Forum of Mathematics, Sigma (2014), Vol. 2, e6, 36 pages
doi:10.1017/fms.2014.8

DEGENERATIONS OF COMPLEX DYNAMICAL SYSTEMS

LAURA DE MARCO[1] and XANDER FABER[2]

[1] Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL, USA;
email: demarco@uic.edu

[2] Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI, USA;
email: xander@math.hawaii.edu

Received 28 May 2013; accepted 17 March 2014

Abstract

We show that the weak limit of the maximal measures for any degenerating sequence of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ must be a countable sum of atoms. For a one-parameter family f_t of rational maps, we refine this result by showing that the measures of maximal entropy have a *unique* limit on $\hat{\mathbb{C}}$ as the family degenerates. The family f_t may be viewed as a single rational function on the Berkovich projective line $\mathbf{P}^1_{\mathbb{L}}$ over the completion of the field of formal Puiseux series in t, and the limiting measure on $\hat{\mathbb{C}}$ is the 'residual measure' associated with the equilibrium measure on $\mathbf{P}^1_{\mathbb{L}}$. For the proof, we introduce a new technique for quantizing measures on the Berkovich projective line and demonstrate the uniqueness of solutions to a quantized version of the pullback formula for the equilibrium measure on $\mathbf{P}^1_{\mathbb{L}}$.

2010 Mathematics Subject Classification: primary 37F10, 37P50; secondary 37F45.

1. Introduction

Let $f_k : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a sequence of endomorphisms of the Riemann sphere of degree $d \geqslant 2$ that diverges in the space of all endomorphisms. Concretely, this means that at least one zero and pole of f_k are colliding in the limit. Our main goal is to understand the degeneration of the dynamical features of f_k and, ultimately, to extract useful information from a 'limit dynamical system'. In this article, we concentrate on the measure of maximal entropy.

The existence and uniqueness of a measure of maximal entropy μ_f for a rational function f of degree $\geqslant 2$ were shown in 1983 [12, 15, 17]. Shortly after, Mañé observed that the measure μ_f moves continuously in families [18], with the weak-$*$ topology of measures and the uniform topology on the space of rational functions. By contrast, the Julia set $J(f) = \operatorname{supp} \mu_f$ fails to move continuously (in the Hausdorff topology) in the presence of bifurcations [16].

The space Rat_d of complex rational functions of degree $d \geqslant 2$ can be identified with the complement of a hypersurface in $\overline{\mathrm{Rat}}_d = \mathbb{P}^{2d+1}$. In [4], the first author showed that for 'most' degenerating sequences $f_k \to \Phi \in \partial\mathrm{Rat}_d$, a limit of the maximal measures μ_{f_k} will exist, depending only on the limit point Φ, and it can be expressed as a countably infinite sum of atoms. (The measures μ_{f_k} themselves are atomless.) There it was also shown that Mañé's continuity property for maximal measures *does not extend* to all of $\overline{\mathrm{Rat}}_d$. Although weak limits of maximal measures for degenerating sequences may not be unique, our first main result shows that every weak limit is purely atomic.

THEOREM A. *Let f_k be a sequence that diverges in the space* Rat_d *of complex rational functions of degree $d \geqslant 2$, and assume that the measures of maximal entropy μ_k converge to a probability measure μ on $\hat{\mathbb{C}}$. Then μ is equal to a countable sum of atoms.*

Our second main result shows that Mañé's continuity property *does extend* to degenerating one-parameter families. Moreover, we are able to give a refined description of the limit measure using an associated dynamical system on the Berkovich projective line.

THEOREM B. *Let $\{f_t : t \in \mathbb{D}\}$ be a meromorphic family of rational functions of degree $d \geqslant 2$ that is degenerate at $t = 0$. The measures of maximal entropy μ_t converge weakly on the Riemann sphere to a limiting probability measure μ_0 as $t \to 0$. The measure μ_0 is equal to the residual equilibrium measure for the induced rational map $f : \mathbf{P}^1_{\mathbb{L}} \to \mathbf{P}^1_{\mathbb{L}}$ on the Berkovich projective line, where $\mathbb{L}$ is the completion of the field of formal Puiseux series in t.*

REMARK 1.1. The continuity of maximal measures on $\hat{\mathbb{C}}$ can fail for degenerating families over a parameter space of dimension two; see [4, Section 5].

REMARK 1.2. While we prefer to work with the more 'geometric' field $\mathbb{L}$, one can replace it with the field of formal Laurent series $\mathbb{C}((t))$ in the statement of the theorem.

One should view the Berkovich dynamical system $(f, \mathbf{P}^1_{\mathbb{L}})$ as the limit of dynamical systems $(f_t, \hat{\mathbb{C}})$ as $t \to 0$. This fruitful perspective was introduced

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

by Jan Kiwi in his work on cubic polynomials and quadratic rational maps; see [13, 14] and [3]. A closely related construction, viewing degenerations of polynomial maps as actions on trees, can be seen in [7]. Charles Favre has recently constructed a compactification of the space of rational maps, where the boundary points are rational maps on a Berkovich $\mathbf{P}^1$ [10]. Our work is very much inspired by these results. The Berkovich space viewpoint allows us to recover the results in [4], and it provides a conceptual explanation for the form of the limiting measures. In a sequel to this article, we describe a countable-state Markov process that allows one to compute the residual measure explicitly [6].

As with nondegenerating families, the Julia sets of f_t may fail to converge to a limit as $t \to 0$. Consider the example of $f_t(z) = t(z + z^{-1})$ in Rat_2. As $t \to 0$ along the real axis, the Julia set of f_t is equal to the imaginary axis, while there is a sequence $t_n \to 0$ (tangent to the imaginary axis) for which $J(f_{t_n}) = \hat{\mathbb{C}}$. Mañé used the continuity of $f \mapsto \mu_f$ to deduce that the Hausdorff dimension of μ_f is a continuous function of f, but this property does not extend to degenerating families; for example, the measures for a flexible Lattès family have dimension two, while the limit measures always have dimension zero.

The measure of maximal entropy μ_f for a rational function f of degree $d \geqslant 2$ is characterized by the conditions that (a) it does not charge exceptional points, and (b) it satisfies the pullback relation

$$\frac{1}{d} f^* \mu_f = \mu_f.$$

To prove Theorem A, we show that any weak limit of measures of maximal entropy on $\hat{\mathbb{C}}$ must satisfy an appropriately defined pullback formula (Theorem 2.4); we then show that any measure satisfying this formula (for all iterates) is atomic. The pullback formula is phrased in terms of 'paired measures', which are ad hoc objects that we introduce to keep track of weak limits of measures in two sets of coordinates simultaneously. This is all accomplished in Section 2.

The proof of Theorem B (which inspired our proof of Theorem A) is more conceptual and can be divided into three parts, each with its own collection of results that are of independent interest. We sketch these results here.

Step 1. Dynamics on a complex surface. In Section 3, we view the holomorphic family $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ as one (meromorphic) dynamical system

$$F : X \dashrightarrow X$$

on the complex surface $X = \mathbb{D} \times \mathbb{P}^1$, given by $(t, z) \mapsto (t, f_t(z))$ for $t \neq 0$. By hypothesis, F will have points of indeterminacy in the central fiber $X_0 = \{0\} \times \mathbb{P}^1$.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

If F collapses X_0 to a point, we let $\pi : Y \to X$ be the (minimal) blow-up of the target surface such that $F : X \dashrightarrow Y$ is nonconstant at $t = 0$; otherwise, set $Y = X$ and $\pi = \mathrm{Id}$. By counting multiplicities at the indeterminacy points of F, we define a notion of pullback F^* from measures on the central fiber of Y to measures on X_0. We prove (Theorem 3.5) that any weak limit ν of the measures μ_t on the central fiber of Y satisfies a pullback relation:

$$\frac{1}{d} F^* \nu = \pi_* \nu. \tag{1.0.1}$$

The proof relies on the Argument Principle for handling the measure at the points of indeterminacy for F.

Step 2. Dynamics and Γ-measures on the Berkovich projective line. Let k be an algebraically closed field of characteristic zero that is complete with respect to a nontrivial non-Archimedean absolute value. The Berkovich analytification of the projective line $\mathbb{P}^1_k$ will be denoted as $\mathbf{P}^1$; it is a compact, Hausdorff, and uniquely arcwise connected topological space. A rational function $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ extends functorially to $\mathbf{P}^1$. If $d = \deg(f) \geqslant 2$, then the equilibrium measure μ_f may be characterized as in the complex case by the conditions that (a) it does not charge exceptional points of $\mathbb{P}^1(k)$, and (b) it satisfies the pullback relation $\frac{1}{d} f^* \mu_f = \mu_f$ [11]. See [1] for a reference specific to the dynamics on $\mathbf{P}^1$, or see [2] for the more general theory of non-Archimedean analytic spaces.

The goal of Section 4 is to define a notion of pullback f^* on a new space of quantized measures relative to a finite set Γ of vertices in $\mathbf{P}^1$. Every Borel probability measure ν on $\mathbf{P}^1$ gives rise to one of these 'Γ-measures' ν_Γ. And if ν is a solution to the standard pullback formula $\frac{1}{d} f^* \nu = \nu$, then ν_Γ will satisfy a quantized version:

$$\frac{1}{d} f^* \nu_\Gamma = \pi_* \nu_\Gamma. \tag{1.0.2}$$

(One must push ν_Γ forward by a certain map π in order to have a meaningful equation since $f^* \nu_\Gamma$ lies in the space of Γ'-measures for a potentially different vertex set Γ'.) A solution to the pullback formula (1.0.2) is typically far from unique. However, we will show (Theorem 4.10) that uniqueness is restored if one considers simultaneous solutions to pullback equations for all iterates of f, after ruling out measures supported on classical exceptional cycles.

Step 3. A transfer principle. Now, let $k = \mathbb{L}$ be the completion of the field of formal Puiseux series in t, equipped with the non-Archimedean absolute value that measures the order of vanishing at $t = 0$. (See [14, Section 3].) On viewing the parameter t as an element of $\mathbb{L}$, the family f_t defines a single rational function f with coefficients in $\mathbb{L}$. We define a vertex set $\Gamma \subset \mathbf{P}^1$ consisting of one

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

vertex only, the Gauss point. In Section 5.2, we define a correspondence between measures on the central fiber of our surface X with Γ-measures on $\mathbf{P}^1$. From Step 1, any weak limit ν of the measures μ_t will satisfy the pullback relation (1.0.1). The corresponding Γ-measure ν_Γ must satisfy the non-Archimedean pullback relation (1.0.2) on $\mathbf{P}^1$, by Proposition 5.1. We repeat the argument for all iterates f_t^n. From Step 2, we deduce that ν_Γ is the equilibrium Γ-measure, and consequently, the limit measure ν is the 'residual' equilibrium measure. See Section 5.

2. The space of rational maps: complex-analytic arguments

In this section we prove Theorem A along with a number of preliminary results that will be used in the first step of the proof of Theorem B. In [4], the first author obtained a version of Theorem A under an additional hypothesis. The observations of Lemma 2.1 and (the more refined result in) Lemma 2.5 concerning Möbius rescalings allow us to obtain the complete statement. These lemmas were inspired by the Berkovich space structure appearing in the proof of Theorem B.

2.1. The space of rational maps. We will let Rat_d denote the set of all complex rational functions of degree d. It can be viewed as an open subset of the complex projective space $\mathbb{P}^{2d+1}$, by identifying a function

$$f(z) = \frac{a_0 z^d + a_1 z^{d-1} + \cdots + a_d}{b_0 z^d + b_1 z^{d-1} + \cdots + b_d}$$

with its coefficients in homogeneous coordinates

$$(a_0 : a_1 : \cdots : a_d : b_0 : b_1 : \cdots : b_d) \in \mathbb{P}^{2d+1}.$$

In fact, any point $\Phi \in \mathbb{P}^{2d+1}$ determines a pair (P, Q) of homogeneous polynomials in two variables, and $\mathrm{Rat}_d = \mathbb{P}^{2d+1} \setminus \{\mathrm{Res}(P, Q) = 0\}$. We set $\overline{\mathrm{Rat}}_d = \mathbb{P}^{2d+1}$ so that $\partial \mathrm{Rat}_d = \{\mathrm{Res} = 0\}$. For each $\Phi = (P, Q) \in \partial \mathrm{Rat}_d$, we let $H = \gcd(P, Q)$, and let ϕ be the induced rational function of degree $<d$ defined by the ratio P/Q. To match the algebraic language of the later sections, we refer to the map ϕ as the **reduction** of Φ.

A one-parameter **holomorphic family** $\{f_t : t \in U\}$ is a holomorphic map from a domain $U \subset \mathbb{C}$ to Rat_d. A **meromorphic family** is a holomorphic map from U to $\overline{\mathrm{Rat}}_d$ with image not contained in $\partial \mathrm{Rat}_d$. A meromorphic family is **degenerate** at $u \in U$ if the image of u lies in $\partial \mathrm{Rat}_d$.

LEMMA 2.1. *Let f_k be a sequence in Rat_d converging to a point $\Phi \in \partial \mathrm{Rat}_d$. After passing to a subsequence if necessary, there is a sequence of Möbius*

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

transformations $A_k \in \mathrm{Rat}_1$ *such that* $A_k \circ f_k$ *converges in* $\overline{\mathrm{Rat}}_d$ *to a point with nonconstant reduction. If* B_k *is any other such sequence in* Rat_1*, then* $M_k = A_k \circ B_k^{-1}$ *converges in* Rat_1 *as* $k \to \infty$ *(along the subsequence determined by* A_k*). If the* f_k *lie in a meromorphic family* $\{f_t : t \in \mathbb{D}\}$*, then the sequence* A_k *may be chosen to lie in a meromorphic family* $\{A_t : t \in \mathbb{D}\}$.

