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ON PRODUCTS OF PSEUDO-ANOSOV MAPS AND DEHN TWISTS OF RIEMANN SURFACES WITH PUNCTURES

C. ZHANG

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Abstract

Let S be a Riemann surface of type (p, n) with $3p + n > 4$ and $n \geq 1$. We investigate products of some pseudo-Anosov maps θ and Dehn twists t_α on S, and prove that under certain conditions the products $t_\alpha^k \circ \theta$ are pseudo-Anosov for all integers k. We also give examples that show that $t_\alpha^k \circ \theta$ are not pseudo-Anosov for some integers k.

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1. Introduction

Let S be a Riemann surface of type (p, n), where p is the genus and n is the number of punctures of S. Assume that $3p + n > 3$. A nonperiodic map of S onto itself is called reducible if it is isotopic to a map that keeps a system $\{c_1, \ldots, c_s\}$ of disjoint and independent simple curves on S invariant. A map f is called pseudo-Anosov [5, 14] if it leaves invariant a pair of transverse measured foliations $\{\mathcal{F}_+, \mathcal{F}_-\}$ such that $f(\mathcal{F}_+) = \lambda \mathcal{F}_+$ and $f(\mathcal{F}_-) = (1/\lambda)\mathcal{F}_-$ for a fixed real number $\lambda > 1$. See also [1, 12, 13] for constructions and more properties of pseudo-Anosov maps. By the Nielsen–Thurston classification of surface homeomorphisms [5, 14], a nonperiodic map f is either isotopic to a reducible map, or isotopic to a pseudo-Anosov map.

The simplest nontrivial reducible map is the Dehn twist t_α along a simple closed geodesic α that is obtained by cutting S along α, rotating one of the copies of α by 360 degrees and then gluing the two copies back together. The problem of determining whether a finite product of a pseudo-Anosov map f and a power of t_α is still isotopic to a pseudo-Anosov map was extensively studied in [4, 8, 9]. By an abuse of language we call a map f pseudo-Anosov if f is isotopic to a pseudo-Anosov map, and we call a mapping class θ pseudo-Anosov if one of its representatives is a pseudo-Anosov map. In [9] Long and Morton proved that $t_\alpha^k \circ f$ are pseudo-Anosov for all but at most a finite number of integer values of k. Later, Fathi [4] found that $t_\alpha^k \circ f$ are pseudo-Anosov for all but at most seven consecutive integer values of k. In [3] Boyer *et al.* improved the number 'seven' to 'six'.

The main purpose of this paper is to identify certain classes of pseudo-Anosov maps f on a Riemann surface S with punctures so that $t_\alpha^k \circ f$ are pseudo-Anosov for all integers k. Throughout the paper we assume that S is of type (p, n) with $3p + n > 4$ and $n \geq 1$. Let a denote a puncture of S. Write $\tilde{S} = S \cup \{a\}$. Let Mod_S^a be the group that consists of isotopy classes of self-maps of S fixing the puncture a. Each element of Mod_S^a projects to an element of the mapping class group $\mathrm{Mod}_{\tilde{S}}$ under the map defined by neglecting the puncture a, thus defining a natural group epimorphism

$$i : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\tilde{S}}.$$

Let $\mathbf{H}$ be the hyperbolic plane, and $\varrho : \mathbf{H} \to \tilde{S}$ the universal covering with a group G of deck transformations. Since $\tilde{S}$ is a compact surface with a finite number of points removed, the group G is finitely generated, torsion free, and of the first kind. Following Bers [2], there is a fiber space $F(\tilde{S})$, referred to in the literature as a Bers fiber space, over the Teichmüller space $T(\tilde{S})$ such that $F(\tilde{S})$ is isomorphic (via an isomorphism φ) to the Teichmüller space $T(S)$. Furthermore, $\mathrm{Mod}_{\tilde{S}}$ extends to a group $\mathrm{mod}(\tilde{S})$ of fiber-preserving holomorphic automorphisms of $F(\tilde{S})$ and the isomorphism φ induces a group isomorphism $\varphi^* : \mathrm{mod}(\tilde{S}) \to \mathrm{Mod}_S^a$. Since G can be regarded as a normal subgroup of $\mathrm{mod}(\tilde{S})$, the group $\varphi^*(G)$ is a normal subgroup of Mod_S^a that consists of mapping classes θ with $i(\theta) = \mathrm{id}$.

By [11, Theorem 2] or [7, Theorem 2], for any primitive simple hyperbolic element $h \in G$, its φ^*-image $\varphi^*(h)$ is represented by a spin $t_{\alpha_1}^{-1} \circ t_{\alpha_0}$, where the pair $\{\alpha_1, \alpha_0\}$ bounds an a-punctured cylinder on S. Denote

$$\mathcal{F}(S, a) = \{\theta \in \mathrm{Mod}_S^a : \theta \text{ is pseudo-Anosov and } i(\theta) = \mathrm{id}\}. \tag{1.1}$$

