{R}$ and $q_i > 0$ for any $1 \le i \le d\}$. It is easy to check the following lemma.

LEMMA 3.13. *Let $b_i = p^i/q^i \in F^1$ for $i = 1, 2$.*

(1) *If $b_1 = b_2$, then $(\theta p^1 + (1-\theta)p^2)/(\theta q^1 + (1-\theta)q^2) = b_1 = b_2$ for any $\theta \in [0, 1]$.*

(2) *If $b_1 \ne b_2$, then $(\theta p^1 + (1-\theta)p^2)/(\theta q^1 + (1-\theta)q^2)$ is strictly monotonic on $\theta \in [0, 1]$.*

We can obtain the following result using mathematical induction.

LEMMA 3.14. *Let $d \in \mathbb{N}$ and $a = (p_1/q_1, \ldots, p_d/q_d) \in F^d$. If $\{b_\xi = (p_1^\xi/q_1^\xi, \ldots, p_d^\xi/q_d^\xi)\}_{\xi\in\{+,-\}^d} \subseteq F^d$ are 2^d numbers satisfying $b_\xi \in a^\xi$ for any $\xi \in \{+, -\}^d$, then there are 2^d numbers $\{\theta_\xi\}_{\xi\in\{+,-\}^d} \subseteq [0, 1]$ such that $\sum_{\xi\in\{+,-\}^d} \theta_\xi = 1$ and*

$$\frac{\sum_{\xi\in\{+,-\}^d} \theta_\xi p_i^\xi}{\sum_{\xi\in\{+,-\}^d} \theta_\xi q_i^\xi} = \frac{p_i}{q_i} \quad \text{for any } 1 \le i \le d.$$

From Lemma 3.14, we have the following corollary.

COROLLARY 3.15. *Suppose that $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is a continuous flow on a compact metric space (M, d), and $\Lambda \subset M$ is a compact invariant set such that Φ is expansive on Λ. Let $d \in \mathbb{N}$ and $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$, satisfying equation (3.7). Then for any $\alpha \in \mathrm{Int}(L_{A,B})$, and 2^d invariant measures $\{\mu_\xi\}_{\xi\in\{+,-\}^d}$ with*

$$\mathcal{P}_{A,B}(\mu_\xi) \in \alpha^\xi \quad \text{for any } \xi \in \{+, -\}^d,$$

there are 2^d *numbers* $\{\theta_\xi\}_{\xi\in\{+,-\}^d} \subseteq [0, 1]$ *such that*

$$\sum_{\xi\in\{+,-\}^d} \theta_\xi = 1 \quad \textit{and} \quad \mathcal{P}_{A,B}\Bigg(\sum_{\xi\in\{+,-\}^d} \theta_\xi \mu_\xi\Bigg) = \alpha.$$

LEMMA 3.16. *Suppose that* $\Phi = (\phi_t)_{t\in\mathbb{R}}$ *is a continuous flow on a compact metric space* (M, d), *and* $\Lambda \subset M$ *is a compact invariant set such that* Φ *is expansive on* Λ. *Let* $d \in \mathbb{N}$ *and* $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$ *satisfying equation (3.7). Then for any* $\alpha \in \mathrm{Int}(L_{A,B})$, *any* $\mu \in M_{A,B}(\alpha)$, *and any* $\eta, \zeta > 0$, *there are* 2^d *invariant measures* $\{\mu_\xi\}_{\xi\in\{+,-\}^d}$ *such that for any* $\xi \in \{+, -\}^d$,

$$\mathcal{P}_{A,B}(\mu_\xi) \in \alpha^\xi, h_{\mu_\xi}(\Phi) > h_\mu(f) - \eta \quad \textit{and} \quad d^*(\mu_\xi, \mu) < \zeta.$$

Proof. By $\alpha \in \mathrm{Int}(L_{A,B})$, there is $\nu_\xi \in \mathcal{M}(\Phi, \Lambda)$ such that $\mathcal{P}_{A,B}(\nu_\xi) \in a^\xi$ for any $\xi \in \{+, -\}^d$. We choose $\tau_\xi \in (0, 1)$ close to 1 such that $\mu_\xi = \tau_\xi \mu + (1 - \tau_\xi)\nu_\xi$ satisfies

$$h_{\mu_\xi}(\Phi) > h_\mu(\Phi) - \eta \quad \text{and} \quad d^*(\mu_\xi, \mu) < \zeta \quad \text{for any } \xi \in \{+, -\}^d.$$

Then we have $\mathcal{P}_{A,B}(\mu_\xi) \in \alpha^\xi$ by $\tau^\xi > 0$ and Lemma 3.13(2). □

3.4.2. *Proof of Theorem 3.9*. Fix $\alpha \in \mathrm{Int}(L_{A,B})$, $\mu_0 \in M_{A,B}(\alpha)$, and $\eta, \zeta > 0$. Since Φ is expansive on Λ, the metric entropy $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous. Hence, there is $0 < \zeta' < \zeta$ such that for any $\omega \in \mathcal{M}(\Phi, \Lambda)$ with $d^*(\mu_0, \omega) < \zeta'$, we have

$$h_\omega(\Phi) < h_{\mu_0}(\Phi) + \frac{3\eta}{8} \quad \text{and} \quad |\chi(\omega) - \chi(\mu_0)| < \frac{\eta}{2}. \tag{3.12}$$

By Lemma 3.16, there are 2^d invariant measures $\{\mu_\xi\}_{\xi\in\{+,-\}^d}$ such that for any $\xi \in \{+, -\}^d$,

$$\mathcal{P}_{A,B}(\mu_\xi) \in \alpha^\xi, \quad h_{\mu_\xi}(\Phi) > h_{\mu_0}(\Phi) - \frac{\eta}{8} \quad \text{and} \quad d^*(\mu_\xi, \mu_0) < \frac{\zeta'}{2}. \tag{3.13}$$

Since the map $\mathcal{P}_{A,B}$ is continuous, there is $0 < \zeta'' < \zeta'$ such that for any $\omega_\xi \in \mathcal{M}(\Phi, \Lambda)$ with $d^*(\omega_\xi, \mu_\xi) < \zeta''$, one has

$$\mathcal{P}_{A,B}(\omega_\xi) \in \alpha^\xi. \tag{3.14}$$

For the 2^d invariant measures $\{\mu_\xi\}_{\xi\in\{+,-\}^d}$, there are compact invariant subsets $\Lambda_\xi \subseteq \Theta \subsetneq \Lambda$ such that for each $\xi \in \{+, -\}^d$:

(1) for any $a \in \mathrm{Int}(\mathcal{P}_{A,B}(\mathcal{M}(\Phi, \Theta)))$ and any $\varepsilon > 0$, there exists an ergodic measure μ_a supported on Θ with $\mathcal{P}_{A,B}(\mu_a) = a$ such that $|h_{\mu_a}(\Phi) - H(\Phi, a, \Theta)| < \varepsilon$;
(2) $h_{\mathrm{top}}(\Lambda_\xi) > h_{\mu_\xi}(\Phi) - \eta/8$;
(3) $d_H(\mathrm{cov}\{\mu_\xi\}_{\xi\in\{+,-\}^d}, \mathcal{M}(\Phi, \Theta)) < \zeta''/2$, $d_H(\mu_\xi, \mathcal{M}(\Phi, \Lambda_\xi)) < \zeta''/2$.