Proof. The proof of existence is carried out, algorithmically, in [19, Prop. 2.4] and appears also in [14, Lemma 3.7] for when the sequence lies in a holomorphic family; the strategy is as follows.

At each step of this argument, we may pass to a subsequence. Write

$$f_k(z, w) = (P_k(z, w) : Q_k(z, w)),$$

normalized such that $(P_k, Q_k) \to (P, Q)$ in $\overline{\mathrm{Rat}}_d$. Note that at least one of P and Q is nonzero. By replacing f_k with $S_k \circ f_k$, where $S_k(z) = \alpha_k z$ with $\alpha_k > 0$, it can be arranged that the limiting P and Q are both nonzero. If P is not a scalar multiple of Q, we are done.

Suppose $P = c_0 Q$ for some constant $c_0 \in \mathbb{C}^*$. If $m = \deg_z P = \deg_z Q$, write

$$f_k(z, w) = (a_k z^m w^{d-m} + \hat{P}_k(z, w) : b_k z^m w^{d-m} + \hat{Q}_k(z, w))$$

where $\hat{P}_k$ and $\hat{Q}_k$ have no term involving $z^m w^{d-m}$. Now, postcompose f_k with a translation by $a_k/b_k = c_0 + o(1)$, replacing f_k with

$$f_k(z, w) = (P_k(z, w) - a_k b_k^{-1} Q_k(z, w) : Q_k(z, w)).$$

If P and Q are not monomials, then we are done; the new limit has nonconstant reduction. If P and Q are monomials, the resulting limit in $\overline{\mathrm{Rat}}_d$ will have constant reduction (=0); we rescale and repeat the initial argument. It follows that the new P cannot be a scalar multiple of Q because it has no term involving $z^m w^{d-m}$. This completes the proof of existence of $\{A_k\}$.

If the given f_k lies in a meromorphic family $f_t = (P_t, Q_t)$, then the scaling and translation maps can be chosen meromorphic in t, since they are built from the coefficients of f_t.

Now suppose that $A_k \circ f_k \to \Phi_A$ and $B_k \circ f_k \to \Phi_B$ in $\overline{\mathrm{Rat}}_d$, with nonconstant reductions ϕ_A and ϕ_B. Set $M_k = A_k \circ B_k^{-1}$. Again passing to a subsequence, M_k converges to $M_0 \in \overline{\mathrm{Rat}}_1$. Away from finitely many points in $\mathbb{P}^1$, we have

$$\phi_A(p) = \lim_{k\to\infty} A_k \circ f_k(p) = \lim_{k\to\infty} M_k \circ B_k \circ f_k(p) = M_0 \circ \phi_B(p).$$

As ϕ_A is nonconstant, so is M_0, and therefore $M_0 \in \mathrm{Rat}_1$. This also shows that M_0 is uniquely determined, so the full sequence M_k converges. □

2.2. Counting preimages. Fix a sequence f_k in Rat_d, and assume that f_k converges to a degenerate point $\Phi \in \partial\mathrm{Rat}_d$ with gcd H and *nonconstant* reduction ϕ. For each point $x \in \mathbb{P}^1$, we define multiplicities

$$m(x) = \deg_x \phi \quad \text{and} \quad s(x) = \mathrm{ord}_x H. \tag{2.2.1}$$

The quantity $m(x)$ is the local degree of ϕ, and the quantity $s(x)$ will be called the **surplus multiplicity** at x.

Let η be a small loop around $\phi(x)$ bounding a disk D, and let γ_x be the small loop around x sent with degree $m(x)$ onto η by ϕ. Choose γ_x small enough that it does not contain any roots of H, except possibly x itself. Because f_k converges locally uniformly to ϕ on $\mathbb{P}^1 \setminus \{H = 0\}$, for each $k \gg 0$ there is a small loop γ_k around x that is mapped by f_k with degree $m(x)$ onto η. Let U_k be the domain bounded by γ_k.

PROPOSITION 2.2. *Assume that f_k converges to $\Phi \in \partial\mathrm{Rat}_d$ with nonconstant reduction. Fix $x \in \mathbb{P}^1$. For all k sufficiently large,*

$$\#(f_k^{-1}(z_0) \cap \overline{U}_k) = m(x) + s(x)$$

and

$$\#(f_k^{-1}(p_0) \cap \overline{U}_k) = s(x),$$

for all points z_0 in $\overline{D}$ and all points p_0 in $\mathbb{P}^1 \setminus \overline{D}$.

Proof. The proof is an application of the Argument Principle from complex analysis. Assume first that $z_0 = 0 \in D$ and $p_0 = \infty \notin \overline{D}$. Then

$$\#(f_k^{-1}(z_0) \cap U_k) = \#(f_k^{-1}(z_0) \cap \overline{U}_k) = \#\,\mathrm{Zeroes}(f_k) \text{ inside } U_k,$$

and

$$\#(f_k^{-1}(p_0) \cap U_k) = \#(f_k^{-1}(p_0) \cap \overline{U}_k) = \#\,\mathrm{Poles}(f_k) \text{ inside } U_k.$$

By the Argument Principle, for all large k we have

$$\#(f_k^{-1}(z_0) \cap U_k) - \#(f_k^{-1}(p_0) \cap U_k) = \int_{\gamma_k} \frac{f_k'}{f_k} = m(x).$$

On the other hand, we may compute directly that

$$s(x) = \#\,\mathrm{Poles}(f_k) \text{ inside } U_k$$

for all sufficiently large k, since $f_k \to \Phi$. Indeed, $H(x) = 0$ with multiplicity $s(x)$ (and $\phi(x) \neq \infty$), so there are exactly $s(x)$ poles converging to x as $k \to \infty$. (Compare [4, Lemma 4.1].) It remains to handle the case where $z_0 \in \eta = \partial D$. By construction, the boundary γ_k of U_k is mapped with degree $m(x)$ over η; and by viewing z_0 as the point ∞, we see that there must be $s(x)$ preimages of z_0 converging to x as $k \to \infty$. $\square$

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

2.3. Pullback by a degenerate map. Let Φ be an element of $\overline{\mathrm{Rat}}_d$ with *nonconstant* reduction ϕ. Exactly as in [4, Section 3], we define the pullback of a measure μ on $\mathbb{P}^1$ by the formula

$$\Phi^*\mu := \phi^*\mu + \sum_{x\in\mathbb{P}^1} s(x)\delta_x. \tag{2.3.1}$$

Recall that $s(x)$ is defined in (2.2.1).

LEMMA 2.3. *For any probability measure μ and $\Phi \in \overline{\mathrm{Rat}}_d$ with nonconstant reduction, the measure $\Phi^*\mu$ has total mass d.*

Proof. The proof is a simple degree count:

$$\Phi^*\mu\,(\mathbb{P}^1) = \deg(\phi) + \sum_{x\in\mathbb{P}^1} s(x) = \deg(\phi) + \deg(H) = d. \qquad \square$$

2.4. Paired measures and weak limits. Let C, E denote two copies of $\mathbb{P}^1$. A **paired measure** (μ_C, μ_E) is a pair of Borel probability measures μ_C on C and μ_E on E. Let $\{A_k\}$ be a sequence of Möbius transformations in Rat_1. We say that a sequence of Borel probability measures $\{\mu_k\}$ on $\mathbb{P}^1$ **converges $\{A_k\}$-weakly** to the paired measure (μ_C, μ_E) if

$$\mu_k \to \mu_C \qquad \text{and} \qquad A_{k*}\mu_k \to \mu_E$$

weakly. Note that, if $A_k \to A \in \mathrm{Rat}_1$, then necessarily $A_*\mu_C = \mu_E$. It will often be the case that A_k diverges in Rat_1, as in Lemma 2.1. In Section 3, C and E will represent components of the 'central fiber' of a complex surface.

THEOREM 2.4. *Let $\{f_k\}$ be a sequence in Rat_d and $\{A_k\}$ a sequence in Rat_1 such that $A_k \circ f_k$ converges to $\Phi \in \overline{\mathrm{Rat}}_d$ with nonconstant reduction. Any $\{A_k\}$-weak limit (μ_C, μ_E) of the maximal measures μ_{f_k} will satisfy the pullback formula*

$$\frac{1}{d}\,\Phi^*\mu_E = \mu_C.$$

Proof. Without loss of generality, we may replace f_k with a subsequence in order to allow the assumption that μ_{f_k} converges $\{A_k\}$-weakly to (μ_C, μ_E). For simplicity, we will write μ_k for μ_{f_k}. By the definition of $\{A_k\}$-weak convergence, and because $d^{-1} f_k^*\mu_k = \mu_k$ for all k, we know that

$$d^{-1} f_k^*\mu_k \to \mu_C \quad \text{as} \quad k \to \infty. \tag{2.4.1}$$

We need to show that the weak limit of $f_k^*\mu_k$ can also be expressed as $\Phi^*\mu_E$.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

Let H be the gcd of Φ, and let ϕ be the reduction map of degree > 0. Let $I(\Phi)$ denote the union of the roots of H. Let U be a small neighborhood of $I(\Phi)$ in $\mathbb{P}^1$. Choose a partition of unity

$$b_r + b_s \equiv 1,$$

subordinate to the open cover $\{\mathbb{P}^1 \setminus I(\Phi), U\}$ such that $b_r \equiv 1$ on $\mathbb{P}^1 \setminus U$ and $b_s \equiv 1$ on a small neighborhood of $I(\Phi)$ inside U; as usual, b_r and b_s are nonnegative continuous functions.

Fix a nonnegative continuous function ψ on $\mathbb{P}^1$. Recall that the pushforward of ψ by $f \in \mathrm{Rat}_d$ can be defined by

$$f_*\psi(y) = \sum_{f(x)=y} \psi(x),$$

where preimages are counted with multiplicity. Because b_r vanishes near $I(\Phi)$, and because $A_k \circ f_k$ converges uniformly to ϕ on compact sets outside $I(\Phi)$, we have uniform convergence of functions

$$(A_k \circ f_k)_*(b_r\psi) \to \phi_*(b_r\psi),$$

and therefore

$$\begin{aligned} \int b_r\psi\ (f_k^*\mu_k) &= \int b_r\psi\ \ ((A_k \circ f_k)^* A_{k*}\mu_k) \\ &= \int (A_k \circ f_k)_*\,(b_r\psi)\ A_{k*}\mu_k \to \int \phi_*(b_r\psi)\ \mu_E \\ &= \int b_r\psi\ \Phi^*\mu_E, \end{aligned} \tag{2.4.2}$$

by the weak convergence of $A_{k*}\mu_k$ to μ_E. Upon shrinking the neighborhood U, (2.4.1) and (2.4.2) together will show that

$$\int_{\mathbb{P}^1 \setminus I(\Phi)} \psi\ \mu_C = \frac{1}{d} \int_{\mathbb{P}^1 \setminus I(\Phi)} \psi\ \Phi^*\mu_E \tag{2.4.3}$$

for any test function ψ.

Fix $x \in I(\Phi)$. As in Section 2.2, let η be a small loop around $\phi(x)$ that bounds an open disk D, and let γ_x be the small loop around x sent with degree $m(x)$ onto η by ϕ. Choose γ_x small enough that it does not contain any point in $I(\Phi)$ other than x itself; we shall further assume that it is contained in the neighborhood where $b_s \equiv 1$. Because $A_k \circ f_k$ converges locally uniformly to ϕ on $\mathbb{P}^1 \setminus I(\Phi)$, for each $k \gg 0$ there is a small loop γ_k around x that is mapped by f_k with degree $m(x)$ onto η; for large k, this γ_k is also contained in the region where $b_s \equiv 1$. Let $U_{x,k}$ be the domain bounded by γ_k.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

We now apply Proposition 2.2 to the sequence $A_k \circ f_k$. For $x \in I(\Phi)$, let $\psi_{\inf}(x)$ denote the infimum of ψ on the component of U containing x. For all k sufficiently large,

$$\begin{aligned}
\int_{\mathbb{P}^1} b_s \psi \, (f_k^* \mu_k) &\geqslant \sum_{x \in I(\Phi)} \psi_{\inf}(x) \int_{\overline{U}_{x,k}} (A_k \circ f_k)^* A_{k*}\mu_k \\
&= \sum_{x \in I(\Phi)} \psi_{\inf}(x) \int_{\mathbb{P}^1} \# \left((A_k \circ f_k)^{-1}(y) \cap \overline{U}_{x,k} \right) \, A_{k*}\mu_k(y) \\
&= \sum_{x \in I(\Phi)} \psi_{\inf}(x) \left[s(x) A_{k*}\mu_k(\mathbb{P}^1 \smallsetminus \overline{D}) + (m(x) + s(x)) A_{k*}\mu_k(\overline{D}) \right] \\
&= \sum_{x \in I(\Phi)} \psi_{\inf}(x) \left[s(x) + m(x) A_{k*}\mu_k(\overline{D}) \right].
\end{aligned}$$

Letting $k \to \infty$, the $\{A_k\}$-weak convergence of measures gives

$$\liminf_{k \to \infty} A_{k*}\mu_k(\overline{D}) \geqslant \mu_E(\{\phi(x)\}).$$

Because $d^{-1} f_k^* \mu_k$ converges weakly to μ_C, we deduce that

$$\int b_s \psi \, \mu_C \geqslant \frac{1}{d} \sum_{x \in I(\Phi)} [s(x) + m(x) \mu_E(\{\phi(x)\})] \, \psi_{\min}(x).$$

Shrinking the neighborhood U of $I(\Phi)$, we obtain

$$\int_{I(\Phi)} \psi \, \mu_C \geqslant \frac{1}{d} \sum_{x \in I(\Phi)} [s(x) + m(x) \, \mu_E(\{\phi(x)\})] \, \psi(x) = \frac{1}{d} \int_{I(\Phi)} \psi \, \Phi^* \mu_E. \tag{2.4.4}$$

As ψ was arbitrary, adding (2.4.3) to (2.4.4) yields the inequality of positive measures

$$\mu_C \geqslant \frac{1}{d} \Phi^* \mu_E.$$

But both are probability measures (by Lemma 2.3), so we must have equality. □

2.5. Proof of Theorem A. Let f_k be a sequence in Rat_d converging to $f_0 \in \partial \mathrm{Rat}_d$ and with maximal measures μ_k converging to a measure μ. From Lemma 2.1, there is a sequence $A_k \in \mathrm{Rat}_1$ such that $A_k \circ f_k$ converges to $\Phi \in \overline{\mathrm{Rat}}_d$ with reduction ϕ of positive degree. Passing to subsequences for each iterate n and applying a diagonalization argument, we choose sequences $\{A_{n,k} : k \in \mathbb{N}\}$ in Rat_1 such that

$$A_{n,k} \circ f_k^n \to \Phi_n \quad \text{as } k \to \infty$$

in $\overline{\mathrm{Rat}}_{d^n}$ with reduction ϕ_n such that $\deg \phi_n > 0$ for every iterate n. By sequential compactness of the space of probability measures on $\mathbb{P}^1$ (and another diagonalization argument, if necessary), we may assume that μ_k converges $\{A_{n,k}\}$-weakly to a paired measure (μ, μ_{E_n}) as $k \to \infty$ for each $n \geqslant 1$.