By [7, Theorem 2], elements of $\mathcal{F}(S, a)$ are φ^*-images of essential hyperbolic elements g of G, where g is called essential if its axis c_g projects to a geodesic $\varrho(c_g)$ on $\tilde{S}$ that intersects every simple closed geodesic on $\tilde{S}$. For each simple closed geodesic $\tilde{\alpha}$ on $\tilde{S}$, let $t_{\tilde{\alpha}}$ denote the Dehn twist along $\tilde{\alpha}$. Let $\hat{\alpha}$ be a geodesic in $\mathbf{H}$ with $\varrho(\hat{\alpha}) = \tilde{\alpha}$. Let $U_{\hat{\alpha}}$ be a component of $\mathbf{H} - \{\hat{\alpha}\}$. Associated with $U_{\hat{\alpha}}$ there is a lift $\tau_{\hat{\alpha}} : \mathbf{H} \to \mathbf{H}$ of $t_{\tilde{\alpha}}$ that in turn determines a disjoint union of half-planes U_i, $i = 1, 2, \ldots,$ so that the region

$$\Sigma_{\hat{\alpha}} = \mathbf{H} - \bigcup U_i \subset \mathbf{H} - U_{\hat{\alpha}} \tag{1.2}$$

is not empty and $\tau_{\hat{\alpha}}|_{\Sigma_{\hat{\alpha}}} = \mathrm{id}$. We call those half-planes U_i maximal elements of $\tau_{\hat{\alpha}}$.

The lift $\tau_{\hat{\alpha}}$ defines an element $[\tau_{\hat{\alpha}}]$ of $\mathrm{mod}(\tilde{S})$. By [15, Lemma 3.2], we assert that $\varphi^*([\tau_{\hat{\alpha}}]) \in \mathrm{Mod}_S^a$ is represented by the Dehn twist t_α along a simple closed geodesic α on S that is freely homotopic to $\tilde{\alpha}$ on $\tilde{S}$ (as a is filled in). The main result of this paper is the following.

THEOREM 1.1. *Let $g \in G$ be an essential hyperbolic element, and let c_g be the axis of g. Assume that $c_g \cap \Sigma_{\hat{\alpha}} \neq \emptyset$. Then for any integer k, the mapping class $t_\alpha^k \circ \varphi^*(g)$ is pseudo-Anosov.*

REMARK. Theorem 1.1 is no longer valid if we drop the assumption that $c_g \cap \Sigma_\alpha \neq \emptyset$. In Section 8, we show that for any geodesic $\alpha \subset S$, there exists an essential hyperbolic element $g \in G$ with $c_g \cap \Sigma_{\hat{\alpha}} = \emptyset$ such that the product $t_\alpha^k \circ \varphi^*(g)$ fails to be pseudo-Anosov for some integer k.

From Theorem 1.1, we can prove for certain reducible mapping classes $\theta \in \mathrm{Mod}_S^a$ that there exists a simple closed geodesic α on S such that $t_\alpha \circ \theta$ is pseudo-Anosov. To state our next result, we call an element $h \in G$ a simple hyperbolic if it is hyperbolic and its axis c_h projects to a simple closed geodesic $\varrho(c_h)$ on $\tilde{S}$. We will prove the following corollary.

COROLLARY 1.2. *Let $h, g \in G$ be hyperbolic elements such that h is simple and hg is essential. Write $\varphi^*(h) = t_\alpha^{-1} \circ t_{\alpha_0}$. If c_g intersects c_h, then the mapping classes $t_\alpha^{-1} \circ \varphi^*(g)$ and $t_{\alpha_0} \circ \varphi^*(g)$ are both pseudo-Anosov.*

Since for every essential hyperbolic element $g \in G$ there is an essential hyperbolic element g_0 in its conjugacy class in G such that the axis c_{g_0} of g_0 meets $\Sigma_{\hat{\alpha}}$, Theorem 1.1 has another immediate corollary.

COROLLARY 1.3. *For any essential hyperbolic element $g \in G$ and any simple closed geodesic $\alpha \subset S$ that is nontrivial on $\tilde{S}$, there is an element $e \in G$ such that for any integer k, the mapping class $t_\alpha^k \circ \varphi^*(ege^{-1})$ is pseudo-Anosov.*

The argument of Theorem 1.1 can be used to prove that any finite product

$$\prod_i (t_\alpha^{k_i} \circ f),$$

where $k_i \in \mathbf{Z}$, is pseudo-Anosov for $f = \varphi^*(g) \in \mathcal{F}(S, a)$ provided that the axis of g meets $\Sigma_{\hat{\alpha}}$.

Now we briefly discuss a generalization of Theorem 1.1 that is related to a problem posed in [6]. By Fathi's result [4], for any geodesic α and any pseudo-Anosov map f of S, the products $t_\alpha^k \circ f$ are pseudo-Anosov for all but at most seven consecutive integers. The question asks if it is possible to replace t_α^k by a multi-twist $\prod_i t_{\alpha_i}^{k_i}$ for an integer tuple $(k_1, \ldots k_m)$ and a collection $(\alpha_1, \ldots, \alpha_m)$ of m disjoint and independent simple closed geodesics on S, where $1 \leq m \leq 3p - 3 + n$, possibly at the cost of replacing the number seven by an undetermined but universal number N. Let

$$\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\} \subset \tilde{S}, \quad 1 \leq m \leq 3p - 4 + n,$$

be a collection of disjoint simple closed geodesics, and let $[\tau_i] \in \mathrm{mod}(\tilde{S})$, $1 \leq i \leq m$, be the lifts of $t_{\tilde{\alpha}_i}$ such that $\varphi^*([\tau_i]) = t_{\alpha_i}$. Then $\alpha_1, \ldots, \alpha_m$ may not be disjoint. But if the region

$$\Sigma_0 = \mathbf{H} - \{\text{all maximal elements of } \tau_1, \tau_2, \ldots, \tau_m\}$$

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is not empty, then $\alpha_1, \ldots, \alpha_m$ are mutually disjoint. Again, the methods of Theorem 1.1 can be employed to conclude that for any essential element $g \in G$ with $c_g \cap \Sigma_0 \neq \emptyset$, the mapping class

$$\left(\prod_i t_{\alpha_i}^{k_i}\right) \circ f, \tag{1.3}$$

where $k_i \in \mathbf{Z}$, is pseudo-Anosov for $f = \varphi^*(g) \in \mathcal{F}(S, a)$.