By item (2) and the variational principle, there is $\nu_\xi \in \mathcal{M}(\Phi, \Lambda_\xi)$ such that

$$h_{\nu_\xi}(\Phi) > h_{\mathrm{top}}(\Lambda_\xi) - \frac{\eta}{8} > h_{\mu_\xi}(\Phi) - \frac{2\eta}{8} > h_{\mu_0}(\Phi) - \frac{3\eta}{8}.$$

Then by item (3) and equation (3.14), we have $\mathcal{P}_{A,B}(\nu_\xi) \in \alpha^\xi$. By Corollary 3.15, there are 2^d numbers $\{\theta_\xi\}_{\xi\in\{+,-\}^d} \subseteq [0, 1]$ such that $\sum_{\xi\in\{+,-\}^d} \theta_\xi = 1$ and $\mathcal{P}_{A,B}(\nu') = \alpha$, where

$\nu' = \sum_{\xi \in \{+,-\}^d} \theta_\xi \nu_\xi$. Then on one hand, we have

$$H(\Phi, \alpha, \Theta) \geq h_{\nu'}(\Phi) \geq \min\{h_{\nu_\xi}(\Phi) : \xi \in \{+, -\}^d\} > h_{\mu_0}(\Phi) - \frac{3\eta}{8}. \tag{3.15}$$

On the other hand, by item (3), and equations (3.13) and (3.12), we have

$$H(\Phi, \alpha, \Theta) < h_{\mu_0}(\Phi) + \frac{3\eta}{8}. \tag{3.16}$$

Now by item (1), there exists an ergodic measure ν supported on Θ with $\mathcal{P}_{A,B}(\nu) = \alpha$ such that $|h_\nu(\Phi) - H(\Phi, \alpha, \Theta)| < \eta/8$. Then $\nu \in M^{\text{erg}}_{A,B}(\alpha)$, and by equations (3.15) and (3.16), we have $|h_\nu(\Phi) - h_{\mu_0}(\Phi)| < \eta/2$. By item (3) and equation (3.13), we have $d^*(\nu, \mu_0) < \zeta' < \zeta$. Finally, by equation (3.12), we have $|\chi(\nu) - \chi(\mu_0)| < \eta/2$ and thus $|P(\Phi, \chi, \nu) - P(\Phi, \chi, \mu_0)| < \eta$.

3.5. *Proof of Theorem 3.10*

3.5.1. *Some lemmas.*

LEMMA 3.17. *Suppose that $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is a continuous flow on a compact metric space (M, d), and $\Lambda \subset M$ is a compact invariant set. Let V be a convex subset of $\mathcal{M}(\Phi, \Lambda)$. If there is an invariant measure $\mu_V \in V$ with $S_{\mu_V} = \Lambda$, then $\{\mu \in V : S_\mu = \Lambda\}$ is residual in V.*

Proof. Since $\{\mu \in \mathcal{M}(\Phi, \Lambda) : S_\mu = \Lambda\}$ is either empty or a dense G_δ subset of $\mathcal{M}(\Phi, \Lambda)$ from [**14**, Proposition 21.11], if there is an invariant measure $\mu_V \in V$ with $S_{\mu_V} = \Lambda$, then $\{\mu \in \mathcal{M}(\Phi, \Lambda) : S_\mu = \Lambda\}$ is a dense G_δ subset of $\mathcal{M}(\Phi, M)$. Thus, $\{\mu \in V : S_\mu = \Lambda\}$ is a G_δ subset of V. In addition, for any $\nu \in V$ and $\theta \in (0, 1)$, we have $\nu_\theta = \theta\nu + (1-\theta)\mu_V \in V$ and $S_{\nu_\theta} = \Lambda$. So $\{\mu \in V : S_\mu = \Lambda\}$ is dense in V, and thus is residual in V. □

LEMMA 3.18. *Suppose that $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is a continuous flow on a compact metric space (M, d), and $\Lambda \subset M$ is a compact invariant set such that Φ is expansive on Λ. Let $d \in \mathbb{N}$ and $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$, satisfying equation (3.7). If $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ is a continuous function satisfying equations (3.10) and (3.11), then for any $\alpha \in \operatorname{Int}(L_{A,B})$ and any $\max_{\mu\in M_{A,B}(\alpha)} \chi(\mu) \leq c < H_{A,B}(\Phi, \chi, \alpha)$, the following properties hold:*

(1) *if $\{\mu \in M^{\text{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$ is dense in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$, then $\{\mu \in M^{\text{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$ is residual in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$;*
(2) *if there is an invariant measure $\tilde{\mu}$ with $S_{\tilde{\mu}} = \Lambda$, then $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c, S_\mu = \Lambda\}$ is residual in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$;*
(3) *if $\{\mu \in \mathcal{M}(\Phi, \Lambda) : h_\mu(\Phi) = 0\}$ is dense in $\mathcal{M}(\Phi, \Lambda)$, then $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c\}$ is residual in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$.*

Proof. (1) From [**14**, Proposition 5.7], $\mathcal{M}_{\text{erg}}(\Phi, \Lambda)$ is a G_δ subset of $\mathcal{M}(\Phi, \Lambda)$. Then $\{\mu \in M^{\text{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$ is a G_δ subset of $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$. If $\{\mu \in M^{\text{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$ is dense in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$, then $\{\mu \in M^{\text{erg}}_{A,B} y(\alpha) : P(\Phi, \chi, \mu) \geq c\}$ is residual in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$.

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(2) Since $\{\mu \in \mathcal{M}(\Phi, \Lambda) : S_\mu = \Lambda\}$ is either empty or a dense G_δ subset of $\mathcal{M}(\Phi, \Lambda)$ from [14, Proposition 21.11], if there is an invariant measure $\tilde{\mu}$ with $S_{\tilde{\mu}} = \Lambda$, then $\{\mu \in \mathcal{M}(\Phi, \Lambda) : S_\mu = \Lambda\}$ is a dense G_δ subset of $\mathcal{M}(\Phi, \Lambda)$.