Since the measures μ_k are also the measures of maximal entropy for iterates f_k^n, Theorem 2.4 implies that

$$\mu(\{p\}) = \frac{1}{d^n} \Phi_n^* \mu_{E_n}(\{p\}) \geqslant \frac{s_{\Phi_n}(p)}{d^n}$$

for any iterate n and any point $p \in \mathbb{P}^1$; recall that the integers $s_{\Phi_n}(p)$ are defined in (2.2.1). Degree counting shows that $\sum_{p\in\mathbb{P}^1} s_{\Phi_n}(p) = d^n - \deg \phi_n$, which yields

$$1 \geqslant \sum_{p\in\mathbb{P}^1} \mu(\{p\}) \geqslant 1 - \frac{\deg \phi_n}{d^n}.$$

If $\deg \phi_n = o(d^n)$ as $n \to \infty$, then we see immediately that μ is a countable sum of atoms. It remains to treat the case where $\deg \phi_n \neq o(d^n)$.

The next lemma shows that the reduction maps ϕ_n are not unrelated.

LEMMA 2.5. *The reduction maps ϕ_n form a composition sequence. That is, there exist rational functions $\phi_{n+1,n}$ of positive degrees $\leqslant d$ such that*

$$\phi_{n+1} = \phi_{n+1,n} \circ \phi_n$$

for each $n \geqslant 1$. Moreover, $A_{n+1,k} \circ f_k \circ A_{n,k}^{-1}$ converges to $\phi_{n+1,n}$ away from finitely many points in $\mathbb{P}^1$.

Proof. This lemma follows from uniqueness in Lemma 2.1. Write $\Phi = H\phi$ for any $\Phi \in \overline{\mathrm{Rat}}_d$, where H is the gcd of the two polynomials defining Φ and ϕ is the reduction. As $k \to \infty$, we have $A_{n,k} \circ f_k^n \to H_n\phi_n$ and $A_{n+1,k} \circ f_k^{n+1} \to H_{n+1}\phi_{n+1}$. Consider the sequence $f_k \circ A_{n,k}^{-1}$ in Rat_d. Passing to a subsequence, there exists a sequence C_k of Möbius transformations such that $C_k \circ f_k \circ A_{n,k}^{-1} \to H\phi$ with $\deg \phi > 0$. But then, by the continuity of degenerate composition (exactly as in [5, Lemma 2.6]), we have

$$(C_k \circ f_k \circ A_{n,k}^{-1}) \circ (A_{n,k} \circ f_k^n) = C_k \circ f_k^{n+1} \to (H_n^d \cdot (H \circ \phi_n))\, \phi \circ \phi_n.$$

But uniqueness in Lemma 2.1 then implies that there exists a Möbius transformation $B = \lim_{k\to\infty} A_{n+1,k} \circ C_k^{-1}$ such that $\phi_{n+1} = B \circ \phi \circ \phi_n$. We set $\phi_{n+1,n} = B \circ \phi$. $\square$

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

Lemma 2.5 implies that the degree of ϕ_n may be computed by using

$$\deg \phi_n = \deg \phi_1 \cdot \prod_{j=1}^{n-1} \deg \phi_{j+1,j}.$$

In particular, $\deg \phi_n \neq o(d^n)$ implies that there exists $n_0 > 0$ such that $\deg \phi_{n+1,n} = d$ for all $n \geqslant n_0$. For the remainder of the proof, we will operate under this assumption.

Suppose for the moment that there exist nonnegative integers $m > n \geqslant n_0$ such that

$$A_{n,k} \circ A_{m,k}^{-1} \to L \in \mathrm{Rat}_1 \quad \text{as } k \to \infty \tag{2.5.1}$$

(after passing to a subsequence, if necessary). From Lemma 2.5 and the continuity of composition,

$$A_{n,k} \circ f_k^{m-n} \circ A_{n,k}^{-1} = A_{n,k} \circ A_{m,k}^{-1} \circ A_{m,k} \circ f_k^{m-n} \circ A_{n,k}^{-1} \longrightarrow L \circ \phi_{m,m-1} \circ \cdots \circ \phi_{n+1,n},$$

and the limiting function has degree d^{m-n}. In other words, the sequence of conjugates $A_{n,k} \circ f_k^{m-n} \circ A_{n,k}^{-1}$ will converge in $\mathrm{Rat}_{d^{m-n}}$. But properness of the iteration map $\mathrm{Rat}_d \to \mathrm{Rat}_{d^{m-n}}$ [4, Corollary 0.3] implies that the sequence $A_{n,k} \circ f_k \circ A_{n,k}^{-1}$ must also converge uniformly to some rational function $g \in \mathrm{Rat}_d$. The continuity of measures within Rat_d then implies that $\mu = \lim_{k\to\infty} (A_{n,k}^{-1})_* \mu_g$. The sequence $\{A_{n,k}\}$ must diverge in Rat_1 (because the sequence $\{f_k\}$ diverges in Rat_d), so the limiting measure μ will be concentrated at a single point.

It remains to treat the case where

$$A_{m,k} \circ A_{n,k}^{-1}$$

diverges in Rat_1 for all $m > n \geqslant n_0$. A diagonalization argument allows us to assume that the limit exists in $\overline{\mathrm{Rat}}_1$, and we set

$$a_{m,n} := \lim_{k\to\infty} A_{m,k} \circ A_{n,k}^{-1}(p)$$

for all but one point p in $\mathbb{P}^1$, say $p = h_{m,n}$. Recall that we continue to assume that $\deg \phi_n \neq o(d^n)$ as $n \to \infty$, so there is a constant $0 < \kappa < 1$ such that $\deg \phi_n = \kappa d^n$ for all $n \geqslant n_0$. We wish to show that $\mu = \lim \mu_k$ is purely atomic. For the sake of a contradiction, we suppose otherwise and write

$$\mu = \nu + \tilde{\nu},$$

where $\tilde{\nu}$ is a countable sum of atoms and $\nu = \mu - \tilde{\nu}$ is a nonzero positive measure with no atoms. Similarly, write $\mu_{E_n} = \nu^n + \tilde{\nu}^n$, where ν^n and $\tilde{\nu}^n$ are the ‘diffuse

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

part' and the 'atomic part' of μ_{E_n}, respectively. Applying Theorem 2.4 to the nth iterates f_k^n and comparing diffuse parts, we find that

$$\nu = \frac{1}{d^n}\phi_n^*\nu^n \Rightarrow 0 < \nu(\mathbb{P}^1) = \frac{\deg \phi_n}{d^n}\nu^n(\mathbb{P}^1) = \kappa \cdot \nu^n(\mathbb{P}^1)$$

for all $n \geqslant n_0$. Hence, there exists N such that

$$\sum_{n=n_0}^{N} \nu^n(\mathbb{P}^1) \geqslant 2,$$

Fix a small $\varepsilon > 0$. For each pair $n_0 \leqslant m, n \leqslant N$ with $m \neq n$, choose small pairwise disjoint closed disks $D_{m,n}$ and $D'_{m,n}$ around $a_{m,n}$ and $h_{m,n}$, respectively. Let U be the complement of all of these disks in $\mathbb{P}^1$. Since ν^n is atomless, by shrinking $D_{m,n}$ and $D'_{m,n}$ as needed we may assume that

$$\nu^n(U) > \nu^n(\mathbb{P}^1) - \frac{\varepsilon}{2^n} \quad (n_0 \leqslant n \leqslant N).$$

Weak convergence of measures $(A_{n,k})_*\mu_k \to \mu_{E_n} = \nu^n + \tilde{\nu}^n$ implies that

$$(A_{n,k})_*\mu_k(U) > \nu^n(\mathbb{P}^1) - \frac{\varepsilon}{2^n}$$

for all sufficiently large k and all $n_0 \leqslant n \leqslant N$. (Restricting to finitely many n allows us to do this uniformly.)

For distinct indices $n_0 \leqslant m, n \leqslant N$, we have constructed U to be disjoint from $D'_{m,n}$. It follows that $A_{m,k} \circ A_{n,k}^{-1}(U) \subset D_{m,n}$ for all $k \gg 0$, and hence $U \cap (A_{m,k} \circ A_{n,k}^{-1}(U)) = \varnothing$ for all sufficiently large k. Therefore, the sets

$$A_{n_0,k}^{-1}(U),\ A_{n_0+1,k}^{-1}(U), \ldots, A_{N,k}^{-1}(U)$$

are pairwise disjoint for all $k \gg 0$. (Again, restricting to finitely many sets allows us to do this uniformly.) But then

$$\mu_k(\mathbb{P}^1) \geqslant \sum_{n=n_0}^{N} \mu_k\left(A_{n,k}^{-1}(U)\right) > \sum_{n=n_0}^{N}\left(\nu^n(\mathbb{P}^1) - \varepsilon/2^n\right) > 2 - \varepsilon > 1,$$

contradicting the fact that μ_k is a probability measure. This completes the proof of Theorem A.

REMARK 2.6. In the case where the sequence f_k lies in a meromorphic family f_t, the condition that $\deg \phi_n \neq o(d^n)$ is characterized in the proof of Proposition 4.7(2), in terms of dynamics on the Berkovich $\mathbf{P}^1$.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

3. One-parameter families and complex surfaces

In this section, we carry out Step 1 in the proof of Theorem B. To start, we consider a meromorphic family $\{f_t : t \in \mathbb{D}\}$ of rational functions of degree $d \geqslant 2$ and set up a geometric framework in which to talk about pullback of measures when $t = 0$. Under the hypothesis of Theorem B, the family f_t defines a holomorphic disk in $\overline{\mathrm{Rat}}_d$ with $f_0 \in \partial\mathrm{Rat}_d$. It is convenient to package the given one-parameter family into one map on the complex surface $X = \mathbb{D} \times \mathbb{P}^1$, as

$$F : X \dashrightarrow X,$$

defined by $F(t, x) = (t, f_t(x))$ for $t \neq 0$. The map F extends to a meromorphic map on the surface X with a finite set of indeterminacy points in the **central fiber** $X_0 := \{0\} \times \mathbb{P}^1$. The indeterminacy points coincide with roots of the polynomial gcd H_{f_0} defined in Section 2.1. On any compact subset of $\mathbb{P}^1 \setminus \{H_{f_0} = 0\}$, the functions f_t converge uniformly to the reduction ϕ_{f_0} as $t \to 0$.

3.1. The modified surface Y.

We now construct the surface Y that was alluded to in the introduction.

PROPOSITION 3.1. *There is a minimal complex modification $\pi : Y \to X$ such that the induced rational map*

$$F : X \dashrightarrow Y$$

is nonconstant on the central fiber X_0. The minimal modification is unique up to isomorphism of fibered surfaces, possibly after shrinking the base $\mathbb{D}$. If ϕ_{f_0} is nonconstant, then $Y = X$. Otherwise, the central fiber of the surface Y is reduced and has exactly two irreducible components.

REMARK 3.2. The surface Y may be singular. We give a necessary and sufficient condition for nonsingularity in the course of the proof of the proposition.

EXAMPLE 3.3. Take $f_t(z) = tz^2$ for t in the unit disk, and let $A_t(z) = t\frac{1}{z}$, so $A_t \circ f_t$ has nonconstant reduction. The rational map $F : X \dashrightarrow X$ collapses the central fiber X_0 to the origin of X_0. The minimal modification Y has local coordinates (t, w) near the exceptional curve, where $w = t\frac{1}{z}$. Equivalently, $t = zw$, which is the standard equation for a blow-up at the origin of the surface $X = \mathbb{D} \times \mathbb{P}^1$ with coordinates (t, z).

Proof of the Proposition. Evidently the surface $Y = X$ satisfies all of the desired properties if ϕ_{f_0} is nonconstant.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

For the remainder of the proof, we assume that ϕ_{f_0} is constant. Lemma 2.1 asserts that there is a Möbius transformation $A_t(z)$ with meromorphic coefficients such that $A_t \circ f_t$ converges to a point of $\overline{\mathrm{Rat}}_1$ with nonconstant reduction. If we take (t, z) to be coordinates on $X = \mathbb{D} \times \mathbb{P}^1$ near a point of the central fiber X_0, then we may construct the surface Y locally as a subvariety of $(\mathbb{D} \times \mathbb{P}^1) \times \mathbb{P}^1$ using the equation $A_t(z) = w$, where w is an affine coordinate for the final $\mathbb{P}^1$ factor. Define $\pi : Y \to X$ to be projection on the $(\mathbb{D} \times \mathbb{P}^1)$ factor of the ambient space.