This paper is organized as follows. Section 2 is background material that is needed in the proof of Theorem 1.1. In Section 3, we study those reducible mapping classes on S isotopic to a Dehn twist on $\tilde{S}$, and investigate the properties of the corresponding curve systems. In Section 4, we interpret those reducible mapping classes as elements of $\mathrm{mod}(\tilde{S})$, and study their actions on $\mathbf{S}^1$. In Section 5, we prove that if $\theta = t_\alpha^k \circ f$, where $f \in \mathcal{F}(S, a)$, is reducible by a curve system $\{c_1, \ldots, c_s\}$, then all c_i are nontrivial on $\tilde{S}$. In Section 6, we prove Theorem 1.1. Section 7 is devoted to the proof of Corollary 1.2. In Section 8, we present an example to illustrate that Theorem 1.1 will not be true if the condition that $c_g \cap \Sigma_{\hat{\alpha}} \neq \emptyset$ is dropped.

2. Preliminaries

We fix a Riemann surface $\tilde{S}$ as introduced above, and consider all possible pairs $(\tilde{S}_1, w_1)$ where $\tilde{S}_1$ is a Riemann surface of the same type (p, n) and $w_1 : \tilde{S} \to \tilde{S}_1$ is a quasiconformal homeomorphism. Two pairs $(\tilde{S}_1, w_1)$ and $(\tilde{S}_2, w_2)$ are equivalent if the map $w_2 \circ w_1^{-1} : \tilde{S}_1 \to \tilde{S}_2$ is isotopic to a conformal map. The Teichmüller space $T(\tilde{S})$ is defined as the set of pairs $(\tilde{S}_1, w_1)$ modulo the equivalence relation. Note that every pair $(\tilde{S}_1, w_1)$ defines a new conformal structure μ_1 on $\tilde{S}$ via pullbacks. Two conformal structures μ_1 and μ_2 are equivalent if $(\tilde{S}_1, w_1)$ is equivalent to $(\tilde{S}_2, w_2)$. Denote by $[\mu]$ the equivalence class of μ.

Associated with each point $[\mu] \in T(\tilde{S})$ there is a Jordan domain $w^\mu(\mathbf{H})$ depending holomorphically on $[\mu]$, where $w^\mu : \mathbf{C} \to \mathbf{C}$ is a quasiconformal map that satisfies $w^\mu(0) = 0$, $w^\mu(1) = 1$, w^μ is conformal off $\mathbf{H}$, and $\partial_{\bar{z}} w^\mu(z)/\partial_z w^\mu(z) = \mu(z)$ for all $z \in \mathbf{H}$. We then form the Bers fiber space

$$F(\tilde{S}) = \{([\mu], z) : [\mu] \in T(\tilde{S}) \text{ and } z \in w^\mu(\mathbf{H})\}.$$

The Bers isomorphism theorem [2, Theorem 9] states that there is an isomorphism $\varphi : F(\tilde{S}) \to T(S)$.

By definition, $\mathrm{Mod}_{\tilde{S}}$ consists of isotopy classes of self-maps of $\tilde{S}$. Let $\zeta \in \mathrm{Mod}_{\tilde{S}}$ be induced by a self-map w of $\tilde{S}$. The map w can be lifted to an automorphism $\omega : \mathbf{H} \to \mathbf{H}$ under the universal covering $\varrho : \mathbf{H} \to \tilde{S}$. We call $\omega, \omega' : \mathbf{H} \to \mathbf{H}$ equivalent if $\omega g \omega^{-1} = \omega' g \omega'^{-1}$ for every element $g \in G$. The equivalence class of ω is denoted by $[\omega]$. The group $\mathrm{mod}(\tilde{S})$ is a collection of $[\omega]$ for all maps $w : \tilde{S} \to \tilde{S}$. The Bers isomorphism $\varphi : F(\tilde{S}) \to T(S)$ induces an isomorphism $\varphi^* : \mathrm{mod}(\tilde{S}) \to \mathrm{Mod}_S^a$ defined by conjugation. Since the covering group G is regarded as a normal subgroup of $\mathrm{mod}(\tilde{S})$, the image group $\varphi^*(G)$ is a normal subgroup of Mod_S^a consisting of elements θ such that $i(\theta)=$id. Let $[\omega] \in \mathrm{mod}(\tilde{S})$ be such that $\varphi^*([\omega]) = \theta$. Then $[\omega] \in G$. There are three cases to consider.

Case 1. $[\omega] \in G$ is a simple hyperbolic element. Let c be its axis and write $\tilde{c} = \varrho(c)$. By [11, Theorem 2] or [7, Theorem 2], we see that $\varphi^*([\omega])$ can be represented in the form $t_\alpha^{-k} \circ t_{\alpha_0}^k$ for an integer k, where $\{\alpha, \alpha_0\}$ bounds an a-punctured cylinder on S. Both α and α_0 are simple closed geodesics homotopic to $\tilde{c}$ on $\tilde{S}$ as a is filled in.