Now we show that $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c, \quad S_\mu = \Lambda\}$ is non-empty. By Lemma 3.16, there exist 2^d invariant measures $\{\mu_\xi\}_{\xi \in \{+,-\}^d}$ such that

$$\mathcal{P}_{A,B}(\mu_\xi) \in \alpha^\xi \quad \text{for any } \xi \in \{+, -\}^d.$$

Since $\{\mu \in \mathcal{M}(\Phi, \Lambda) : S_\mu = \Lambda\}$ is dense in $\mathcal{M}(\Phi, \Lambda)$ and the map $\mathcal{P}_{A,B}$ is continuous, then there exists $\omega_\xi \in \mathcal{M}(\Phi, \Lambda)$ close to μ_ξ such that

$$\mathcal{P}_{A,B}(\omega_\xi) \in \alpha^\xi, \quad S_{\omega_\xi} = \Lambda \quad \text{for any } \xi \in \{+, -\}^d.$$

By Corollary 3.15, there are 2^d numbers $\{\theta_\xi\}_{\xi \in \{+,-\}^d} \subseteq [0, 1]$ such that $\sum_{\xi \in \{+,-\}^d} \theta_\xi = 1$ and $\mathcal{P}_{A,B}(\omega) = \alpha$, where $\omega = \sum_{\xi \in \{+,-\}^d} \theta_\xi \mu_\omega$. Then $S_\omega = \Lambda$. Since $c < H_{A,B}(\Phi, \chi, \alpha)$, there is $\nu \in M_{A,B}(\alpha)$ such that $P(\Phi, \chi, \nu) > c$. By equation (3.10), we can choose $\theta \in (0, 1)$ close to 1 such that $\mu' = \theta\nu + (1 - \theta)\omega$ satisfies $P(\Phi, \chi, \mu') > c$. Then $\mu' \in \{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c, \ S_\mu = \Lambda\}$. Note that $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$ is a convex set by equations (3.10), (3.11), and Lemma 3.13. So by Lemma 3.17, we complete the proof of item (2).

(3) Fix $\mu_0 \in M_{A,B}(\alpha)$ with $P(\Phi, \chi, \mu_0) \geq c$ and $\zeta > 0$. By Lemma 3.16, there exist 2^d invariant measures $\{\mu_\xi\}_{\xi \in \{+,-\}^d}$ such that

$$\mathcal{P}_{A,B}(\mu_\xi) \in \alpha^\xi \quad \text{and} \quad d^*(\mu_\xi, \mu_0) < \frac{\zeta}{2} \quad \text{for any } \xi \in \{+, -\}^d. \tag{3.17}$$

Since $\{\mu \in \mathcal{M}(\Phi, \Lambda) : h_\mu(\Phi) = 0\}$ is dense in $\mathcal{M}(\Phi, \Lambda)$ and the function $\mathcal{P}_{A,B}$ is continuous, then there exists $\nu_\xi \in \mathcal{M}(\Phi, \Lambda)$ close to μ_ξ such that

$$\mathcal{P}_{A,B}(\nu_\xi) \in \alpha^\xi, \quad h_{\nu_\xi}(\Phi) = 0 \quad \text{and} \quad d^*(\nu_\xi, \mu_\xi) < \frac{\zeta}{2} \quad \text{for each } \xi \in \{+, -\}^d. \tag{3.18}$$

By Corollary 3.15, there are 2^d numbers $\{\theta_\xi\}_{\xi \in \{+,-\}^d} \subseteq [0, 1]$ such that $\sum_{\xi \in \{+,-\}^d} \theta_\xi = 1$ and $\mathcal{P}_{A,B}(\nu') = \alpha$, where $\nu' = \sum_{\xi \in \{+,-\}^d} \theta_\xi \nu_\xi$. Then by equation (3.18), $h_{\nu'}(\Phi) = 0$. By equations (3.17) and (3.18), we have $d^*(\nu', \mu_0) < \zeta$. Now by equation (3.10), we choose $\theta \in [0, 1]$ such that $\nu = \theta\mu_0 + (1 - \theta)\nu'$ satisfies $P(\Phi, \chi, \nu) = c$. Then by Lemma 3.13(1),

$$\mathcal{P}_{A,B}(\nu) = \alpha \quad \text{and} \quad d^*(\nu, \mu_0) < \zeta. \tag{3.19}$$

So $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c\}$ is dense in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$. Since Φ is expansive on Λ, the metric entropy $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous. Hence, $\{\mu \in \mathcal{M}(\Phi, \Lambda) : P(\Phi, \chi, \mu) \in [c, c + 1/n)\}$ is open in $\{\mu \in \mathcal{M}(\Phi, \Lambda) : P(\Phi, \chi, \mu) \geq c\}$ for any $n \in \mathbb{N}^+$. Then

$$\left\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \in \left[c, c + \frac{1}{n}\right)\right\}$$
$$\text{is open and dense in } \{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\},$$

and thus $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c\}$ is residual in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$. $\square$

3.5.2. *Proof of Theorem 3.10.* (1) Fix $\alpha \in \mathrm{Int}(L_{A,B})$, $\mu_0 \in M_{A,B}(\alpha)$, $\max_{\mu \in M_{A,B}(\alpha)} \alpha(\mu) \leq c \leq P(\Phi, \chi, \mu_0)$, and $\eta, \zeta > 0$. By Lemma 3.18(3), there exists $\nu' \in M_{A,B}(\alpha)$ such that $P(\Phi, \chi, \nu') = c$ and $d^*(\nu', \mu_0) < \zeta/2$. For $\alpha \in \mathrm{Int}(L_{A,B})$, $\nu' \in M_{A,B}(\alpha)$, and $\eta, \zeta/2 > 0$, there is $\nu \in M^{\mathrm{erg}}_{A,B}(\alpha)$ such that $d^*(\nu, \nu') < \zeta/2$ and $|P(\Phi, \chi, \nu) - P(\Phi, \chi, \nu')| < \eta$. Then we complete the proof of item (1).

(2) Fix $\alpha \in \mathrm{Int}(L_{A,B})$ and $\max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \leq c < H_{A,B}(\Phi, \chi, \alpha)$. First we show that

$$\{\mu \in M^{\mathrm{erg}}_{A,B}(\alpha) : P(f, \alpha, \mu) \geq c\} \text{ is dense in } \{\mu \in M_{A,B}(\alpha) : P(f, \alpha, \mu) \geq c\}. \tag{3.20}$$

Let $\mu_0 \in M_{A,B}(\alpha)$ be an invariant measure with $P(\Phi, \chi, \mu_0) \geq c$ and $\zeta > 0$. If $P(\Phi, \chi, \mu_0) > c$, then there is $\eta > 0$ such that $c < c + \eta < P(\Phi, \chi, \mu_0)$. For $\alpha \in \mathrm{Int}(L_{A,B})$, $\mu_0 \in M_{A,B}(\alpha)$, and $\eta, \zeta > 0$, there exists an ergodic measure $\nu \in M^{\mathrm{erg}}_{A,B}(\alpha)$ such that $d^*(\nu, \mu_0) < \zeta$ and $|P(\Phi, \chi, \nu) - P(\Phi, \chi, \mu_0)| < \eta$. If $P(\Phi, \chi, \mu_0) = c$, then we can pick an invariant measure $\mu' \in M_{A,B}(\alpha)$ such that $c < P(\Phi, \chi, \mu') \leq H_{A,B}(\Phi, \chi, \alpha)$, and next pick a sufficiently small number $\theta \in (0, 1)$ such that $d^*(\mu_0, \mu'') < \zeta/2$, where $\mu'' = (1 - \theta)\mu_0 + \theta\mu'$. By equation (3.10), we have $P(\Phi, \chi, \mu'') > c$. By the same argument, there exists an ergodic measure $\nu \in M^{\mathrm{erg}}_{A,B} y(\alpha)$ such that $d^*(\nu, \mu'') < \zeta/2$ and $P(\Phi, \chi, \nu) > c$. So $d^*(\nu, \mu_0) < \zeta$.