Note that A_t is invertible for t sufficiently small and nonzero, so π is an isomorphism away from the central fibers of X and Y. As f_t has constant reduction, A_t must have constant reduction too. Without loss of generality, we can assume that $A_t \to 0$ as $t \to 0$ away from finitely many points of $\mathbb{P}^1$. Thus we can write

$$A_t(z) = t^n \cdot \frac{\alpha z + \beta}{\gamma z + \delta},$$

where $\alpha, \beta, \gamma, \delta$ are meromorphic functions of t, holomorphic at $t = 0$, and they satisfy $\alpha\delta - \beta\gamma \not\to 0$ as $t \to 0$. Thus, a local equation for the surface Y is

$$(\gamma z + \delta)w = t^n(\alpha z + \beta).$$

Setting $t = 0$ exhibits the two rational components of the central fiber of Y—one parameterized by z and the other by w—and also shows that the central fiber is reduced. (Moreover, Y is nonsingular if and only if $n = 1$.)

We now prove uniqueness and minimality of this modification, simultaneously. Suppose that $\pi' : Y' \to X$ is a minimal modification of X such that the induced rational map $F : X \dashrightarrow Y'$ is nonconstant. Then Y' must have exactly two distinct components in its central fiber: the proper transform of the central fiber of X and the image of the central fiber of X under F. For if there were another component, we could blow it down and violate minimality. Working in coordinates at a smooth point of the exceptional divisor of Y', we observe that $F(t, z) = (t, g_t(z))$ for some meromorphic family of rational functions g_t. For t away from 0, the function g_t must agree with f_t up to a coordinate change on the target, so there is a meromorphic family of Möbius transformations B_t such that $B_t \circ f_t = g_t$. That is, the surface Y' is constructed locally via the equation $w' = B_t(z)$ as in the previous paragraphs. Since g_t has nonconstant reduction, Lemma 2.1 implies that $A_t \circ B_t^{-1}$ converges in Rat_1 as $t \to 0$. In particular, this means that Y' and Y are isomorphic via the fibered isomorphism $A_t \circ B_t^{-1}$. We conclude that $\pi : Y \to X$ is minimal and unique up to fibered isomorphism. □

For the remainder of the paper, we fix the following notation. We choose a family A_t of Möbius transformations such that $A_t \circ f_t$ converges to a point $\Phi \in \overline{\mathrm{Rat}}_d$ with gcd H and reduction ϕ of degree > 0. If the reduction of f_0

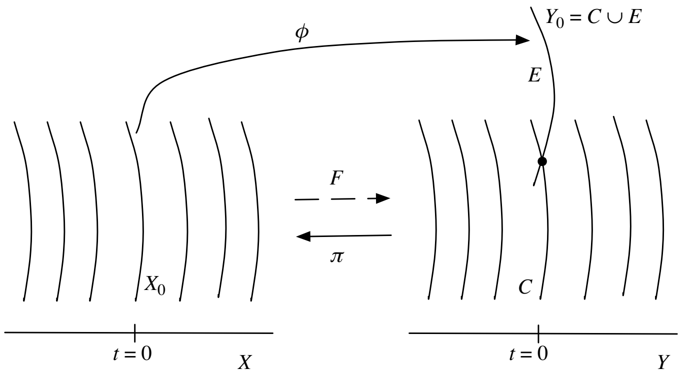

Figure 1: The surface map $F : X \dashrightarrow Y$ when the given reduction ϕ_{f_0} is constant.

is nonconstant, we let A_t be the identity for all t. This choice gives rise to a minimal modification $\pi : Y \to X$ as in the above proof. If the central fiber Y_0 has two components, we view Y as the blow-up of X along an ideal supported at a single point in the central fiber of X_0. Let E denote the exceptional curve of the projection $\pi : Y \to X$ and let C be the other component of Y_0 (identified with the original fiber X_0); see Figure 1. There are two natural projections from Y_0 to its irreducible components: the modification π induces a projection $\pi : Y_0 \to C$ collapsing E to a point, and $\pi_E : Y_0 \to E$, which is the unique continuous projection that is the identity on E.

3.2. Weak limits of measures on Y. Suppose that μ_t is a family of probability measures on the fibers Y_t on the surface Y. We say μ_0 on Y_0 is a weak limit of the measures μ_t if there is a sequence $t_n \to 0$ such that

$$\int_{Y_{t_n}} \psi \, \mu_{t_n} \to \int_{Y_0} \psi \, \mu_0$$

for every continuous function ψ on Y. If $Y = X = \mathbb{D} \times \mathbb{P}^1$, this notion of weak limit agrees with the usual notion for measures on a single $\mathbb{P}^1$. If $Y \neq X$, this notion of convergence coincides with $\{A_{t_n}\}$-weak convergence to the paired measure $(\pi_*\mu_0, (\pi_E)_*\mu_0)$ on $C \cup E = Y_0$, as the following lemma shows; recall the definitions from Section 2.4.

LEMMA 3.4. *Suppose that $Y \neq X$, and so $Y_0 = C \cup E$. A measure μ_0 on Y_0 is a weak limit of measures μ_t on Y_t if and only if there is a sequence t_n of parameters such that μ_{t_n} converges $\{A_{t_n}\}$-weakly to the paired measure $(\pi_*\mu_0, (\pi_E)_*\mu_0)$ on C, E.*

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

Proof. Assume first that μ_0 is the weak limit of measures μ_{t_n} on fibers Y_{t_n}. We identify the curves C and E with abstract copies of $\mathbb{P}^1$, with coordinates $z \in C$ and $w \in E$. Let ψ_C be any continuous function on C, and let ψ_E be any continuous function on E. We define continuous functions on Y by

$$\tilde{\psi}_C(y) = \psi_C(z)$$

when $\pi(y) = (t, z) \in X = \mathbb{D} \times \mathbb{P}^1$, and

$$\tilde{\psi}_E(y) = \begin{cases} \psi_E(w) & \text{if } \pi(y) = (t, A_t^{-1}(w)), t \neq 0, \text{ on } X = \mathbb{D} \times \mathbb{P}^1 \\ \psi_E(w) & \text{for } w \in E \\ \psi_E(p) & \text{on } C \end{cases}$$

where $p \in Y_0$ denotes the point of intersection of C and E.

Then

$$\begin{aligned} \int_C \psi_C \, \mu_{t_n} &= \int_{Y_{t_n}} \tilde{\psi}_C \, \mu_{t_n} \\ &\longrightarrow \int_{Y_0} \tilde{\psi}_C \, \mu_0 \\ &= \int_{C \setminus \{p\}} \tilde{\psi}_C \, \mu_0 + \tilde{\psi}_C(p)\mu_0(\{p\}) + \int_{E \setminus \{p\}} \tilde{\psi}_C \, \mu_0 \\ &= \int_{C \setminus \{p\}} \psi_C \, \mu_0 + \psi_C(p)\mu_0(E) = \int_C \psi_C \, (\pi_* \mu_0), \end{aligned}$$

demonstrating the weak convergence of $\mu_{t_n} \to \pi_* \mu_0$.

Also, we have

$$\begin{aligned} \int_E \psi_E(w) \, A_{t_n *} \mu_{t_n}(w) &= \int_{Y_{t_n}} \tilde{\psi}_E(t_n, A_{t_n}^{-1}(w)) \, A_{t_n *} \mu_{t_n}(w) \\ &= \int_{Y_{t_n}} \tilde{\psi}_E(t_n, z) \, \mu_{t_n}(z) \\ &\longrightarrow \int_{Y_0} \tilde{\psi}_E \, \mu_0 \\ &= \int_{E \setminus \{p\}} \tilde{\psi}_E \, \mu_0 + \tilde{\psi}_E(p)\mu_0(\{p\}) + \int_{C \setminus \{p\}} \tilde{\psi}_E \, \mu_0 \\ &= \int_{E \setminus \{p\}} \psi_E \, \mu_0 + \psi_E(p)\mu_0(C) = \int_E \psi_E \, (\pi_{E*} \mu_0), \end{aligned}$$

demonstrating the weak convergence of $A_{t_n *} \mu_{t_n} \to \pi_{E*} \mu_0$.

For the reverse implication, assume that μ_0 is a probability measure on Y_0 such that $\{\mu_{t_n}\}$ converges $\{A_{t_n}\}$-weakly to the paired measure $(\pi_*\mu_0, (\pi_E)_*\mu_0)$. Let ψ be any continuous function on Y and fix $\varepsilon > 0$. With a partition of unity, we may write $\psi = \psi_C + \psi_E + \psi_p$, where $\psi_C \equiv 0$ on E, $\psi_E \equiv 0$ on C, and ψ_p is supported on a small neighborhood of $\{p\} = C \cap E$. We choose the neighborhood of p small enough that

$$\left| \int_{Y_0} \psi_p \, \mu_0 - \psi(p)\mu_0(\{p\}) \right| < \varepsilon. \tag{3.2.1}$$

By uniform continuity of ψ_C near the central fiber Y_0, the weak convergence of μ_{t_n} to $\pi_*\mu_0$ guarantees that

$$\left| \int_{Y_{t_n}} \psi_C \, \mu_{t_n} - \int_C \psi_C \, (\pi_*\mu_0) \right| < \varepsilon$$

for all n sufficiently large. Note that

$$\int_C \psi_C \, (\pi_*\mu_0) = \int_{Y_0} \psi_C \, \mu_0$$

because $\psi_C \equiv 0$ on E. Now let $\phi_E = \psi_E | E$. By continuity of ψ_E, with coordinate $w = A_t(z)$ near E, we see that

$$|\psi_E(t, z) - \phi_E(A_t(z))| < \varepsilon$$

on Y_t, uniformly for all t small. Then the convergence of $A_{t_n *}\mu_{t_n}$ to $\pi_{E*}\mu_0$ implies that

$$\left| \int_{Y_{t_n}} \psi_E \, \mu_{t_n} - \int \phi_E \, (\pi_{E*}\mu_0) \right| < 2\varepsilon$$

for all n sufficiently large. Note that

$$\int \phi_E \, (\pi_{E*}\mu_0) = \int_{Y_0} \psi_E \, \mu_0,$$

because $\psi_E \equiv 0$ on C. At the intersection point $\{p\} = C \cap E$, we have

$$\mu_0(\{p\}) = \pi_*\mu_0(\{p\}) - \pi_{E*}\mu_0(E \smallsetminus \{p\}) = \pi_{E*}\mu_0(\{p\}) - \pi_*\mu_0(C \smallsetminus \{p\}).$$

It follows from weak convergence to the paired measure and (3.2.1) that

$$\left| \int_{Y_{t_n}} \psi_p \, \mu_{t_n} - \int_{Y_0} \psi_p \, \mu_0 \right| < 2\varepsilon$$

for all n large. Putting the pieces together, and shrinking the neighborhood of p, we conclude that

$$\int_{Y_{t_n}} \psi \, \mu_{t_n} = \int_{Y_{t_n}} (\psi_C + \psi_E + \psi_p) \, \mu_{t_n} \longrightarrow \int_{Y_0} \psi \, \mu_0. \qquad \square$$

3.3. Pullback of measures from Y_0 to X_0. For any Borel probability measure μ on the central fiber Y_0 of Y, we can define a measure $F^*\mu$ on the central fiber X_0 of X of total mass d. We use the degenerate pullbacks defined in Section 2.3. If $Y = X$, we simply set

$$F^*\mu := \Phi^*\mu = \phi^*\mu + \sum_{x\in\mathbb{P}^1} s(x)\delta_x. \tag{3.3.1}$$

For the case $Y \neq X$, recall that the projection $\pi_E : Y_0 \to E$ collapses C to a point. We define

$$F^*\mu := \Phi^*(\pi_{E*}\mu) = \phi^*(\pi_{E*}\mu) + \sum_{x\in\mathbb{P}^1} s(x)\delta_x. \tag{3.3.2}$$

We already know that weak limits of maximal measures satisfy a paired-measure pullback formula (Theorem 2.4). Translating into our surface framework with Lemma 3.4, we immediately obtain the main result of this section:

THEOREM 3.5. *Any weak limit μ_0 of the maximal measures μ_t on the central fiber Y_0 of Y will satisfy the pullback equation*

$$\frac{1}{d}F^*\mu_0 = \pi_*\mu_0$$

on the central fiber X_0 of X.

4. Dynamics and Γ-measures on the Berkovich projective line

In this section, we quantize a dynamical system f on the Berkovich projective line and describe the solutions to a system of pullback formulas, thereby completing Step 2 of our program outlined in the introduction. Throughout, we let k be an algebraically closed field of characteristic zero that is complete with respect to a nontrivial non-Archimedean absolute value. Only the case where k has residue characteristic zero is necessary for our application; however, with essentially no extra work, we obtain a more general result. The Berkovich projective line over k will be denoted as $\mathbf{P}^1$ for brevity (as opposed to $\mathbf{P}^{1,\mathrm{an}}_k$) .

4.1. Basic background. The Berkovich projective line is a complicated object from an analytic viewpoint; a careful development is given in [1]. However, the arguments in the next few subsections will typically only require basic topological properties of $\mathbf{P}^1$ and of its self-maps. For a self-contained and detailed summary with complete references, see [8, Sections 2, 3]. To aid the reader, we give our own intuitive summary now.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

The Berkovich projective line $\mathbf{P}^1$ over k is a connected compact Hausdorff topological space with a 'tree structure': it is uniquely arcwise connected. It contains a canonical homeomorphic copy of the space $\mathbb{P}^1(k)$ with the metric topology induced from k; we refer to these as the classical or type I points of $\mathbf{P}^1$. These points all lie at the 'ends' of the tree. The branch points of $\mathbf{P}^1$ are called type II points; they correspond bijectively to closed disks in k with radius in the value group $|k^\times|$. In particular, we will make use of the 'Gauss point' of $\mathbf{P}^1$ corresponding to the closed unit disk in k. For each type II point ζ, the connected components of $\mathbf{P}^1 \smallsetminus \{\zeta\}$ are examples of 'open Berkovich disks'.