Case 2. $[\omega] \in G$ is parabolic. In this case, the mapping class $\varphi^*([\omega])$ is represented by an ordinary power of a Dehn twist along a curve c, where c bounds a twice punctured disk on S enclosing the puncture a and another puncture of $\tilde{S}$ corresponding to the conjugacy class of the parabolic element.

Case 3. $[\omega] \in G$ is essential hyperbolic. In this case, the mapping class $\varphi^*([\omega])$ is pseudo-Anosov.

We proceed to investigate some special elements $[\omega]$ in $\mathrm{mod}(\tilde{S}) - G$ as well as its image in Mod_S^a under the Bers isomorphism $\varphi^* : \mathrm{mod}(\tilde{S}) \to \mathrm{Mod}_S^a$. Let $\tilde{\alpha}$ be a simple closed geodesic on $\tilde{S}$ and $\hat{\alpha} \in \mathbf{H}$ a geodesic such that $\tilde{\alpha} = \varrho(\hat{\alpha})$. Let U and U^* be the components of $\mathbf{H} - \hat{\alpha}$. As mentioned earlier, the Dehn twist $t_{\tilde{\alpha}}$ can be lifted to a quasiconformal homeomorphism $\tau : \mathbf{H} \to \mathbf{H}$ with respect to U in the following way. Let $h \in G$ be a primitive simple hyperbolic element such that $h(U) = U$. We take an earthquake h-shift on U and leave U^* fixed. We then define τ via G-invariance.

Obviously, the map τ gives rise to a collection $\mathcal{U}$ of layered half-planes in $\mathbf{H}$ in a partial order defined by inclusion. There are infinitely many disjoint maximal elements of $\mathcal{U}$ such that the complement Σ of all maximal elements in $\mathbf{H}$ is nonempty and simply connected. The map τ keeps each maximal element of $\mathcal{U}$ invariant and the restriction $\tau|_\Sigma$ is the identity.

Let $\mathbf{Q} \subset \mathbf{S}^1$ denote the dense subset consisting of points covered by finitely many elements of $\mathcal{U}$. Choose $z \in \mathbf{Q}$ and let $U = U_0 \supset U_1 \supset \cdots \supset U_m$, $U_i \in \mathcal{U}$, cover z. Let h_i, $i = 0, 1, \ldots m$, denote the primitive simple hyperbolic elements of G that keep U_i invariant and take the same orientation as $h_0 = h$. Then $\tau(z)$ is defined as

$$\tau(z) = h_0 h_1 \cdots h_m(z). \tag{2.1}$$

For a point $z \in \mathbf{S}^1$ not covered by any elements of $\mathcal{U}$, we have $\tau_i(z) = z$. Now for any other point $z \in \mathbf{S}^1 - \mathbf{Q}$, we choose a sequence $\{z_j\} \subset \mathbf{Q}$ with $z_j \to z$. We have $\tau(z) = \lim_{j\to\infty} \tau(z_j)$.

The equivalence class of τ determines an element $[\tau]$ of $\mathrm{mod}(\tilde{S})$. By [15, Lemma 3.3], the mapping class $\varphi^*([\tau])$ is represented by a Dehn twist t_α for a simple closed geodesic $\alpha \subset S$. For this reason, in the rest of this paper we use the symbols τ_α, $\mathcal{U}_\alpha$ and Σ_α to denote τ, $\mathcal{U}$ and Σ, respectively. Observe that if τ_α is a lift of $t_{\tilde{\alpha}}$ with respect to U, then $h^{-1}\tau_\alpha$ is also a lift of $t_{\tilde{\alpha}}$ with respect to U^*. But $\varphi^*([h^{-1}\tau_\alpha])$ is represented by t_{α_0}, where α_0 together with α bounds an a-punctured cylinder on S.

By [15, Lemma 3.3] again, for every simple closed geodesic $\alpha \subset S$, there exists a lift τ_α of $t_{\tilde{\alpha}}$ such that $\varphi^*([\tau_\alpha]) = t_\alpha$.

Let $[\omega] \in \mathrm{mod}(\tilde{S}) - G$ be a lift of $t_{\tilde{\alpha}}$. Then $[\omega]$ is of the form $[\tau_\alpha]h$ for some $h \in G$. In this case, the mapping class $\varphi^*([\omega])$ is the product of t_α and $\varphi^*(h)$. Suppose that G has a parabolic fixed point x, and that $T \in G$ is the parabolic element so that $T(x) = x$. By [15, Lemma 3.1], we have $x \in \mathbf{Q}$. Hence there are only finitely many elements of $\mathcal{U}_\alpha$ that cover x. It follows that every parabolic fixed point x of G is associated with a positive integer $\epsilon(\tau_\alpha, x)$ that is the number of elements of $\mathcal{U}_\alpha$ containing x. It is evident that $\epsilon(\tau_\alpha, x) = \epsilon(\tau_\alpha, \tau_\alpha(x)) \neq 0$ if x is covered by a maximal element of $\mathcal{U}_\alpha$, and $\epsilon(\tau_\alpha, x) = 0$ if and only if x lies outside of all maximal elements of $\mathcal{U}_\alpha$. In the latter case, the parabolic element T commutes with τ_α and the geodesic α on S determined by $t_\alpha = \varphi^*([\tau_\alpha])$ is disjoint from the boundary of the twice punctured disk determined by $\varphi^*(T)$ [11, Theorem 2].