By equation (3.20) and Lemma 3.18(1),

$$\{\mu \in M^{\mathrm{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\} \text{ is residual in } \{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}. \tag{3.21}$$

By Lemma 3.18(3),

$$\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c\} \text{ is residual in } \{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}. \tag{3.22}$$

If there is an invariant measure $\tilde{\mu}$ with $S_{\tilde{\mu}} = \Lambda$, then by Lemma 3.18(2), we have

$$\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c,\ S_\mu = \Lambda\} \text{ is residual in } \{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}. \tag{3.23}$$

So by equations (3.21), (3.22), and (3.23), we complete the proof of item (2).

(3) Fix $\alpha \in \mathrm{Int}(L_{A,B})$ and $\mu_0 \in M_{A,B}(\alpha)$ with $\max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \leq P(\Phi, \chi, \mu_0) < H_{A,B}(\Phi, \chi, \alpha)$. Then by item (2), the set $\{\mu \in M^{\mathrm{erg}}_{A,B} y(\alpha) : P(\Phi, \chi, \mu) = P(\Phi, \chi, \mu_0)\}$ is residual in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq P(\Phi, \chi, \mu_0)\}$. In particular, there is $\mu_\alpha \in M^{\mathrm{erg}}_{A,B}(\alpha)$ so that $P(\Phi, \chi, \mu_\alpha) = P(\Phi, \chi, \mu_0)$.

4. *Proofs of Theorems A and B*

Now we use 'multi-horseshoe' dense property and the results of asymptotically additive families obtained in §§2 and 3 to give a more general result than Theorems A and B.

THEOREM 4.1. *Let $X \in \mathscr{X}^1(M)$ and Λ be a basic set. Let $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ be a continuous function satisfying equations (3.10) and (3.11). Let $d \in \mathbb{N}$ and $(A, B) \in AA(\Phi, \Lambda)^d \times AA(\Phi, \Lambda)^d$ satisfying equation (3.7). Then:*

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(i) *for any* $\alpha \in \mathrm{Int}(L_{A,B})$, *any* $\mu_0 \in M_{A,B}(\alpha)$, *any* $\max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \le c \le P(\phi, \chi, \mu_0)$, *and any* $\eta, \zeta > 0$, *there is* $\nu \in M^{\mathrm{erg}}_{A,B}(\alpha)$ *such that* $d^*(\nu, \mu_0) < \zeta$ *and* $|P(\Phi, \chi, \nu) - c| < \eta$;

(ii) *for any* $\alpha \in \mathrm{Int}(L_{A,B})$ *and* $\max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \le c < H_{A,B}(\Phi, \chi, \alpha)$, *the set* $\{\mu \in M^{\mathrm{erg}}_{A,B} y(\alpha) : P(\Phi, \chi, \mu) = c, \quad S_\mu = \Lambda\}$ *is residual in* $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \ge c\}$;

(iii) *the set* $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}(\Phi, \Lambda), \quad \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}), \max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H_{A,B}(\Phi, \chi, \alpha)\}$ *coincides with* $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda), \quad \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}), \max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H_{A,B}(\Phi, \chi, \alpha)\}$.

If further $(A, B) \in A(\Phi, \Lambda)^d \times A(\Phi, \Lambda)^d$ *such that B satisfies equation (3.4),* a^i, b^i *has bounded variation and* $\sup_{t \in [0,s]} \|a^i_t\|_\infty < \infty$, $\sup_{t \in [0,s]} \|b^i_t\|_\infty < \infty$ *for some* $s > 0$ *and for any* $1 \le i \le d$. *Then:*

(iv) *the following two set are equal*

$$\{(\mathcal{P}_{A,B}(\mu), h_\mu(\Phi)) : \mu \in \mathcal{M}(\Phi, \Lambda),\ \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B})\},$$

$$\{(\mathcal{P}_{A,B}(\mu), h_\mu(\Phi)) : \mu \in \mathcal{M}_{\mathrm{erg}} yy(\Phi, \Lambda),\ \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B})\}.$$

Proof. It is known that each basic set is expansive, there is an invariant measure with full support, the set of periodic measures supported on in Λ is a dense subset of $\mathcal{M}(\Phi, \Lambda)$, and thus $\{\mu \in \mathcal{M}(\Phi, M) : h_\mu(\Phi) = 0\}$ is dense in $\mathcal{M}(\Phi, M)$. Then we obtain items (i)–(iii) by Theorems 2.5, 3.7, 3.9, and 3.10.

Let $\chi = 0$, then by item (iii), we have $\{(\mathcal{P}_{A,B}(\mu), h_\mu(\Phi)) : \mu \in \mathcal{M}(\Phi, \Lambda), \quad \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}), \quad 0 \le h_\mu(\Phi) < H_{A,B}(\Phi, \alpha)\}$ coincides with $\{(\mathcal{P}_{A,B}(\mu), h_\mu(\Phi)) : \mu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda), \quad \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}), \quad 0 \le h_\mu(\Phi) < H_{A,B}(\Phi, \alpha)\}$. Combining with Theorems 3.4 and 3.5, we obtain item (iv). □

Now we give the proofs of Theorems A and B.

For a continuous function g, let $g_t = \int_0^t g(\phi_\tau(x))\, d\tau$, then $\sup_{t \in [0,1]} \|g_t\|_\infty < \infty$, $(g_t)_{t \ge 0}$ is an additive family of continuous functions. So letting $\chi \equiv 0$ and $d = 1$ in Theorem 4.1(ii), we obtain Theorem A.

Let $\chi \equiv 0$ and $d = 2$ in Theorem 4.1(ii) for any $\alpha \in \mathrm{Int}(L_{g,h})$ and

$$0 \le c < \max\{h_\nu(X) : \nu \in M_{g,h}(\alpha)\},$$

the set $\{\mu \in M^{\mathrm{erg}}_{g,h} y(\alpha) : h_\mu(X) = c,\ S_\mu = \Lambda\}$ is residual in $\{\mu \in M_{g,h}(\alpha) : h_\mu(X) \ge c\}$. Take $c = 0$, then the set $\{\mu \in M^{\mathrm{erg}}_{g,h}(\alpha) : S_\mu = \Lambda\}$ is dense in $M_{g,h}(\alpha)$. From [14, Proposition 5.7], $\mathcal{M}_{\mathrm{erg}} yy(\Phi, \Lambda)$ is a G_δ subset of $\mathcal{M}(\Phi, \Lambda)$. Then $M^{\mathrm{erg}}_{g,h}(\alpha)$ is a G_δ subset of $M_{g,h}(\alpha)$, and thus $M^{\mathrm{erg}}_{g,h} y(\alpha)$ is residual in $M_{g,h}(\alpha)$. Since $M_{g,h}(\alpha)$ is convex, by Lemma 3.17, the set $\{\mu \in M_{g,h}(\alpha) : S_\mu = \Lambda\}$ is residual in $M_{g,h}(\alpha)$. So $\{\mu \in M^{\mathrm{erg}}_{g,h}(\alpha) : S_\mu = \Lambda\}$ is residual in $M_{g,h}(\alpha)$ and we obtain Theorem B.