A rational function $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ extends uniquely to a morphism $f : \mathbf{P}^1 \to \mathbf{P}^1$. If f is nonconstant, there is a multiplicity (or local degree) function $m_f : \mathbf{P}^1 \to \{1, \ldots, \deg(f)\}$ that extends the usual algebraic multiplicity for $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$. As one expects, 'most' points $x \in \mathbf{P}^1$ have $m_f(x) = 1$, and each point $y \in \mathbf{P}^1$ has exactly $\deg(f)$ preimages under f when counted with multiplicities.

4.2. Vertex sets and measures. A **vertex set** for $\mathbf{P}^1$ is a finite nonempty set of type II points, which we denote by Γ. The connected components of $\mathbf{P}^1 \smallsetminus \Gamma$ will be referred to as **Γ-domains**. When a Γ-domain has only one boundary point, we call it a **Γ-disk**. Write $\mathcal{S}(\Gamma)$ for the partition of $\mathbf{P}^1$ consisting of the elements of Γ and all of its Γ-domains.

Let $(\mathbf{P}^1, \Gamma)$ be the measurable space structure on $\mathbf{P}^1$ equipped with the σ-algebra generated by the power set of $\mathcal{S}(\Gamma)$. A measurable function on $(\mathbf{P}^1, \Gamma)$ will be called **Γ-measurable**. The space of complex measures on $(\mathbf{P}^1, \Gamma)$ will be denoted as $M(\Gamma)$, and we call any such measure a **Γ-measure**. We write $M^\ell(\Gamma)$ for the convex subspace of $M(\Gamma)$ consisting of positive measures of volume ℓ.

REMARK 4.1. A function $\phi : \mathbf{P}^1 \to \mathbb{C}$ is Γ-measurable if and only if it is constant on elements of $\mathcal{S}(\Gamma)$.

Suppose that $\Gamma \subset \Gamma'$ are two vertex sets. If we write $\pi : \mathbf{P}^1 \to \mathbf{P}^1$ for the identity morphism, then $\pi : (\mathbf{P}^1, \Gamma') \to (\mathbf{P}^1, \Gamma)$ is a measurable morphism. In particular, the projection

$$\pi_* : M(\Gamma') \to M(\Gamma)$$

is $\mathbb{C}$-linear and preserves positivity and volume of measures.

4.3. Pulling back measures by a rational function. Throughout this section we assume that $f : \mathbf{P}^1 \to \mathbf{P}^1$ is a rational function of degree $d \geqslant 2$. Suppose that $\Gamma = \{\zeta\}$ is a singleton vertex set, and let $\Gamma' = \{\zeta, f(\zeta)\}$ be a second vertex set.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

For the applications in this article, we will only need to consider vertex sets of cardinality one or two.

Now we define a pullback map $f^* : M(\Gamma') \to M(\Gamma)$. As a first step, we define certain multiplicities $m_{U,V} \in \{0, 1, \ldots, d\}$ for each $U \in \mathcal{S}(\Gamma')$ and $V \in \mathcal{S}(\Gamma)$. If $V = \{\zeta\}$, set $m_{U,V} = m_f(\zeta)$, the usual local degree of f at ζ. For a Γ-disk V, we may write $V = D(\vec{v})$ for some tangent vector $\vec{v} \in T\mathbf{P}^1_\zeta$. Set $\bar{f}(V) = D(Tf(\vec{v}))$. Write $m_f(V)$ and $s_f(V)$ for the directional and surplus multiplicities for f associated with V. (See [8, Section 3].) By definition, we have

$$\#\left(f^{-1}(y) \cap V\right) = \begin{cases} m_f(V) + s_f(V) & \text{if } y \in \bar{f}(V) \\ s_f(V) & \text{if } y \notin \bar{f}(V). \end{cases}$$

Here we count each preimage x with multiplicity $m_f(x)$. Since $\bar{f}(V)$ is a union of elements of $\mathcal{S}(\Gamma')$, the function $y \mapsto \#\left(f^{-1}(y) \cap V\right)$ is constant on elements of $\mathcal{S}(\Gamma')$. For each $U \in \mathcal{S}(\Gamma')$, define $m_{U,V}$ to be this constant value. The following lemma gives a compatibility relation among the multiplicities $m_{U,V}$.

LEMMA 4.2. *For each $U \in \mathcal{S}(\Gamma')$, we have*

$$\sum_{V \in \mathcal{S}(\Gamma)} m_{U,V} = \deg(f).$$

Proof. Choose a point $y \in U$. For each $V \in \mathcal{S}(\Gamma)$, we have that $m_{U,V} = \#\left(f^{-1}(y) \cap V\right)$. Since f is everywhere $\deg(f)$-to-one, the result follows. □

REMARK 4.3. The notation $m_{U,V}$ is backward from the point of view of mappings: it is the local degree of the map $\left(f^{-1}(U) \cap V\right) \xrightarrow{f} U$. However, in the sequel [6], we interpret this quantity in terms of a transition probability for passing from 'state U' to 'state V' in a certain random process. Specifically, the fraction $p_{U,V} := m_{U,V}/d$ is the probability that a randomly chosen preimage of a point $y \in U$ lies in V. The preceding lemma shows that these probabilities sum to 1 when U is fixed and V varies.

For a measurable function $\phi : (\mathbf{P}^1, \Gamma) \to \mathbb{C}$, we define a Γ'-measurable function $f_*\phi$ by

$$f_*\phi(U) = \sum_{W \in \mathcal{S}(\Gamma)} m_{U,W} \cdot \phi(W) \qquad (U \in \mathcal{S}(\Gamma')).$$

Here we have abused notation by writing $f_*\phi(U)$ for the constant value of $f_*\phi$ on U, and similarly for $\phi(W)$. Note that the sum defining $f_*\phi(U)$ is finite by Lemma 4.2.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

If ϕ is a bounded Γ-measurable function, then $\|f_*\phi\| \leqslant d\|\phi\|$, where we have written $\|\cdot\|$ for the sup norm. For each $\nu \in M(\Gamma')$, the linear functional $\phi \mapsto \int f_*\phi\ \nu$ is bounded, and by duality there exists a Γ-measure $f^*\nu$ satisfying $\int \phi\ f^*\nu = \int f_*\phi\ \nu$ for all bounded Γ-measurable functions ϕ. Evidently $f^* : M(\Gamma') \to M(\Gamma)$ preserves positivity of measures, and Lemma 4.2 shows that f^* carries $M^\ell(\Gamma')$ into $M^{\ell d}(\Gamma)$ for each $\ell \in \mathbb{C}$. In particular, $\frac{1}{d} f^*$ maps probability measures to probability measures.

4.4. The equilibrium and exceptional Γ-measures. For a given rational function $f : \mathbf{P}^1 \to \mathbf{P}^1$ of degree $d \geqslant 2$ and finite vertex set Γ, there are two distinguished Γ-measures that will play a key role in our theory.

Write μ_f for the equilibrium measure on $\mathbf{P}^1$ relative to f [11]. (Another common name in the literature is 'canonical measure' [1, Section 10].) It is the unique Borel probability measure ν that satisfies $f^*\nu = d \cdot \nu$ and that does not charge classical points of $\mathbf{P}^1$ [11, Thm. A]. Here f^* is the usual pullback operator for Borel measures on $\mathbf{P}^1$—not the one defined in Section 4.3. For a vertex set Γ, we define the **equilibrium Γ-measure** $\omega_{f,\Gamma}$ by the formula

$$\omega_{f,\Gamma}(U) := \mu_f(U)$$

for each $U \in \mathcal{S}(\Gamma)$.

By construction, the equilibrium measure $\omega_{f,\Gamma}$ must be supported on a countable subset of $\mathcal{S}(\Gamma)$. Also, the measure $\omega_{f,\Gamma}$ has total mass 1. Indeed, the Julia set of f on $\mathbf{P}^1$ is precisely the support of the measure μ_f [1, Section 10.5]. For each Γ-domain U that intersects the Julia set, there is an iterate f^n that maps U over Γ. So only countably many Γ-domains can intersect the Julia set. In other words, there cannot be a collection X of Γ-domains which has positive measure for $\omega_{f,\Gamma}$ but with $\omega_{f,\Gamma}(U) = 0$ for all $U \in X$. (Note that $\omega_{f,\Gamma}(\Gamma) > 0$ if and only if the Julia set of f lies in Γ [1, 11].)

LEMMA 4.4. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational function of degree $d \geqslant 2$, let $\Gamma = \{\zeta\}$ be a singleton vertex set, let $\Gamma' = \{\zeta, f(\zeta)\}$, and let π_* and f^* be the operators defined in the previous section. Then $\pi_*\omega_{f,\Gamma'} = \omega_{f,\Gamma}$ and $f^*\omega_{f,\Gamma'} = d \cdot \pi_*\omega_{f,\Gamma'}$.*

Proof. The statement about π_* is immediate from the definitions.

Let $\phi : \mathbf{P}^1 \to \mathbb{C}$ be a Γ-measurable function. It is also Borel measurable on $\mathbf{P}^1$ since each element of $\mathcal{S}(\Gamma)$ is either an open set or a point. The definitions of the multiplicities $m_{U,V}$ show that

$$f_*\phi(y) = \sum_{f(x)=y} m_f(x)\phi(x) \qquad (y \in \mathbf{P}^1),$$

which agrees with the formula for the pushforward of Borel measurable functions. Since $f^*\mu_f = d \cdot \mu_f$ as Borel measures on $\mathbf{P}^1$, we find that

$$\int \phi \, f^*\omega_{f,\Gamma'} = \int f_*\phi \, \omega_{f,\Gamma'} = \int f_*\phi\mu_f$$
$$= \int \phi \, f^*\mu_f = d \int \phi \, \mu_f = d \int \phi \, \pi_*\omega_{f,\Gamma'}.$$

Hence $f^*\omega_{f,\Gamma'} = d \cdot \pi_*\omega_{f,\Gamma'}$ as elements of $M(\Gamma)$. □

Suppose now that the rational function $f : \mathbf{P}^1 \to \mathbf{P}^1$ has an exceptional orbit $\mathcal{E}$. The **exceptional Γ-measure** associated with the orbit $\mathcal{E}$ is defined to be the probability measure $\delta_{\mathcal{E}} \in M(\Gamma)$ given by

$$\delta_{\mathcal{E}}(U) = \frac{\#(\mathcal{E} \cap U)}{\#\mathcal{E}}.$$

REMARK 4.5. Recall that an exceptional orbit $\mathcal{E}$ is finite and $f^{-1}(\mathcal{E}) = \mathcal{E}$. Since k has characteristic zero, the function f admits at most two classical exceptional points and at most one exceptional point in $\mathbf{P}^1 \smallsetminus \mathbb{P}^1(k)$ (necessarily of type II).

LEMMA 4.6. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational function of degree $d \geqslant 2$, let $\Gamma = \{\zeta\}$ be a singleton vertex set, and let $\Gamma' = \{\zeta, f(\zeta)\}$. Suppose that $\mathcal{E}$ is an exceptional orbit for f. Write $\delta_{\mathcal{E}}$ and $\delta'_{\mathcal{E}}$ for the associated probability measures with respect to Γ and Γ', respectively. Then $\pi_*\delta'_{\mathcal{E}} = \delta_{\mathcal{E}}$ and $f^*\delta'_{\mathcal{E}} = d \cdot \pi_*\delta'_{\mathcal{E}}$.*

Proof. Since exceptional measures count the number of exceptional points, we evidently have $\pi_*\delta'_{\mathcal{E}} = \delta_{\mathcal{E}}$. For the other equality, let $U \in \mathcal{S}(\Gamma)$. Then

$$f^*\delta'_{\mathcal{E}}(U) = \sum_{\substack{V \in \mathcal{S}(\Gamma') \\ V \subset f(U)}} m_{V,U} \frac{\#(\mathcal{E} \cap V)}{\#\mathcal{E}}.$$

The quantity $m_{V,U}$ is the constant value of $\#\left(f^{-1}(y) \cap U\right)$ for $y \in V$, counted with multiplicities. In particular, if $c \in \mathcal{E} \cap V$, then $m_{V,U} = 0$ or d, depending on whether $f^{-1}(c) \cap U$ is empty or not. Note also that $\#(\mathcal{E} \cap U) = \#(\mathcal{E} \cap f(U))$, since $\mathcal{E}$ is a totally invariant set. Hence,

$$f^*\delta'_{\mathcal{E}}(U) = \sum_{\substack{V \in \mathcal{S}(\Gamma') \\ V \subset f(U)}} d \frac{\#(\mathcal{E} \cap V)}{\#\mathcal{E}} = d \frac{\#(\mathcal{E} \cap U)}{\#\mathcal{E}} = d \cdot \pi_*\delta'_{\mathcal{E}}(U).$$ □

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

4.5. Surplus equidistribution and surplus estimates. We now give two technical results that will be used to prove the main result in the next section. The first is of interest in its own right: it describes how surplus multiplicities of disks behave under iteration. The second gives a lower bound for the mass of a Γ-disk in terms of its surplus multiplicity.