3. Reducible mapping classes and curves

Let $\theta \in \mathcal{F}(S, a)$. Let $g \in G$ be such that $\varphi^*(g) = \theta$. Let $\alpha \subset S$ be a simple closed geodesic so that α is also nontrivial on $\tilde{S}$. Then α is not a geodesic on $\tilde{S}$ when a is filled in. In what follows, we use $\tilde{\alpha}$ to denote the geodesic homotopic to α on $\tilde{S}$. Assume that $\theta = t_\alpha^k \circ \varphi^*(g)$ is not pseudo-Anosov. Then there is a system

$$\mathcal{C} = \{c_1, \ldots, c_s\}, \tag{3.1}$$

where $s \geq 1$, of disjoint simple closed geodesics on S that is invariant under a suitable representative of θ. We assume that every curve in $\mathcal{C}$ is also nontrivial on $\tilde{S}$. The case in which $\mathcal{C}$ contains a curve c that is trivial on $\tilde{S}$ will be handled in Section 5. We can write

$$\theta(\mathcal{C}) = \mathcal{C}.$$

Let Λ be the set of simple closed geodesics c on S that project to nontrivial simple closed geodesics $\tilde{c}$ so that $\tilde{c} = \tilde{\alpha}$ or $\tilde{c}$ is disjoint from $\tilde{\alpha}$. Let Λ_1 be the subset of Λ consisting of geodesics c such that near the puncture a, the geodesics c and α bound a bigon B enclosing a. Let $\Lambda_2 = \Lambda - \Lambda_1$. Then $\Lambda_1 \cup \Lambda_2 = \Lambda$ and Λ_2 consist of geodesics c on S that are nontrivial on $\tilde{S}$ and are equal to or disjoint from α.

LEMMA 3.1. $\mathcal{C} \subset \Lambda$.

PROOF. By taking a suitable power of c_i, we may assume that $\theta(c_i) = c_i$ for every $i = 1, \ldots, s$. Assume that there is a $c_1 \in \mathcal{C}$, say, so that $\tilde{c}_1 \subset \tilde{S}$ is nontrivial and intersects $\tilde{\alpha}$.

Since $i(\theta) = t_{\tilde{\alpha}}^k$, the Dehn twist $t_{\tilde{\alpha}}^k$ keeps $\tilde{c}_1$ invariant. By hypothesis, the curve $\tilde{c}_1 \subset \tilde{S}$ intersects $\tilde{\alpha}$, which means that the image loop $t_{\tilde{\alpha}}^k(\tilde{c}_1)$ intersects $\tilde{c}_1$. It follows that $t_{\tilde{\alpha}}^k$ sends $\tilde{c}_1$ to a different homotopy class. This is a contradiction. □

Note that θ may not keep each element of $\mathcal{C}$ invariant. Let $\mathcal{C}_0$ be the subset of $\mathcal{C}$ consisting of curves in $\mathcal{C}$ with $\theta(c) \neq c$.

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LEMMA 3.2. *$\mathcal{C}_0$ contains at most two curves. In other words, the mapping class θ^2 keeps each element of $\mathcal{C}$ invariant. Further, if $\{c_1, c_2\} = \mathcal{C}_0$, then $\{c_1, c_2\}$ bounds an a-punctured cylinder on S.*

PROOF. Suppose that $\mathcal{C}_0$ consists of at least three curves c_1, c_2 and c_3. Since there are at most two disjoint curves c and c' on S so that $\tilde{c} = \tilde{c}'$, we may assume that $\theta(c_1) = c_2$ and $\{c_1, c_2\}$ does not bound an a-punctured cylinder on S. That is, the geodesic $\tilde{c}_1$ is disjoint from $\tilde{c}_2$. Since $\theta(c_1) = c_2$, by filling in the puncture a, we obtain

$$i(\theta)(\tilde{c}_1) = \tilde{c}_2. \tag{3.2}$$

On the other hand, we recall that $i(\theta) = t_{\tilde{\alpha}}$. From Lemma 3.1, $\mathcal{C}_0 \subset \Lambda$. We see that both $\tilde{c}_1$ and $\tilde{c}_2$ are disjoint from $\tilde{\alpha}$. This implies that $t_{\tilde{\alpha}}(\tilde{c}_1) = \tilde{c}_1$ and $t_{\tilde{\alpha}}(\tilde{c}_2) = \tilde{c}_2$. This contradicts (3.2). □

LEMMA 3.3. *Suppose $\mathcal{C} \cap \Lambda_1$ is empty. Then θ is not reduced by the system $\mathcal{C}$.*

PROOF. By Lemma 3.1, we have $\mathcal{C} \subset \Lambda$. So if $\mathcal{C} \cap \Lambda_1$ is empty, then every curve in $\mathcal{C}$ must be in Λ_2. Therefore t_α commutes with t_j for $1 \leq j \leq s$, where for simplicity $t_j = t_{c_j}$. Suppose that $\theta = t_\alpha^k \circ \varphi^*(g)$ is reduced by $\mathcal{C}$. Then

$$(t_1 \circ \cdots \circ t_s) \circ (t_\alpha^k \circ \varphi^*(g)) = (t_\alpha^k \circ \varphi^*(g)) \circ (t_1 \circ \cdots \circ t_s).$$