5. *Singular hyperbolic attractors*

In this section, we consider singular hyperbolic attractors and give corresponding results on Question 1.2.

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5.1. *Singular hyperbolicity and geometric Lorenz attractors.* First, we recall the definition of singular hyperbolicity which was introduced by Morales, Pacifico, and Pujals [29] to describe the geometric structure of Lorenz attractors and these ideas were extended to higher dimensional cases in [25, 28].

Definition 5.1. Given a vector field $X \in \mathscr{X}^1(M)$, a compact and invariant set Λ is *singular hyperbolic* if it admits a continuous $\mathrm{D}\phi_t$-invariant splitting $T_\Lambda M = E^{ss} \oplus E^{cu}$ and constants $C, \eta > 0$ such that, for any $x \in \Lambda$ and any $t \geq 0$:

- $E^{ss} \oplus E^{cu}$ is a *dominated splitting*: $\|\mathrm{D}\phi_t|_{E^{ss}(x)}\| \cdot \|\mathrm{D}\phi_{-t}|_{E^{cu}(\phi_t(x))}\| < Ce^{-\eta t}$;
- E^{ss} is uniformly contracted by $\mathrm{D}\phi_t$: $\|\mathrm{D}\phi_t(v)\| < Ce^{-\eta t}\|v\|$ for any $v \in E^{ss}(x) \setminus \{0\}$;
- E^{cu} is *sectionally expanded* by $\mathrm{D}\phi_t$: $|\det \mathrm{D}\phi_t|_{V_x}| > Ce^{\eta t}$ for any two-dimensional subspace $V_x \subset E^{cu}_x$.

LEMMA 5.2. [30, Theorem A and Lemma 2.9] *Given a vector field $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, if Λ is sectional hyperbolic, all the singularities in Λ are hyperbolic, then Φ is entropy expansive on Λ and thus the metric entropy function $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous.*

We recall the concept of a homoclinic class of a hyperbolic periodic orbit.

Definition 5.3. Given a vector field $X \in \mathscr{X}^1(M)$, an invariant compact subset $\Lambda \subset M$ is a *homoclinic class* if there exists a hyperbolic periodic point $p \in \Lambda \cap \mathrm{Per}(X)$ so that

$$\Lambda : = \overline{W^s(\mathrm{Orb}(p)) \pitchfork W^u(\mathrm{Orb}(p))},$$

that is, it is the closure of the points of transversal intersection between stable and unstable manifolds of the periodic orbit $\mathrm{Orb}(p)$ of p. We say a homoclinic class is *non-trivial* if it is not reduced to a single hyperbolic periodic orbit.

From [2, Theorem 2.17], any non-empty homoclinic class Λ contains a dense set of periodic orbits. Then there is an invariant measure $\tilde{\mu}$ with $S_{\tilde{\mu}} = \Lambda$ by [14, Proposition 21.12].

LEMMA 5.4. *Let $X \in \mathscr{X}^1(M)$ and Λ be a homoclinic class of X. Then there is an invariant measure $\tilde{\mu}$ with $S_{\tilde{\mu}} = \Lambda$.*

Now we give the definition of geometric Lorenz attractors following Guckenheimer and Williams [18, 19, 39] for vector fields on a closed 3-manifold M^3.

Definition 5.5. We say $X \in \mathscr{X}^r(M^3)$ $(r \geq 1)$ exhibits a *geometric Lorenz attractor* Λ if X has an attracting region $U \subset M^3$ such that $\Lambda = \bigcap_{t>0} \phi_t^X(U)$ is a singular hyperbolic attractor and satisfies the following (see Figure 2).

- There exists a unique singularity $p \in \Lambda$ with three exponents $\lambda_1 < \lambda_2 < 0 < \lambda_3$, which satisfy $\lambda_1 + \lambda_3 < 0$ and $\lambda_2 + \lambda_3 > 0$.
- Λ admits a C^r-smooth cross-section S which is C^1-diffeomorphic to $[-1, 1] \times [-1, 1]$ such that $\Gamma = \{0\} \times [-1, 1] = W^s_{\mathrm{loc}}(\sigma) \cap S$, and for every $z \in U \setminus W^s_{\mathrm{loc}}(\sigma)$, there exists $t > 0$ such that $\phi_t^X(z) \in S$.

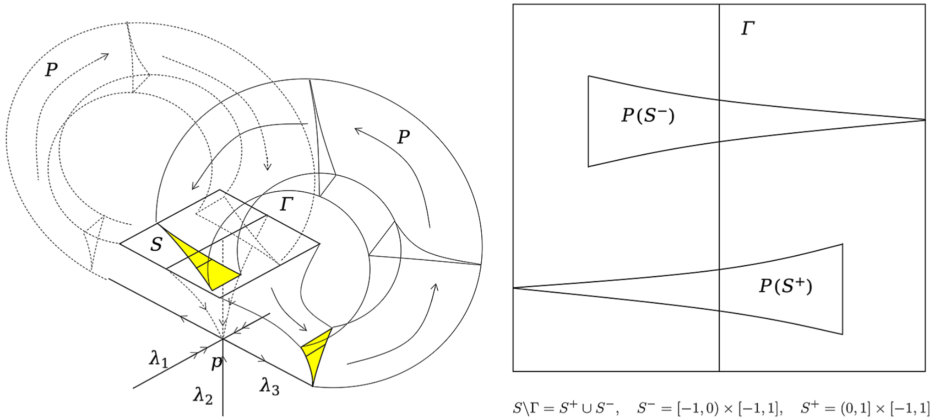

FIGURE 2. Geometric Lorenz attractor and return map.

- Up to the previous identification, the Poincaré map $P : S \setminus \Gamma \to S$ is a skew-product map

$$P(x, y) = \big(f(x)\,,\ g(x, y)\big) \quad \text{for all } (x, y) \in [-1, 1]^2 \setminus \Gamma.$$

  Moreover, it satisfies:
  - $g(x, y) < 0$ for $x > 0$, and $g(x, y) > 0$ for $x < 0$;
  - $\sup_{(x,y)\in S\setminus\Gamma} \left|\partial g(x, y)/\partial y\right| < 1$ and $\sup_{(x,y)\in S\setminus\Gamma} \left|\partial g(x, y)/\partial x\right| < 1$;
  - the one-dimensional quotient map $f : [-1, 1] \setminus \{0\} \to [-1, 1]$ is C^1-smooth and satisfies $\lim_{x\to 0^-} f(x) = 1$, $\lim_{x\to 0^+} f(x) = -1$, $-1 < f(x) < 1$, and $f'(x) > \sqrt{2}$ for every $x \in [-1, 1] \setminus \{0\}$.

It has been proved that the geometric Lorenz attractor is a homoclinic class [**2**, Theorem 6.8] and C^2-robust [**34**, Proposition 4.7].

PROPOSITION 5.6. *Let $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and $X \in \mathscr{X}^r(M^3)$. If X exhibits a geometric Lorenz attractor Λ with attracting region U, then there exists a C^r-neighborhood $\mathcal{U}$ of X in $\mathscr{X}^r(M^3)$ such that for every $Y \in \mathcal{U}$, U is an attracting region of Y, and the maximal invariant set $\Lambda_Y = \bigcap_{t>0} \phi_t^Y(U)$ is a geometric Lorenz attractor. Moreover, the geometric Lorenz attractor is a singular hyperbolic homoclinic class, and every pair of periodic orbits are homoclinic related.*

For singular hyperbolic attractors, we have the following result.