PROPOSITION 4.7 (Surplus Equidistribution). *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational function of degree $d \geqslant 2$ with associated equilibrium measure μ_f. Let U be an open Berkovich disk with boundary point ζ. Suppose that the Julia set of f is not equal to $\{\zeta\}$. Then exactly one of the following is true:*

(1) *The iterated surplus multiplicities of U satisfy*

$$s_{f^n}(U) = \mu_f(U) \cdot d^n + o(d^n).$$

(2) *The orbit $\mathcal{O}_f(\zeta)$ converges along the locus of total ramification to a classical exceptional orbit (of length 1 or 2), and*

$$s_{f^n}(U) = 0 \quad \text{and} \quad \mu_f(f^n(U)) = 1 \quad \text{for all } n \geqslant 1.$$

Proof. The two cases of the proposition are mutually exclusive. For if (2) holds, then $s_f(U) = 0$, so $f(U) \neq \mathbf{P}^1$. The relation $\mu_f = \frac{1}{d} f^* \mu_f$ of Borel measures yields

$$\mu_f(U) = \frac{m_f(U)}{d} \mu_f(f(U)) = \frac{m_f(U)}{d} > 0.$$

But then (1) is contradicted.

In the remainder of the proof, let us assume that case (1) of the proposition does not hold. The equilibrium measure μ_f does not charge ζ by hypothesis on the Julia set of f. Let y be an arbitrary point of $\mathbf{P}^1$ that is not a classical exceptional point. Using equidistribution of iterated preimages [**11**, Thm. A], we find that

$$\mu_f(U) = \lim_{n \to \infty} \frac{\#\left(f^{-n}(y) \cap U\right)}{d^n} = \lim_{n \to \infty} \frac{\varepsilon(y, n, U) \cdot m_{f^n}(U) + s_{f^n}(U)}{d^n},$$

where $\varepsilon(y, n, U) = 1$ if $y \in \overline{f^n}(U)$ and 0 otherwise. We conclude that $m_{f^n}(U) \neq o(d^n)$; for otherwise, we are in case (1) of the proposition.

Let $\zeta_0 = \zeta$ and set $\zeta_n = f(\zeta_{n-1})$ for each $n \geqslant 1$. We can write $\vec{v}_n \in T\mathbf{P}^1_{\zeta_n}$ for the tangent vector such that $D(\vec{v}_0) = U$ and $Tf^n(\vec{v}_0) = \vec{v}_n$. Then

$$m_{f^n}(U) = \prod_{i=0}^{n-1} m_f(D(\vec{v}_i)).$$

Each factor in the product is an integer in the range $1, \ldots, d$. If infinitely many of the multiplicities $m_f(D(\vec{v}_i))$ are strictly smaller than d, then $m_{f^n}(U) = o(d^n)$. Thus $m_f(D(\vec{v}_n)) = d$ for all $n \gg 0$. As multiplicities are upper semicontinuous, this shows that $m_f(\zeta_n) = d$ for all n sufficiently large, so the orbit $\mathcal{O}_f(\zeta)$ eventually lies in the locus of total ramification $\mathcal{R}_f^{\mathrm{tot}}$.

We now show that $\mathcal{O}_f(\zeta)$ converges to a classical exceptional orbit. Let n_0 be such that $\zeta_n \in \mathcal{R}_f^{\mathrm{tot}}$ for all $n \geqslant n_0$. The locus of total ramification is connected [8, Thm. 8.2], and any pair of points in $\mathcal{R}_f^{\mathrm{tot}}$ lie at finite hyperbolic distance to each other unless one is a classical critical point. So it suffices to prove that the hyperbolic distance $\rho_{\mathbf{H}}(\zeta_{n_0}, \zeta_n)$ grows without bound as $n \to \infty$. For ease of notation, let us assume that $n_0 = 0$. Since $\zeta_n, \zeta_{n+1} \in \mathcal{R}_f^{\mathrm{tot}}$, the entire segment connecting them must lie in the locus of total ramification as well. Hence f maps $[\zeta_n, \zeta_{n+1}]$ injectively onto $[\zeta_{n+1}, \zeta_{n+2}]$. Moreover, $\rho_{\mathbf{H}}(\zeta_{n+1}, \zeta_{n+2}) = d \cdot \rho_{\mathbf{H}}(\zeta_n, \zeta_{n+1})$. By induction, we see that

$$\rho_{\mathbf{H}}(\zeta_{n+\ell}, \zeta_{n+\ell+1}) = d^{\ell} \cdot \rho_{\mathbf{H}}(\zeta_n, \zeta_{n+1}), \quad \ell = 0, 1, 2, \ldots,$$

so the locus of total ramification has infinite diameter. The locus of total ramification has at most two classical points in it; hence, some classical totally ramified point c is an accumulation point of $\mathcal{O}_f(\zeta)$. By (weak) continuity of f, we find that $f(c) \in \mathcal{R}_f^{\mathrm{tot}}$. So c is exceptional of period one or two. The orbit $\mathcal{O}_f(\zeta)$ must actually converge to the orbit of c since the latter is attractive.

Since f has a classical exceptional point, it is conjugate either to a polynomial or to $z \mapsto z^{-d}$. We treat the former case and leave the latter to the reader. Without loss of generality, we now assume that f is a polynomial and that $f^n(\zeta)$ converges to ∞ along the locus of total ramification. As k has characteristic zero, the ramification locus near ∞ is contained in a strong tubular neighborhood of finite radius around $(\zeta_{0,R}, \infty)$ for some $R > 1$ [9, Thm. F]. Since hyperbolic distance is expanding on the ramification locus, we see that $f^n(\zeta)$ converges to infinity along the segment $(\zeta_{0,R}, \infty)$. In particular, since the Julia set of f is bounded away from ∞, and since f preserves the partial ordering of points in the tree $\mathbf{P}^1$ (relative to the maximal point ∞), we see that ζ must lie above the entire Julia set. That is, every segment from a Julia point to ∞ must pass through ζ.

Recall that $\Gamma = \{\zeta\}$, and that we are looking at a distinguished Γ-disk U. The previous paragraph shows that either U does not meet infinity or it does not meet the Julia set (or both). In particular, since the Julia set and the point at infinity are totally invariant for f, we have $f^n(U) \neq \mathbf{P}^1$, and so the surplus multiplicities satisfy $s_{f^n}(U) = 0$ for all $n \geqslant 1$. In fact, U must meet the Julia set; otherwise, $\mu_f(U) = 0$, and we are in case (1) of the proposition. Observe that $\zeta \in f^n(U)$ for each $n \geqslant 1$, so the entire Julia set of f is contained in $f^n(U)$. This shows that $\mu_f(f^n(U)) = 1$, and we are in case (2) of the proposition as desired. $\square$

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

LEMMA 4.8 (Surplus Estimate). *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational function of degree $d \geqslant 2$, and let $\Gamma = \{\zeta\}$ be a singleton vertex set. Set $\Gamma' = \{\zeta, f(\zeta)\}$. (Note that $\Gamma = \Gamma'$ is allowed.) For any Γ-disk U and any Γ'-measure solution ν to the equation $f^*\nu = d \cdot \pi_*\nu$, we find that*

$$\nu(U) \geqslant \frac{s_f(U)}{d}.$$

Proof. For ease of notation, let us write $m = m_f(U)$ and $s = s_f(U)$. We may explicitly compute the multiplicities appearing in the pullback operator to be

$$m_{V,U} = \begin{cases} m + s & \text{if } V \subset \bar{f}(U) \\ s & \text{if } V \not\subset \bar{f}(U). \end{cases}$$

Then for χ_U the characteristic function on the Γ-disk U,

$$\begin{aligned} d \cdot \pi_*\nu(U) = f^*\nu(U) &= \int_{\bar{f}(U)} f_*\chi_U \ \nu + \int_{\mathbf{P}^1 \smallsetminus \bar{f}(U)} f_*\chi_U \ \nu \\ &= (m+s) \cdot \nu\left(\bar{f}(U)\right) + s \cdot \nu\left(\mathbf{P}^1 \smallsetminus \bar{f}(U)\right) \\ &= s + m \cdot \nu\left(\bar{f}(U)\right) \geqslant s. \end{aligned}$$

Dividing by d gives the result. □

4.6. Simultaneous solutions to iterated pullback formulas. The equation $f^*\nu = d \cdot \pi_*\nu$ does not necessarily have a unique solution $\nu \in M^1(\Gamma)$ as one might expect by analogy with the standard setting. However, the solution does become essentially unique if we impose all pullback relations $(f^n)^*\nu = d^n \cdot \pi_{n*}\nu$ for $n = 1, 2, 3, \ldots$

Let $\Gamma = \{\zeta\}$ be a singleton vertex set for $\mathbf{P}^1$. Let $\Gamma_n = \{\zeta, f^n(\zeta)\}$ for each $n \geqslant 1$, and write $(f^n)^*$ and π_{n*} for the pullback and pushforward operators relative to Γ and Γ_n, respectively. We define a set of Γ-measures $\Delta_f \subset M^1(\Gamma)$ by

$$\Delta_f = \bigcap_{n \geqslant 1} \pi_{n*} \left\{\omega \in M^1(\Gamma_n) : (f^n)^*\omega = d^n \cdot \pi_{n*}\omega\right\}.$$

Each element of Δ_f is the projection of a solution to a pullback formula for each iterate of f, although we do not require any compatibility among these solutions. Linearity of the pullback and pushforward operators shows that Δ_f is a convex polyhedral set in the space $M^1(\Gamma)$. Note that Δ_f is nonempty: since $\omega_{f,\Gamma} = \omega_{f^n,\Gamma}$, the set Δ_f must contain the equilibrium Γ-measure $\omega_{f,\Gamma}$ (Lemma 4.4).

REMARK 4.9. The intersected sets that define Δ_f are typically not nested.

THEOREM 4.10. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational function of degree $d \geqslant 2$, and let $\Gamma = \{\zeta\}$ be a singleton vertex set. Suppose that the Julia set of f is not equal*

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

to $\{\zeta\}$. With the above notation, Δ_f is the convex hull of the equilibrium Γ-measure $\omega_{f,\Gamma}$ and at most one probability measure $\delta_{\mathcal{E}}$ supported on a classical exceptional orbit $\mathcal{E}$. Moreover, if $\Delta_f \neq \{\omega_{f,\Gamma}\}$, then $f^n(\zeta)$ converges to an exceptional orbit along the locus of total ramification for f.

REMARK 4.11. For our application to complex dynamics, it is sufficient to restrict to countably supported measures in the definition of Δ_f. But the theorem shows that this hypothesis is unnecessary: an arbitrary Γ-measure satisfying all pullback formulas is countably supported.

REMARK 4.12. With a little more work, one can show that this result continues to hold when k has positive characteristic provided that $\mathcal{O}_f(\zeta)$ does not converge to a wildly ramified exceptional orbit.

COROLLARY 4.13. *With the hypotheses of Theorem 4.10, no measure in Δ_f charges ζ.*

Proof. The hypothesis on the Julia set guarantees that ζ is not exceptional and that μ_f does not charge ζ. □

Proof of Theorem 4.10. Suppose that $f^n(\zeta)$ does not converge along the locus of total ramification to a classical exceptional periodic orbit for f. Let U be any Γ-domain for $\Gamma = \{\zeta\}$. If $\nu \in \Delta_f$, Proposition 4.7 and the Surplus Estimate applied to f^n and U show that

$$\nu(U) \geqslant \frac{s_{f^n}(U)}{d^n} = \mu_f(U) + o(1).$$

Since this is true for any Γ-disk U, and since μ_f is a probability measure with no support at ζ, we conclude that $\nu(U) = \mu_f(U)$ for every $U \in \mathcal{S}(\Gamma)$.

Now suppose that $f^n(\zeta)$ converges along the locus of total ramification to the orbit of a classical exceptional point. Without loss of generality, we may assume that the exceptional point is fixed by replacing f with f^2. After conjugating the exceptional fixed point to ∞, we may assume that f is a polynomial. As in the proof of Proposition 4.7, we find that $f^n(\zeta)$ converges to ∞ along the segment $(\zeta_{0,R}, \infty)$ for some $R > 1$, and ζ lies above the entire Julia set.

Suppose that U is a Γ-domain that meets the Julia set. Then $f(U)$ contains the entire Julia set, and the standard pullback formula $f^*\mu_f = d \cdot \mu_f$ on $\mathbf{P}^1$ shows that

$$d \cdot \mu_f(U) = m_f(U) \cdot \mu_f(f(U)) = m_f(U) \quad \Rightarrow \quad \mu_f(U) = \frac{m_f(U)}{d}. \tag{4.6.1}$$

In particular, only finitely many Γ-disks may meet the Julia set.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

Fix any $\nu \in \Delta_f$. Write U_∞ for the unique Γ-domain containing infinity; write $U_1, \dots, U_r$ for the Γ-domains that meet the Julia set; write U_0 for the union of the remaining elements of $\mathcal{S}(\Gamma)$. Note that since we are in case (2) of Proposition 4.7, the surplus multiplicity satisfies $s_{f^n}(U) = 0$ for all $n \geqslant 1$ and $U \in \mathcal{S}(\Gamma)$. Furthermore, we observe that $f(U_\infty) \subset U_\infty$ and $m_{f^n}(U_\infty) = d^n$, and that f^n maps U_0 onto $f^n(U_0) \subset U_\infty$ in everywhere d^n-to-one fashion.

First we show that $\nu(U_0) = 0$. For each $n \geqslant 1$, there exists $\nu_n \in M^1(\Gamma_n)$ such that $(f^n)^* \nu_n = d^n \cdot \pi_{n*}\nu_n = d^n \cdot \nu$. Then

$$d^n \cdot \pi_{n*}\nu_n(U_\infty) = (f^n)^*\nu_n(U_\infty) = \int (f^n)_* \chi_{U_\infty}\ \nu_n = d^n \cdot \nu_n\left(f^n(U_\infty)\right). \quad (4.6.2)$$

Thus $\nu(U_\infty) = \nu_n\left(f^n(U_\infty)\right)$ for any $n \geqslant 1$. Write A for the annulus with boundary points ζ and $f^n(\zeta)$. By the definition of the pushforward, we see that

$$\nu(U_\infty) = \pi_{n*}\nu_n(U_\infty) = \nu_n(f^n(U_\infty)) + \nu_n(f^n(U_0)) + \nu_n(A).$$

Therefore, $\nu_n(A) = \nu_n(f^n(U_0)) = 0$. But the calculation (4.6.2) applies equally well to U_0, showing that $\nu(U_0) = \nu_n\left(f^n(U_0)\right)$, and so we conclude that $\nu(U_0) = 0$.