Since t_α commutes with each t_j for $1 \leq j \leq s$, we obtain

$$(t_1 \circ \cdots \circ t_s) \circ \varphi^*(g) = \varphi^*(g) \circ (t_1 \circ \cdots \circ t_s). \tag{3.3}$$

Recall that $g \in G$ is essential. We see that (3.3) cannot hold since it says that $\varphi^*(g)$ keeps $c_1, \ldots, c_s$ invariant. □

It follows from Lemma 3.3 that $\mathcal{C} \cap \Lambda_1 \neq \emptyset$. Consequently, we can choose a curve $c \in \mathcal{C} \cap \Lambda_1$. By Lemma 3.2, we can take a square of θ if necessary, and may assume that $\theta(c) = c$. Let $\tau_c : \mathbf{H} \to \mathbf{H}$ be the lift of $t_{\tilde{c}}$ so that $\varphi^*([\tau_c])$ is represented by t_c. We have the following result.

LEMMA 3.4. *The pair (τ_α, τ_c) satisfies the following properties.*

(1) *The geodesic boundary ∂W_0 of any maximal element W_0 of $\mathcal{U}_c$ is disjoint from the geodesic boundary ∂U_0 of any maximal element U_0 of $\mathcal{U}_\alpha$.*
(2) *There exist maximal elements U and W of $\mathcal{U}_\alpha$ and $\mathcal{U}_c$, respectively, such that $U \cap W \neq \emptyset$ and $U \cup W = \mathbf{H}$.*
(3) *For any maximal element $U_0 \neq U$ of $\mathcal{U}_\alpha$, we have $U_0 \subset W$.*

PROOF. Since $c \in \mathcal{C} \cap \Lambda_1 \subset \Lambda$, the geodesic $\tilde{c}$ is disjoint from $\tilde{\alpha}$. So every geodesic in the set $\{\varrho^{-1}(\tilde{c})\}$ of preimages of $\tilde{c}$ is disjoint from any geodesic in the set $\{\varrho^{-1}(\tilde{\alpha})\}$ of preimages of $\tilde{\alpha}$. But the geodesic boundary ∂U is one of the elements in $\{\varrho^{-1}(\tilde{\alpha})\}$ and ∂W is one of the elements in $\{\varrho^{-1}(\tilde{c})\}$. This proves (1).

To prove (2) of the lemma, we suppose that there is no such pair (U, W). That is, for any maximal element $U \in \mathcal{U}_\alpha$ and any maximal element $W \in \mathcal{U}_c$, either $U \subset W$, or $W \subset U$, or U, W are disjoint. Suppose that $U \subset W$. For any hyperbolic element $h \in G$ whose repelling fixed point is contained in $U \cap \mathbf{S}^1$ and whose attracting fixed point lies outside of W, by construction, the region $h(\mathbf{H} - U)$ is contained in a maximal element U' of $\mathcal{U}_\alpha$. By assumption, the half-plane U' is disjoint from W. It follows that $\Sigma_\alpha \cap \Sigma_c \neq \emptyset$ (where Σ_α and Σ_c are defined as in (2)) and the boundary components of $\Sigma_\alpha \cap \Sigma_c$ are either ∂U for some $U \in \mathcal{U}_\alpha$, or ∂W for some $W \in \mathcal{U}_c$. This implies that $[\tau_c]$ commutes with $[\tau_\alpha]$. Via the Bers isomorphism $\varphi^* : \mathrm{mod}(\tilde{S}) \to \mathrm{Mod}_S^a$, we see that t_c commutes with t_α. This implies that c and α are disjoint, which contradicts the fact that $c \in \mathcal{C} \cap \Lambda_1$.

(3) is obvious. If $U_0 \neq U$, then U_0 and U are disjoint. Thus $U_0 \subset \mathbf{H} - U$. From (2), we have $U_0 \subset W$. Lemma 3.4 is proved. □

Suppose that c_g intersects Σ_α. Then there is a maximal element $U \in \mathcal{U}_\alpha$ such that c_g intersects ∂U. Also assume that U contains the repelling (but not attracting) fixed point of g. Let $U_0 \in \mathcal{U}_\alpha$ be another maximal element that contains $g(\mathbf{H} - U)$. Then U_0 must be disjoint from U. Under the circumstances, a slight modification of the argument of Lemma 3.4 leads to the following result.

LEMMA 3.5. *Let U, $U_0 \in \mathcal{U}_\alpha$ be maximal elements defined as above. Then there exists a maximal element $W \in \mathcal{U}_c$ such that either one of (U, W) and (U_0, W) satisfies condition (2) of Lemma 3.4, or both U and U_0 are contained in W.*

PROOF. Choose a maximal element $W \in \mathcal{U}_c$ so that W is not disjoint from U. If (U, W) satisfies condition (2) of Lemma 3.4, we are done. Otherwise, either $U \subset W$, or $W \subset U$.

Suppose that $U \subset W$. Let $U_0 \in \mathcal{U}_\alpha$ be a maximal element that includes $g(\mathbf{H} - U)$. Assume that U_0 is not contained in W. If (U_0, W) satisfies condition (2) of Lemma 3.4, we are done. Otherwise, we see that $\Sigma_\alpha \cap \Sigma_c \neq \emptyset$. Now the argument of Lemma 3.4(2) can be applied to show that τ_c commutes with τ_α. But this would contradict the fact that $c \in \mathcal{C} \cap \Lambda_1$.