THEOREM C. *There exists a Baire residual subset $\mathcal{R}^r \subset \mathscr{X}^r(M^3)$, ($r \in \mathbb{N}_{\geq 2}$) and a Baire residual subset $\mathcal{R} \subset \mathscr{X}^1(M)$ so that if Λ is a geometric Lorenz attractor of $X \in \mathcal{R}^r$ or a singular hyperbolic attractor of $X \in \mathcal{R}$, then for any continuous function g, h on Λ, we have:*

(i) *for any $\alpha \in \mathrm{Int}(L_g)$ and $0 \leq c < \max\{h_\mu(\Phi) : \mu \in M_g(\alpha)\}$, the set $\{\mu \in M_g^{\mathrm{erg}} y(\alpha) : h_\mu(\Phi) = c,\ S_\mu = \Lambda\}$ is residual in $\{\mu \in M_g(\alpha) : h_\mu(\Phi) \geq c\}$;*

(ii) *for any $\alpha \in \mathrm{Int}(L_{g,h})$, the set $\{\mu \in M_{g,h}^{\mathrm{erg}}(\alpha) : S_\mu = \Lambda\}$ is residual in $M_{g,h}(\alpha)$.*

Remark 5.7. We will prove one general result Theorem 5.15 stating that when Λ is a singular hyperbolic homoclinic class such that each pair of periodic orbits are homoclinically related and $\mathcal{M}(\Phi, \Lambda) = \overline{\mathcal{M}_1(\Lambda)} = \overline{\mathcal{M}_{\mathrm{per}}(\Lambda)}$, then the conclusions of Theorem C hold. Then Theorem C is a direct consequence of Theorem 5.15. The reason is that when Λ is a Lorenz attractor of vector fields in a Baire residual subset $\mathcal{R}^r \subset \mathcal{X}^r(M^3)$, ($r \in \mathbb{N}_{\geq 2}$) or Λ is a singular hyperbolic attractor Λ of vector fields in a Baire residual set $\mathcal{R} \subset \mathcal{X}^1(M)$, then Λ is a homoclinic class such that each pair of periodic orbits are homoclinically related (cf. [**2**, Theorem 6.8] for Lorenz attractors and [**12**, Theorem B] for singular hyperbolic attractors) and $\mathcal{M}(\Phi, \Lambda) = \overline{\mathcal{M}_1(\Lambda)} = \overline{\mathcal{M}_{\mathrm{per}}(\Lambda)}$ (cf. [**34**, Theorems A and B]).

5.2. *Proof of Theorem C.* Given a vector field $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, let $\mathcal{N} \subset \mathcal{M}(\Phi, \Lambda)$ be a convex set. We denote $\mathcal{N}_{\mathrm{erg}} y = \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda) \cap \mathcal{N}$. Let $d \in \mathbb{N}$ and $(A, B) \in A(\Phi, \Lambda)^d \times A(\Phi, \Lambda)^d$ such that B satisfies equation (3.4). Denote

$$L^{\mathcal{N}}_{A,B} = \{\mathcal{P}_{A,B}(\mu) : \mu \in \mathcal{N}\}.$$

For any $\alpha \in L^{\mathcal{N}}_{A,B}$, denote

$$M^{\mathcal{N}}_{A,B}(\alpha) = \{\mu \in \mathcal{N} : \mathcal{P}_{A,B}(\mu) = \alpha\}, \quad M^{\mathrm{erg},\mathcal{N}}_{A,B}(\alpha) = \{\mu \in \mathcal{N}_{\mathrm{erg}} y : \mathcal{P}_{A,B}(\mu) = \alpha\}.$$

Let $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ be a continuous function. We define the pressure of χ with respect to μ by $P(\Phi, \chi, \mu) = h_\mu(\Phi) + \chi(\mu)$. For any $\alpha \in L^{\mathcal{N}}_{A,B}$, denote

$$H^{\mathcal{N}}_{A,B}(\Phi, \chi, \alpha) = \sup\{P(\phi, \chi, \mu) : \mu \in M^{\mathcal{N}}_{A,B}(\alpha)\}.$$

In particular, when $\chi \equiv 0$, we write

$$H^{\mathcal{N}}_{A,B}(\Phi, \alpha) = H^{\mathcal{N}}_{A,B}(\Phi, 0, \alpha) = \sup\{h_\mu(\Phi) : \mu \in M^{\mathcal{N}}_{A,B}(\alpha)\}.$$

If we replace $\mathcal{M}(\Phi, \Lambda)$ by $\mathcal{N}$ in §§3.3–3.5, the results of Theorems 3.9, 3.10, and Remark 3.12 also hold. In fact, the convexity of $\mathcal{M}(\Phi, \Lambda)$ is one core property in the proof of Theorems 3.9, 3.10, and Remark 3.12. Since $\mathcal{N}$ is convex, then the arguments of §3.3 are also true. Here, we omit the proof.

THEOREM 5.8. *Given a vector field $X \in \mathscr{X}^1(M)$ and an invariant compact set Λ, assume that the metric entropy $\mathcal{M}(\Phi, \Lambda) \ni \mu \mapsto h_\mu(\Phi)$ is upper semi-continuous. Let $d \in \mathbb{N}$ and $(A, B) \in A(\Phi, \Lambda)^d \times A(\Phi, \Lambda)^d$ such that B satisfies equation (3.4). Let $\mathcal{N} \subset \mathcal{M}(\Phi, \Lambda)$ be a convex set. Assume that the following holds: for any $F = \mathrm{cov}\{\mu_i\}_{i=1}^m \subseteq \mathcal{N}$, and any $\eta, \zeta > 0$, there are compact invariant subsets $\Lambda_i \subseteq \Theta \subsetneq \Lambda$ such that for each $i \in \{1, 2, \ldots, m\}$:*

(1) *for any $a \in \mathrm{Int}(\mathcal{P}_{A,B}(\mathcal{M}(\Phi, \Theta)))$ and any $\varepsilon > 0$, there exists an ergodic measure $\mu_a \in \mathcal{N}$ supported on Θ with $\mathcal{P}_{A,B}(\mu_a) = a$ such that $|h_{\mu_a}(\Phi) - H(\Phi, a, \Theta)| < \varepsilon$, where $H(\Phi, a, \Theta) = \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}(\Phi, \Theta)$ and $\mathcal{P}_{A,B}(\mu) = a\}$;*
(2) *$h_{\mathrm{top}}(\Lambda_i) > h_{\mu_i}(\Phi) - \eta$;*
(3) *$d_H(K, \mathcal{M}(\Phi, \Theta)) < \zeta$, $d_H(\mu_i, \mathcal{M}(\Phi, \Lambda_i)) < \zeta$.*

Let $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ be a continuous function. Then we have the following.