Next we observe that for $i = 1, \dots, r$, we have

$$d^n \cdot \pi_{n*}\nu_n(U_i) = (f^n)^*\nu_n(U_i) = \int (f^n)_* \chi_{U_i}\ \nu_n = m_{f^n}(U_i) \cdot \nu_n\left(f^n(U_i)\right).$$

From (4.6.1), we see that

$$\mu_f(U_i) = \frac{m_f(U_i)}{d} = \frac{m_{f^n}(U_i)}{d^n}, \quad n \geqslant 1, \quad i = 1, \dots, r.$$

Combining the last two displayed equations gives

$$\nu(U_i) = \pi_{n*}\nu_n(U_i) = \mu_f(U_i)\nu_n\left(f^n(U_i)\right).$$

The quantity $a := \nu_n\left(f^n(U_i)\right)$ is independent of n and i since $\mu_f(U_i) > 0$ for $i = 1, \dots, r$ and $f^n(U_1) = \cdots = f^n(U_r)$. Setting $b = \nu(U_\infty)$, we have proved that $\nu = a \cdot \omega_{f,\Gamma} + b \cdot \delta_\infty$. □

5. A transfer principle

In this section, we complete the proof of Theorem B. We explain the transfer of solutions of the pullback formula for the dynamics of our complex surface map $F : X \dashrightarrow Y$ to Γ-measure solutions of the pullback formula for a related function $f : \mathbf{P}^1 \to \mathbf{P}^1$, and vice versa.

In all of the proofs, we must deal with two subcases: $Y = X$ and $Y \neq X$. We encourage the reader to think only of the former case on a first reading. As a further aid, we give a coarse sketch of the proof when $Y = X$ in the next subsection. Afterwards, we give a more general and rigorous description of the transfer process.

5.1. Outline of the transfer process when $Y = X$. As discussed in Section 3, we have a meromorphic one-parameter family of complex rational functions f_t, from which we define a rational self-map of the fibered surface $X = \mathbb{D} \times \mathbb{P}^1$ given by

$$F : X \dashrightarrow X \qquad F(t, z) = (t, f_t(z)).$$

The map F is nonconstant on the central fiber $X_0 = \{0\} \times \mathbb{P}^1$ if and only if the reduction of f_0 is nonconstant; we assume that these equivalent conditions hold.

The field $\mathbb{C}((t))$ of formal Laurent series in the variable t has a natural non-Archimedean absolute value on it that measures the order of vanishing:

$$\left| \sum_{n \geqslant N} a_n t^n \right| = \exp(-N) \qquad (a_N \neq 0).$$

By viewing the parameter t as a formal variable, we may identify the family f_t with a single non-Archimedean rational function f defined over the field $\mathbb{C}((t))$.

Recall that the Berkovich projective line $\mathbf{P}^1$ defined over $\mathbb{C}((t))$ has a natural tree structure with a canonical root ζ, called the Gauss point. More is true: the branches of $\mathbf{P}^1$ at the Gauss point ζ are in canonical bijection with points of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. If we use the vertex set $\Gamma = \{\zeta\}$ on $\mathbf{P}^1$, then this bijection tells us how to relate Borel measures on $\mathbb{P}^1(\mathbb{C})$ with Γ-measures on $\mathbf{P}^1$. Namely, the point measure at $x \in \mathbb{P}^1(\mathbb{C})$ corresponds to the Γ-measure with all of its mass on the corresponding branch U_x, which is a Γ-disk in the sense of Section 4. This correspondence extends by linearity to any purely atomic Borel measure on the Riemann sphere and any countably supported Γ-measure that does not charge the Gauss point. (Borel measures constructed in this way will be called **residual measures**.) Finally, a Borel measure μ of mass M with no atom on $\mathbb{P}^1(\mathbb{C})$ gives rise to the Γ-measure $M \cdot \delta_\zeta$ on $\mathbf{P}^1$, but evidently this part of the correspondence has no inverse.

The Transfer Principle (Proposition 5.1) will show that this correspondence preserves solutions to the pullback equations $F^*\mu = d \cdot \mu$ for Borel measures on X_0 and $f^*\omega = d \cdot \omega$ for Γ-measures on $\mathbf{P}^1$. The argument is essentially formal once one proves equality of the local multiplicities and surplus multiplicities in the two definitions of pullback. It turns out that, even in the complex setting, each is an algebraic quantity that may be deduced from the expression for f_0.

5.2. Reduction and the residual measures. Let $X \to \mathbb{D}$ be a proper fibered surface over a complex disk with generic fiber $\mathbb{P}^1_{\mathbb{C}}$. Assume that the fiber X_0 over

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

the origin is reduced. Let $\mathbb{L}$ be the completion of an algebraic closure of $\mathbb{C}((t))$ endowed with the natural non-Archimedean absolute value, and write $\mathbb{L}^\circ$ for its valuation ring. We claim that X gives rise, canonically, to a vertex set $\Gamma \subset \mathbf{P}^1$. The local ring of $\mathbb{D}$ at the origin is contained inside $\mathbb{L}^\circ$, and hence so is its completion. By completing along the central fiber X_0 and base extending to $\mathbb{L}^\circ$, we obtain an admissible formal scheme $\mathfrak{X}$ over $\mathbb{L}^\circ$ with generic fiber $\mathbf{P}^1 = \mathbf{P}^1_{\mathbb{L}}$. Note that since X_0 is reduced, it may be identified with the special fiber $\mathfrak{X}_s$ as $\mathbb{C}$-schemes. Let

$$\mathrm{red}_X : \mathbf{P}^1 \to X_0$$

be the surjective reduction map [2, 2.4.4]. Let $\eta_1, \ldots, \eta_r$ be the generic points of the irreducible components of the special fiber X_0. There exist unique type II points $\zeta_1, \ldots, \zeta_r \in \mathbf{P}^1$ such that $\mathrm{red}_X(\zeta_i) = \eta_i$ for $i = 1, \ldots, r$. The desired vertex set is $\Gamma = \{\zeta_1, \ldots, \zeta_r\}$.

For each closed point $x \in X_0$, the formal fiber $\mathrm{red}_X^{-1}(x)$ is a Γ-domain, as defined in Section 4.2. The association $x \mapsto \mathrm{red}_X^{-1}(x)$ induces a bijection between points of the scheme X_0 and elements of $\mathcal{S}(\Gamma)$. We obtain a projection of measures,

$$\mathrm{red}_X^* : M^1(X_0) \to M^1(\Gamma),$$

where $M^1(X_0)$ is the space of Borel probability measures on $X_0(\mathbb{C})$ (with its analytic topology) and $M^1(\Gamma)$ is the space of positive Γ-measures of total mass 1 on $\mathbf{P}^1$, defined as follows. Given $\mu \in M^1(X_0)$, let $B = \{x \in X_0(\mathbb{C}) : \mu(\{x\}) > 0\}$. The set B is at most countable. Write $\eta_1, \ldots, \eta_r$ for the generic points of the irreducible components $C_1, \ldots, C_r$ of X_0. Define $\omega = \mathrm{red}_X^*(\mu)$ by

$$\omega(\mathrm{red}_X^{-1}(x)) := \begin{cases} \mu(x) & \text{if } x \in X_0(\mathbb{C}) \\ \mu(C_i \smallsetminus B) & \text{if } x = \eta_i \text{ for some } i = 1, \ldots, r. \end{cases}$$

Evidently, $\omega(\mathbf{P}^1) = \mu(X_0) = 1$.

Now let $M^1(\Gamma)^\dagger \subset M^1(\Gamma)$ be the subset of Γ-measures that are countably supported on elements of $\mathcal{S}(\Gamma)$, and assign no mass to the elements of Γ. The reduction map red_X induces

$$\mathrm{red}_{X*} : M^1(\Gamma)^\dagger \to M^1(X_0)$$

as a partial inverse to red_X^*. Explicitly, the **residual measure** $\mu = \mathrm{red}_{X*}(\omega) \in M^1(X_0)$ is defined by

$$\mu(\{x\}) := \omega(\mathrm{red}_X^{-1}(x)) \quad (x \in X_0(\mathbb{C})).$$

For each $\omega \in M^1(\Gamma)^\dagger$, the residual measure μ is an atomic probability measure on X_0. The terminology is explained by the case where X_0 is irreducible and $\Gamma = \{\zeta_{0,1}\}$ is the Gauss point of $\mathbf{P}^1$; the mass of the residual measure at a closed point $x \in X_0(\mathbb{C})$ is precisely the volume of the residue class $\mathrm{red}_X^{-1}(x) \subset \mathbf{P}^1$.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

5.3. Compatibility of pullbacks. Let f_t be a one-parameter family of dynamical systems of degree $d \geqslant 2$ with t varying holomorphically in a small punctured disk $\mathbb{D}^*$ and extending meromorphically over the puncture. As in Section 3, we let $X = \mathbb{D} \times \mathbb{P}^1(\mathbb{C})$ and write $\pi : Y \to X$ for the minimal modification of X along X_0 such that the induced rational map $F : X \dashrightarrow Y$ is not constant along X_0. The surfaces X and Y induce vertex sets $\Gamma = \{\zeta\}$ and $\Gamma' = \{\zeta, f(\zeta)\}$ on $\mathbf{P}^1 = \mathbf{P}^1_{\mathbb{L}}$, where $\mathbb{L}$ is the completion of an algebraic closure of $\mathbb{C}((t))$ endowed with the natural non-Archimedean absolute value, and the family f_t defines $f : \mathbf{P}^1 \to \mathbf{P}^1$. The pullback F^* from measures on Y_0 to measures on X_0 is given by the formula (3.3.1) or (3.3.2), depending on whether or not f fixes ζ.

PROPOSITION 5.1 (Transfer Principle). *Let $F : X \dashrightarrow Y$, $f : \mathbf{P}^1 \to \mathbf{P}^1$, Γ, and Γ' be as above. The following conclusions hold.*

(1) *If μ is a measure on the central fiber Y_0 such that $F^*\mu = d \cdot \pi_*\mu$, then $\omega = \mathrm{red}_Y^*\mu$ is a Γ'-measure satisfying $f^*\omega = d \cdot \pi_*\omega$.*

(2) *If ω is a countably supported Γ' probability measure satisfying $\omega(\Gamma') = 0$ and $f^*\omega = d \cdot \pi_*\omega$, then the residual measure $\mu = \mathrm{red}_{Y*}(\omega)$ satisfies $F^*\mu = d \cdot \pi_*\mu$.*

Proof. We begin by comparing the notions of multiplicity defined for F (on X_0) and for f (on $\mathbf{P}^1$). Lemma 2.1 gives a meromorphic family of Möbius transformations $A_t \in \mathrm{PGL}_2(\mathbb{C})$ for $t \in \mathbb{D}$, holomorphic away from $t = 0$, such that $A_t \circ f_t$ converges as $t \to 0$ to $\Phi \in \overline{\mathrm{Rat}}_d$ with nonconstant reduction ϕ. On one hand, this implies that ϕ describes the meromorphic map F from the fiber X_0 onto its image component E in Y_0 (or C if $X = Y$). Evidently the local degree $m(x)$ for each point of X_0 may be read off algebraically as the order of vanishing of $\phi(z) - \phi(x)$ at x. On the other hand, we may view A_t as an element $A \in \mathrm{PGL}_2(\mathbb{C}((t)))$. In particular, $A \circ f$ has nonconstant reduction as a rational function on $\mathbf{P}^1$, and the reduction is equal to ϕ. If U_x is a Γ-disk with reduction $x \in X_0$, Rivera-Letelier's Algebraic Reduction Formula [1, Corollary 9.25] shows that the directional multiplicity $m_f(U_x)$ is equal to the order of vanishing of $\phi(z) - \phi(x)$ at x, and so we conclude that

$$m(x) = m_f(U_x).$$

From the description of the surplus multiplicity of the map Φ in (2.2.1) and the corresponding description of the surplus multiplicity in [8, Lemma 3.17], we also see that

$$s(x) = s_f(U_x).$$

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

Finally, the Algebraic Reduction Formula shows that $\deg(\phi) = m_f(\zeta)$, where ζ is the unique vertex in Γ.

Since $\omega = \mathrm{red}_Y^*(\mu)$ is supported on countably many Γ'-domains, to prove the first statement of the Transfer Principle it suffices to show that

$$F^*\mu = d \cdot \pi_*\mu \;\Rightarrow\; f^*\omega(U) = d \cdot \pi_*\omega(U) \text{ for every } U \in \mathcal{S}(\Gamma). \tag{TP1}$$

Under the hypotheses of the second statement of the Transfer Principle, we find that μ is countably supported. Thus it suffices to show that

$$f^*\omega = d \cdot \pi_*\omega \;\Rightarrow\; F^*\mu(\{x\}) = d \cdot \pi_*\mu(\{x\}) \text{ for every closed point } x \in X_0. \tag{TP2}$$

We now prove these statements.