If $W \subset U$, then we consider the set $\mathcal{M}$ of all maximal elements of $\mathcal{U}_c$ that contains $h(\mathbf{H} - W)$, where h runs over all hyperbolic elements whose attracting fixed point lies outside of W and whose repelling fixed point lies in $W \cap \mathbf{S}^1$. If $\mathcal{M}$ contains an element W' such that (U, W') satisfies condition (2) of Lemma 3.4, we are done. Otherwise, the map τ_c commutes with τ_α, in contradiction to $c \in \mathcal{C} \cap \Lambda_1$. □

4. Reducible mapping classes interpreted as elements of $\mathrm{mod}(\tilde{S})$

In this section we discuss certain reducible mapping classes by virtue of elements of $\mathrm{mod}(\tilde{S})$. Let $c \subset S$ be a simple closed geodesic. Let $\chi \in \mathrm{Mod}_S^a$ be a reducible mapping class by a curve system containing c. Let $[\omega] \in \mathrm{mod}(\tilde{S})$ be an element such that $\varphi^*([\omega]) = \chi$.

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LEMMA 4.1. *Suppose that $\tilde{c}$ is nontrivial and that $\chi(c) = c$. Let $[\tau_c] \in \mathrm{mod}(\tilde{S})$ be the element such that $\varphi^*([\tau_c])$ is represented by t_c. Then there exists a map in the equivalence class $[\omega]$ (which is denoted by ω also) such that ω keeps the set of maximal elements of $\mathcal{U}_c$ invariant.*

PROOF. By assumption, we may choose a representative σ for χ so that $\sigma(c) = c$. It is then obvious that σ commutes with the Dehn twist t_c. That is,

$$\sigma \circ t_c \circ \sigma^{-1} = t_c. \tag{4.1}$$

From (4.1) and the Bers isomorphism φ^*, we obtain

$$[\omega][\tau_c][\omega]^{-1} = [\tau_c]. \tag{4.2}$$

Let W be any maximal element of $\mathcal{U}_c$. Choose a representative ω' of $[\omega]$. Obviously, the map τ_c keeps $W \cap \mathbf{S}^1$ invariant, and no points in the interior of $W \cap \mathbf{S}^1$ in $\mathbf{S}^1$ are fixed by τ_c. Then $\omega'\tau_c\omega'^{-1}|_{\mathbf{S}^1}$ sends $\omega'(W) \cap \mathbf{S}^1$ to itself and does not fix any point in $\omega'(W) \cap \mathbf{S}^1$. From (4.2), we see that $\tau_c|_{\mathbf{S}^1}$ sends $\omega'(W) \cap \mathbf{S}^1$ to itself and does not fix any point in $\omega'(W) \cap \mathbf{S}^1$. This implies that there is a representative ω of $[\omega]$ such that $\omega(W)$ is also a maximal element of $\mathcal{U}_c$. □

The following result was proved in [16].

LEMMA 4.2. *If $\tilde{c}$ is trivial, then $\chi(c) = c$ and every representative ω of $[\omega]$ fixes a parabolic fixed point of G.*

For every maximal element $W \in \mathcal{U}_c$, we write $\overline{W} = W \cup \partial W$ and $W^* = \mathbf{H} - W$. Also, the complement of an arc Γ in $\mathbf{S}^1$ is denoted by Γ^c.

LEMMA 4.3. *Let W be a maximal element of $\mathcal{U}_c$. Let $[\omega] \in \mathrm{mod}(\tilde{S})$. If the intersection $\omega(W^*) \cap W^* = \emptyset$ for a representative ω of $[\omega]$, then for any representative ω_0 of $[\omega]$, the region $\omega_0(W)$ is not a maximal element of $\mathcal{U}_c$.*

PROOF. First, observe that ω is a quasiconformal homeomorphism of $\mathbf{H}$. For any representative ω_0 of $[\omega]$ we must have $\omega|_{\mathbf{S}^1} = \omega_0|_{\mathbf{S}^1}$. The hypothesis implies that $(\omega_0(W^*) \cap \mathbf{S}^1) \cap (W^* \cap \mathbf{S}^1) = \emptyset$. It follows that

$$\begin{aligned} \mathbf{S}^1 &= ((\omega_0(W^*) \cap \mathbf{S}^1) \cap (W^* \cap \mathbf{S}^1))^c = (\omega_0(W^*) \cap \mathbf{S}^1)^c \cup (W^* \cap \mathbf{S}^1)^c \\ &= (\overline{\omega_0(W)} \cap \mathbf{S}^1) \cup (\overline{W} \cap \mathbf{S}^1) = (\overline{\omega_0(W)} \cup \overline{W}) \cap \mathbf{S}^1. \end{aligned} \tag{4.3}$$

Hence $\overline{\omega_0(W)}$ and $\overline{W}$ cannot be disjoint. If $\overline{\omega_0(W)} = \overline{W}$, then $\overline{\omega_0(W)} \cup \overline{W} = \overline{W}$. From (4.3), we obtain $\mathbf{S}^1 = \overline{W} \cup \mathbf{S}^1$. But $\overline{W}$ is a closed half-plane. This is absurd. We thus conclude that $\omega_0(W)$ and W cannot be both maximal elements of $\mathcal{U}_c$. Lemma 4.3 is proved. □

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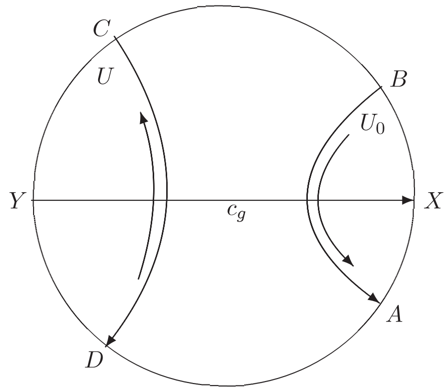

FIGURE 1. The axis c_g and the sets U and U_0.