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(i) *For any* $\alpha \in \mathrm{Int}(L^{\mathcal{N}}_{A,B})$*, any* $\mu_0 \in M^{\mathcal{N}}_{A,B}(\alpha)$*, and any* $\eta, \zeta > 0$*, there is* $\nu \in M^{\mathrm{erg},\mathcal{N}}_{A,B}(\alpha)$ *such that* $d^*(\nu, \mu_0) < \zeta$ *and* $|P(\Phi, \chi, \nu) - P(\Phi, \chi, \mu_0)| < \eta$.

If further $\{\mu \in \mathcal{N} : h_\mu(\Phi) = 0\}$ *is dense in* $\mathcal{N}$*, and* $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ *satisfies equations (3.10) and (3.11), then we have:*

(ii) *for any* $\alpha \in \mathrm{Int}(L^{\mathcal{N}}_{A,B})$*, any* $\mu_0 \in M^{\mathcal{N}}_{A,B}(\alpha)$*, any* $\max_{\mu\in M^{\mathcal{N}}_{A,B}(\alpha)} \chi(\mu) \le c \le P(\phi, \chi, \mu_0)$ *and any* $\eta, \zeta > 0$*, there is* $\nu \in M^{\mathrm{erg},\mathcal{N}}_{A,B}(\alpha)$ *such that* $d^*(\nu, \mu_0) < \zeta$ *and* $|P(\Phi, \chi, \nu) - c| < \eta$;

(iii) *for any* $\alpha \in \mathrm{Int}(L^{\mathcal{N}}_{A,B})$ *and* $\max_{\mu\in M^{\mathcal{N}}_{A,B}(\alpha)} \chi(\mu) \le c < H^{\mathcal{N}}_{A,B}(\Phi, \chi, \alpha)$*, the set* $\{\mu \in M^{\mathrm{erg},\mathcal{N}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c\}$ *is residual in* $\{\mu \in M^{\mathcal{N}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) \ge c\}$*. If further there is an invariant measure* $\tilde{\mu} \in \mathcal{N}$ *with* $S_{\tilde{\mu}} = \Lambda$*, then for any* $\alpha \in \mathrm{Int}(L^{\mathcal{N}}_{A,B})$ *and* $\max_{\mu\in M^{\mathcal{N}}_{A,B}(\alpha)} \chi(\mu) \le c < H^{\mathcal{N}}_{A,B}(\Phi, \chi, \alpha)$*, the set* $\{\mu \in M^{\mathrm{erg},\mathcal{N}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c,\ S_\mu = \Lambda\}$ *is residual in* $\{\mu \in M^{\mathcal{N}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) \ge c\}$;

(iv) *the set* $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{N},\ \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L^{\mathcal{N}}_{A,B}), \max_{\mu\in M^{\mathcal{N}}_{A,B}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H^{\mathcal{N}}_{A,B}(\Phi, \chi, \alpha)\}$ *coincides with* $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{N}_{\mathrm{erg}},\ \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L^{\mathcal{N}}_{A,B}), \max_{\mu\in M^{\mathcal{N}}_{A,B}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H^{\mathcal{N}}_{A,B}(\Phi, \chi, \alpha)\}$.

Denote by $\mathrm{Sing}(\Lambda)$ the set of singularities for the vector field X in Λ, by $\mathcal{M}_{\mathrm{per}}(\Lambda)$ the set of periodic measures supported on Λ, and set

$$\mathcal{M}_1(\Lambda) = \{\mu \in \mathcal{M}(\Phi, \Lambda) : \mu(\mathrm{Sing}(\Lambda)) = 0\} \quad \text{and} \quad \mathcal{M}_0(\Lambda) = \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda) \cap \mathcal{M}_1(\Lambda).$$

PROPOSITION 5.9. [**33**, Proposition 4.12] *Let* $X \in \mathscr{X}^1(M)$ *and* Λ *be a singular hyperbolic homoclinic class of X. Assume each pair of periodic orbits of* Λ *are homoclinic related. Then for each* $\varepsilon > 0$ *and any* $\mu \in \mathcal{M}_1(\Lambda)$*, there exist a basic set* $\Lambda' \subset \Lambda$ *and* $\nu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda')$ *so that*

$$d^*(\nu, \mu) < \varepsilon \quad \text{and} \quad h_\nu(\Phi) > h_\mu(\Phi) - \varepsilon.$$

PROPOSITION 5.10. [**33**, Proposition 4.13] *Let* $X \in \mathscr{X}^1(M)$ *and* Λ *be a singular hyperbolic homoclinic class of X. Assume each pair of periodic orbits of* Λ *are homoclinic related, and* $\mathcal{M}(\Phi, \Lambda) = \overline{\mathcal{M}_1(\Lambda)}$*. Then for each* $\varepsilon > 0$ *and any* $\mu \in \mathcal{M}(\Phi, \Lambda)$*, there exist a basic set* $\Lambda' \subset \Lambda$ *and* $\nu \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda')$ *so that*

$$d^*(\nu, \mu) < \varepsilon \quad \text{and} \quad h_\nu(\Phi) > h_\mu(\Phi) - \varepsilon.$$

LEMMA 5.11. [**33**, Lemma 4.5] *Let* Λ_1 *and* Λ_2 *be two basic sets of* $X \in \mathcal{X}^1(M)$*. Assume there exists hyperbolic periodic points* $p_1 \in \Lambda_1$ *and* $p_2 \in \Lambda_2$ *such that* $\mathrm{Orb}(p_1)$ *and* $\mathrm{Orb}(p_2)$ *are homoclinically related. Then there exists a larger basic set* Λ *that contains both* Λ_1 *and* Λ_2.

Now we state a result on the 'multi-horseshoe' dense property of singular hyperbolic homoclinic classes.

THEOREM 5.12. *Let $X \in \mathscr{X}^1(M)$ and Λ be a singular hyperbolic homoclinic class of X. Assume each pair of periodic orbits of Λ are homoclinic related. Then Λ satisfies the 'multi-horseshoe' dense property on $\mathcal{M}_1(\Lambda)$. Moreover, if $\mathcal{M}(\Phi, \Lambda) = \overline{\mathcal{M}_1(\Lambda)} = \overline{\mathcal{M}_{\mathrm{per}}(\Lambda)}$, then Λ satisfies the 'multi-horseshoe' dense property.*

Proof. Fix $F = \mathrm{cov}\{\mu_i\}_{i=1}^m \subseteq \mathcal{M}_1(\Lambda)(\mathcal{M}(\Phi, \Lambda))$ and $\eta, \zeta > 0$. By Proposition 5.9 (Proposition 5.10), for each $1 \le i \le m$, there exist a basic set $\Lambda_i' \subset \Lambda$ and $\nu_i \in \mathcal{M}_{\mathrm{erg}}(\Phi, \Lambda_i')$ so that

$$d^*(\nu_i, \mu_i) < \frac{\zeta}{2} \quad \text{and} \quad h_{\nu_i}(\Phi) > h_{\mu_i}(\Phi) - \frac{\eta}{2}.$$

By Lemma 5.11, there exists a larger basic set $\tilde{\Lambda}$ that contains every Λ_i'. Applying Theorem 2.5 to $\tilde{\Lambda}$, $\tilde{F} = \mathrm{cov}\{\nu_i\}_{i=1}^m \subseteq \mathcal{M}(\Phi, \tilde{\Lambda})$ and $\eta/2, \zeta/2$, we complete the proof. □

Now we show that the results of Theorem 5.8 hold for singular hyperbolic homoclinic classes.