Case $Y = X$. Let $x \in X_0$ be a closed point, and write $U_x = \mathrm{red}_X^{-1}(x) \in \mathcal{S}(\Gamma)$. Then

$$\begin{aligned} f^*\omega(U_x) &= \sum_{V \in \mathcal{S}(\Gamma)} m_{V,U_x}\omega(V) \\ &= m_{\bar{f}(U_x),U_x}\omega(\bar{f}(U_x)) + \sum_{V \in \mathcal{S}(\Gamma)} s_f(U_x)\omega(V) \\ &= m(x)\mu(\{\phi(x)\}) + s(x) \\ &= F^*\mu(\{x\}), \end{aligned}$$

while $d \cdot \omega(U_x) = d \cdot \mu(\{x\})$ by definition. This immediately implies equation (TP2).

To verify equation (TP1), it remains to consider the mass on $\Gamma = \Gamma'$. Set $B = \{x \in X_0 : \mu(x) > 0\}$. If $F^*\mu = d \cdot \mu$, then $F^*\mu$ has no atoms in $X_0 \smallsetminus B$. From the definition of $F^*\mu$ in (3.3.1), we see that $F^*\mu$ agrees with $\phi^*\mu$ on $X_0 \smallsetminus B$. Thus,

$$\begin{aligned} d \cdot \omega(\zeta) = d \cdot \mu(X_0 \smallsetminus B) &= F^*\mu(X_0 \smallsetminus B) \\ &= \phi^*\mu(X_0 \smallsetminus B) = \deg(\phi) \cdot \mu(X_0 \smallsetminus B) \\ &= m_f(\zeta)\omega(\zeta) = f^*\omega(\zeta). \end{aligned}$$

This proves equation (TP1) for all U in $\mathcal{S}(\Gamma)$.

Case $Y \neq X$. Recall that $Y_0 = C \cup E$, where C is the proper transform of X_0, and E is the exceptional fiber of $\pi : Y \to X$. In Section 3.3, to define $F^*\mu$ we introduced the (continuous) projection $\pi_E : Y_0 \to E$ that collapses C to a point.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

Let U_x be the Γ-disk corresponding to a closed point $x \in X_0$. Recall that we set $\varepsilon(V, U_x) = 1$ or 0 depending on whether $\bar{f}(U_x) = V$ or not. We see that

$$\begin{aligned} f^*\omega(U_x) &= \sum_{V \in \mathcal{S}(\Gamma')} m_{V,U_x}\omega(V) \\ &= \sum_{V \in \mathcal{S}(\Gamma')} \big(s_f(U_x) + m_f(U_x)\varepsilon(V, U_x)\big)\,\omega(V) \\ &= s(x) + m(x)\,(\pi_{E*}\mu)(\{\phi(x)\}) \\ &= F^*\mu\,(\{x\})\,, \end{aligned}$$

while $\pi_*\omega(U_x) = \pi_*\mu\,(\{x\})$ is immediate. Evidently (TP2) follows, and (TP1) holds for all Γ-disks.

To verify (TP1), it remains to check the pullback relation for the mass on vertices. Let $B = \{y \in Y_0 : \mu(\{y\}) > 0\}$, and set $B' = \pi(B \cup E) \subset X_0$. Then

$$d \cdot \pi_*\mu\left(X_0 \smallsetminus B'\right) = d \cdot \mu\left(\pi^{-1}(X_0 \smallsetminus B')\right) = d \cdot \mu(C \smallsetminus B).$$

If $F^*\mu = d \cdot \pi_*\mu$, there are no atoms of $F^*\mu$ outside B'. From the definition of $F^*\mu$ in (3.3.2), the measure $F^*\mu$ must agree with the pullback of $\pi_{E*}\mu$ by ϕ on the set $X_0 \smallsetminus B'$; therefore,

$$F^*\mu(X_0 \smallsetminus B') = \phi^*(\pi_{E*}\mu)(X_0 \smallsetminus B') = \deg(\phi) \cdot \mu(E \smallsetminus B).$$

Putting these observations together yields

$$f^*\omega(\zeta) = m_f(\zeta)\omega(f(\zeta)) = \deg(\phi)\mu(E \smallsetminus B) = d \cdot \mu(C \smallsetminus B) = d \cdot \pi_*\omega(\zeta),$$

so (TP1) is verified for all U in $\mathcal{S}(\Gamma)$. □

5.4. Proof of Theorem B. We retain all of the notation from previous sections.

For each $n \geqslant 1$, let $F_n : X \dashrightarrow Y^n$ be the rational map of surfaces associated with the one-parameter family of nth iterates f_t^n, as constructed in Section 2, and write $\pi_n : Y^n \to X$ for the blowing-up morphism. Define

$$\Delta_0 = \bigcap_{n \geqslant 1} \pi_{n*}\{\mu \in M^1(Y_0^n) : F_n^*\,\mu = d^n \cdot \pi_{n*}\mu\} \subset M^1(X_0).$$

Write $\omega_{f,\Gamma}$ for the equilibrium Γ-measure for $f : \mathbf{P}^1 \to \mathbf{P}^1$ associated with $\Gamma = \{\zeta\}$. Recall that Δ_f was defined in Section 4.6.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

THEOREM 5.2. *Let f_t be a meromorphic one-parameter family of rational functions of degree $d \geqslant 2$. Suppose that the family is not holomorphic at $t = 0$; i.e.,* $\deg(f_0) < d$. *The reduction map induces a bijection*

$$\mathrm{red}_X^* : \Delta_0 \xrightarrow{\sim} \Delta_f,$$

with inverse given by the residual measure construction red_{X*}.

Proof. No measure in Δ_f charges the vertex $\zeta \in \Gamma$, and every measure in Δ_f is countably supported (Theorem 4.10). The Transfer Principle (applied to all iterates of f_t and f) shows that the maps

$$\mathrm{red}_X^* : \Delta_0 \to \Delta_f \quad \text{and} \quad \mathrm{red}_{X*} : \Delta_f \to \Delta_0$$

are well defined. That they are inverse to one another follows from the definitions of red_X^* and red_{X*}. □

COROLLARY 5.3. *With the setup of Theorem 5.2, Δ_0 always contains the residual measure $\mathrm{red}_{X*}(\omega_{f,\Gamma})$, and Δ_0 is either a point or a segment in the space of all probability measures. In the latter case, there exists a point mass $\delta_{p_0} \in \Delta_0$ and a one-parameter family of exceptional periodic points p_t for f_t such that f_0 is constant with value p_0, and p_0 is not an indeterminacy point for the rational map $F : X \dashrightarrow X$.*

Proof. Theorem 5.2 allows us to transfer the statements about Δ_0 to Δ_f. The first statement is immediate from Theorem 4.10. If $\Delta_f \neq \{\omega_{f,\Gamma}\}$, then $f^n(\zeta)$ converges along the locus of total ramification to a classical exceptional orbit $\mathcal{E}$. Replacing f and f_t with their second iterates if necessary, we may assume that $\mathcal{E} = \{p\}$ is a single point. Now $p \in \mathbb{P}^1(\mathbb{C}((t)))$ by completeness. *A priori*, this gives a formal one-parameter family p_t with complex coefficients. Since $f_t(p_t) = p_t$ and $\frac{df_t}{dz}(p_t) \equiv 0$, the implicit function theorem shows that p_t is a meromorphic one-parameter family in a small disk about $t = 0$. That is, $p = p_t$ is a one-parameter family of exceptional fixed points for the family f_t. Since $f^n(\zeta)$ converges to p, and since p is a superattracting fixed point for f, it follows that f_0 is constant with value equal to p_0. If U is the open Γ-disk containing p, then $f(U) \subsetneq U$. In particular, this shows that $s_f(U) = s(p_0) = 0$, so p_0 is not an indeterminacy point for the rational map F. □

We are now ready to prove the second main result of the article. With the terminology that we have set up in the preceding sections, our goal is to show that the family of measures of maximal entropy $\{\mu_t : t \in \mathbb{D}^*\}$ converges weakly to the residual measure $\mathrm{red}_{X*}(\omega_{f,\Gamma})$ as $t \to 0$, where $\Gamma = \{\zeta\}$ is the Gauss point of $\mathbf{P}^1$.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

Proof of Theorem B. Let μ^1 be any weak limit of the family μ_t of maximal measures as $t \to 0$ on the surface Y^1. Fix a subsequence $(t_\ell)_{\ell \geqslant 1}$ such that $t_\ell \to 0$ and $\mu_{t_\ell} \to \mu^1$ weakly on Y^1. Set $\mu_0 = \pi_{1*}\mu^1$; then $\mu_{t_\ell} \to \mu_0$ weakly on X.

For each $n \geqslant 2$, let μ^n be a weak limit of the sequence (μ_{t_ℓ}) on the surface Y^n. Note that $\mu_0 = \pi_{n*}\mu^n$ by construction. Moreover, we have $F_n^* \mu^n = d^n \cdot \pi_{n*}\mu^n$ for all $n \geqslant 1$ (Theorem 3.5). Hence $\mu_0 \in \Delta_0$.

It remains to prove that $\mu_0 = \mathrm{red}_{X*}(\omega_{f,\Gamma})$, the residual measure associated with ω_f and the vertex set Γ. This follows immediately from the preceding corollary unless there exists a family of exceptional periodic points p_t for f_t, the reduction of f_0 is equal to the constant p_0, and p_0 is not indeterminate for the rational map F. In that case, $\mu_0 = a \cdot \mathrm{red}_{X*}(\omega_{f,\Gamma}) + b \cdot \mathrm{red}_{X*}(\delta_{\mathcal{E}})$, for some $a, b \geqslant 0$, where $p_0 \in \mathrm{supp}(\mathrm{red}_{X*}(\delta_{\mathcal{E}}))$. We must prove that $b = 0$.

Since p_0 is not indeterminate, by continuity there exists a neighborhood N of p_0 such that $f_t(N) \subset N$ for all t sufficiently close to zero. Hence, N is contained in the Fatou set of f_t, and μ_t assigns no mass to N. By weak continuity, $\mu_0(N) = 0$. That is, $b = 0$ and $\mu_0 = \mathrm{red}_{X*}(\omega_{f,\Gamma})$ as desired. $\square$

Acknowledgements

We are grateful for the opportunities that allowed these ideas to germinate: the 2010 Bellairs Workshop in Number Theory funded by the CRM in Montreal, and the Spring 2012 semester on Complex and Arithmetic Dynamics at ICERM. We would like to thank Charles Favre, Mattias Jonsson, Jan Kiwi, and Juan Rivera-Letelier for helpful discussions, and we further thank Jonsson for inviting us to speak about this work at the December 2012 RTG workshop at the University of Michigan. Finally, we thank the anonymous referee for useful suggestions for improving the presentation.

References

[1] M. Baker and R. Rumely, *Potential Theory and Dynamics on the Berkovich Projective Line*, Mathematical Surveys and Monographs, 159 (American Mathematical Society, Providence, RI, 2010).

[2] V. G. Berkovich, *Spectral Theory and Analytic Geometry Over Non-Archimedean Fields*, Mathematical Surveys and Monographs, 33 (American Mathematical Society, Providence, RI, 1990).

[3] A. Bonifant, J. Kiwi and J. Milnor, 'Cubic polynomial maps with periodic critical orbit. II. Escape regions', *Conform. Geom. Dyn.* **14** (2010), 68–112.

[4] L. DeMarco, 'Iteration at the boundary of the space of rational maps', *Duke Math. J.* **130** (1) (2005), 169–197.

[5] L. DeMarco, 'The moduli space of quadratic rational maps', *J. Amer. Math. Soc.* **20** (2) (2007), 321–355.

https://doi.org/10.1017/fms.2014.8 Published online by Cambridge University Press

[6] L. DeMarco and X. Faber, 'Degenerations of complex dynamical systems II: Analytic and algebraic stability'. Preprint, arXiv:1309.7103 [math.DS], 2013.
[7] L. G. DeMarco and C. T. McMullen, 'Trees and the dynamics of polynomials', *Ann. Sci. Éc. Norm. Supér. (4)* **41** (3) (2008), 337–382.
[8] X. Faber, 'Topology and geometry of the Berkovich ramification locus for rational functions, I', *Manuscripta Math.* **142** (3–4) (2013), 439–474.
[9] X. Faber, 'Topology and geometry of the Berkovich ramification locus for rational functions, II', *Math. Ann.* **356** (3) (2013), 819–844.
[10] C. Favre, Personal communication.
[11] C. Favre and J. Rivera-Letelier, 'Théorie ergodique des fractions rationnelles sur un corps ultramétrique', *Proc. Lond. Math. Soc. (3)* **100** (1) (2010), 116–154.
[12] A. Freire, A. Lopes and R. Mañé, 'An invariant measure for rational maps', *Bol. Soc. Brasil. Mat.* **14** (1) (1983), 45–62.
[13] J. Kiwi, 'Puiseux series polynomial dynamics and iteration of complex cubic polynomials', *Ann. Inst. Fourier (Grenoble)* **56** (5) (2006), 1337–1404.
[14] J. Kiwi, 'Rescaling limits of complex rational maps'. arXiv:1211.3397 [math.DS]. Preprint, 2012.
[15] M. J. Ljubich, 'Entropy properties of rational endomorphisms of the Riemann sphere', *Ergodic Theory Dynam. Systems* **3** (3) (1983), 351–385.
[16] R. M. P. Sad and D. Sullivan, 'On the dynamics of rational maps', *Ann. Sci. École Norm. Sup. (4)* **16** (2) (1983), 193–217.
[17] R. Mañé, 'On the uniqueness of the maximizing measure for rational maps', *Bol. Soc. Brasil. Mat.* **14** (1) (1983), 27–43.
[18] R. Mañé, 'The Hausdorff dimension of invariant probabilities of rational maps', in *Dynamical systems, Valparaiso 1986*, Lecture Notes in Math., 1331 (Springer, Berlin, 1988), 86–117.
[19] J. R. Letelier, 'Dynamique des fonctions rationnelles sur des corps locaux', *Astérisque* **287** (xv) (2003), 147–230. Geometric methods in dynamics. II.