THEOREM 5.13. *Let $X \in \mathscr{X}^1(M)$ and Λ be a singular hyperbolic homoclinic class of X. Assume each pair of periodic orbits of Λ are homoclinic related. Let $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ be a continuous function satisfying equations (3.10) and (3.11). Let $d \in \mathbb{N}$ and $(A, B) \in A(\Phi, \Lambda)^d \times A(\Phi, \Lambda)^d$ satisfying equation (3.7). Then:*

(i) *for any $\alpha \in \mathrm{Int}(L_{A,B}^{\mathcal{M}_1(\Lambda)})$, any $\mu_0 \in M_{A,B}^{\mathcal{M}_1(\Lambda)}(\alpha)$, any $\max_{\mu \in M_{A,B}^{\mathcal{M}_1(\Lambda)}(\alpha)} \chi(\mu) \le c \le P(\phi, \chi, \mu_0)$, and any $\eta, \zeta > 0$, there is $\nu \in M_{A,B}^{\mathrm{erg}, \mathcal{M}_1(\Lambda)}(\alpha)$ such that $d^*(\nu, \mu_0) < \zeta$ and $|P(\Phi, \chi, \nu) - c| < \eta$;*

(ii) *for any $\alpha \in \mathrm{Int}(L_{A,B}^{\mathcal{M}_1(\Lambda)})$ and $\max_{\mu \in M_{A,B}^{\mathcal{M}_1(\Lambda)}(\alpha)} \chi(\mu) \le c < H_{A,B}^{\mathcal{M}_1(\Lambda)}(\Phi, \chi, \alpha)$, the set $\{\mu \in M_{A,B}^{\mathrm{erg}, \mathcal{M}_1(\Lambda)}(\alpha) : P(\Phi, \chi, \mu) = c\}$ is residual in $\{\mu \in M_{A,B}^{\mathcal{M}_1(\Lambda)}(\alpha) : P(\Phi, \chi, \mu) \ge c\}$;*

(iii) *the set $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}_1(\Lambda),\ \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}^{\mathcal{M}_1(\Lambda)}),\ \max_{\mu \in M_{A,B}^{\mathcal{M}_1(\Lambda)}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H_{A,B}^{\mathcal{M}_1(\Lambda)}(\Phi, \chi, \alpha)\}$ coincides with $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}_0(\Lambda),\ \alpha = \mathcal{P}_{A,B}(\mu) \in \mathrm{Int}(L_{A,B}^{\mathcal{M}_1(\Lambda)}),\ \max_{\mu \in M_{A,B}^{\mathcal{M}_1(\Lambda)}(\alpha)} \chi(\mu) \le P(\Phi, \chi, \mu) < H_{A,B}^{\mathcal{M}_1(\Lambda)}(\Phi, \chi, \alpha)\}$.*

Proof. Note that since Λ is a singular hyperbolic homoclinic class, the vector field X satisfies the star condition in a neighborhood of Λ. More precisely, there exist a neighborhood U of Λ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathscr{X}^1(M)$ such that every critical element contained in U associated to a vector field $Y \in \mathcal{U}$ is hyperbolic. Then by Lemma 5.2, the entropy function is upper semi-continuous. Note that $\mathcal{M}(\Phi, \Theta) \subset \mathcal{M}_1(\Lambda)$ if $\Theta \subset \Lambda$ is a horseshoe. Since the metric entropy of periodic measure is zero, then $\{\mu \in \mathcal{M}_1(\Lambda) : h_\mu(\Phi) = 0\}$ is dense in $\mathcal{M}_1(\Lambda)$ by Theorem 5.12. So we complete the proof by Theorems 3.7, 5.8, and 5.12. □

Combining with Theorem 5.6, the results of Theorem 5.13 hold for geometric Lorenz attractors.

COROLLARY 5.14. *Let $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and $X \in \mathscr{X}^r(M^3)$. If X exhibits a geometric Lorenz attractor Λ, then the results of Theorem 5.13 hold for Λ.*

If further we have $\mathcal{M}(\Phi, \Lambda) = \overline{\mathcal{M}_1(\Lambda)} = \overline{\mathcal{M}_{\text{per}}(\Lambda)}$, then using Theorems 3.7, 3.9, 3.10, Remark 3.12, Theorem 5.12, and Lemma 5.4, we have the following result for singular hyperbolic homoclinic classes.

THEOREM 5.15. *Let $X \in \mathscr{X}^1(M)$ and Λ be a singular hyperbolic homoclinic class of X. Assume each pair of periodic orbits of Λ are homoclinic related, and $\mathcal{M}(\Phi, \Lambda) = \overline{\mathcal{M}_1(\Lambda)} = \overline{\mathcal{M}_{\text{per}}(\Lambda)}$. Let $\chi : \mathcal{M}(\Phi, \Lambda) \to \mathbb{R}$ be a continuous function satisfying equations (3.10) and (3.11). Let $d \in \mathbb{N}$ and $(A, B) \in A(\Phi, \Lambda)^d \times A(\Phi, \Lambda)^d$ satisfying equation (3.7). Then:*

(i) *for any $\alpha \in \text{Int}(L_{A,B})$, any $\mu_0 \in M_{A,B}(\alpha)$, any $\max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \leq c \leq P(\phi, \chi, \mu_0)$, and any $\eta, \zeta > 0$, there is $\nu \in M^{\text{erg}}_{A,B}(\alpha)$ such that $d^*(\nu, \mu_0) < \zeta$ and $|P(\Phi, \chi, \nu) - c| < \eta$;*

(ii) *for any $\alpha \in \text{Int}(L_{A,B})$ and $\max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \leq c < H_{A,B}(\Phi, \chi, \alpha)$, the set $\{\mu \in M^{\text{erg}}_{A,B}(\alpha) : P(\Phi, \chi, \mu) = c,\ S_\mu = \Lambda\}$ is residual in $\{\mu \in M_{A,B}(\alpha) : P(\Phi, \chi, \mu) \geq c\}$;*

(iii) *the set $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}(\Phi, M), \alpha = \mathcal{P}_{A,B}(\mu) \in \text{Int}(L_{A,B}), \max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \leq P(\Phi, \chi, \mu) < H_{A,B}(\Phi, \chi, \alpha)\}$ coincides with $\{(\mathcal{P}_{A,B}(\mu), P(\Phi, \chi, \mu)) : \mu \in \mathcal{M}_{\text{erg}}(\Phi, M),\ \alpha = \mathcal{P}_{A,B}(\mu) \in \text{Int}(L_{A,B}), \max_{\mu \in M_{A,B}(\alpha)} \chi(\mu) \leq P(\Phi, \chi, \mu) < H_{A,B}(\Phi, \chi, \alpha)\}$.*

Proceeding in a similar manner to Theorems A and B, letting $\chi \equiv 0$ and $d = 1, 2$ in Theorem 5.15, we obtain Theorem C by Remark 5.7.

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