5. Boundaries of twice punctured disks on S that enclose a fixed puncture

In this section we handle the case in which the curve system $\mathcal{C}$ defined as in (3.1) contains a curve c that is trivial on $\tilde{S}$. We begin with the following lemma.

LEMMA 5.1. *Suppose that c_g crosses Σ_α. Then as a circle homeomorphism, the map $\tau_\alpha^k g|_{\mathbf{S}^1}$ does not fix any parabolic fixed point of G.*

PROOF. By hypothesis, the axis c_g meets Σ_α. Let U be a maximal element of $\mathcal{U}_\alpha$ such that c_g intersects ∂U. Assume without loss of generality that U covers the repelling fixed point Y of g. There exists another maximal element U_0 of $\mathcal{U}_\alpha$ that contains $g(\mathbf{H} - U)$. Then, of course, the half-plane U_0 covers the attracting fixed point X of g. Write $\{A, B\} = \partial U_0 \cap \mathbf{S}^1$ and $\{C, D\} = \partial U \cap \mathbf{S}^1$, as labeled in Figure 1. Let (AB) denote the circular arc connecting A and B on $\mathbf{S}^1$ without passing through any other labeled points.

Clearly, on $(YC) \cup (CB) \cup (BX)$ the action of g is consistent with the action of τ_α. Hence there are no fixed points of $\tau_\alpha^k g$ there. Let $z \in (DA)$. If z is not covered by any maximal element of $\mathcal{U}_\alpha$, then $\tau_\alpha^{-k}(z) = z$ and thus $g^{-1}\tau_\alpha^{-k}(z) = g^{-1}(z) \neq z$. This implies that $\tau_\alpha^k g(z) \neq z$. If z is covered by a maximal element V of $\mathcal{U}_\alpha$, then V is disjoint from U and U_0, and $\tau_\alpha^{-k}(z) \in V \cap \mathbf{S}^1$. Observe that ∂V also projects to $\tilde{\alpha}$, and $g^{-1}(V) \cap V = \emptyset$. It follows that $g^{-1}\tau_\alpha^k(z) \neq z$. We conclude that there are no fixed points of $\tau_\alpha^k g$ on (DA).

We must show that $\tau_\alpha^k g$ has no fixed points on $(AX) \cup (DY)$. Let $z \in (AX)$. By [15, Lemma 3.1], there exist a finite number of elements $U_0, U_1, \ldots, U_m$ in $\mathcal{U}_\alpha$ such that $U_0 \supset U_1 \supset \cdots \supset U_m \ni z$. This tells us that

$$\epsilon(\tau_\alpha, z) = m + 1. \tag{5.1}$$

Now $g(z)$ is covered by $U_0 \supset g(U_0) \supset g(U_1) \supset \cdots \supset g(U_m) \ni g(z)$. By invariance, all $g(U_i) \in \mathcal{U}_\alpha$. Since $U_0 \in \mathcal{U}_\alpha$, we see that $\epsilon(\tau_\alpha, g(z)) \geq m + 2$. But $g(z) \in (AX)$,

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so $\epsilon(\tau_\alpha, \tau_\alpha^k g(z)) = \epsilon(\tau_\alpha, g(z))$. Hence

$$\epsilon(\tau_\alpha, \tau_\alpha^k g(z)) = \epsilon(\tau_\alpha, g(z)) \geq m + 2. \tag{5.2}$$

Combining (5.1) and (5.2) leads to $\tau_\alpha^k g(z) \neq z$.

Similarly, by considering the inverse of the map $\tau_\alpha^k g$, one can prove that there are no fixed points of $\tau_\alpha^k g$ on (DY). The details are omitted.

Finally, we notice that any labeled point in $\{A, B, C, D, X, Y\}$ is a fixed point of a hyperbolic element of G; it cannot be a fixed point of any parabolic element of G. This proves Lemma 5.1. □

Assume that for some integer k, the mapping class $\theta = \varphi^*([\tau_\alpha^k g])$ is reducible by a curve system (3.1). Let f be a representative of θ such that

$$f(\{c_1, \ldots, c_s\}) = \{c_1, \ldots, c_s\}.$$

PROPOSITION 5.2. *The system $\mathcal{C}$ does not contain any curve c that is trivial on $\tilde{S}$.*

PROOF. If $\tilde{S}$ is compact, then there is nothing to prove. We assume henceforth that $\tilde{S}$ contains at least one puncture.

Suppose on the contrary that $\mathcal{C}$ contains a curve c that is trivial on $\tilde{S}$. Then c is the boundary of a twice punctured disk Δ enclosing a and a puncture of $\tilde{S}$. We observe that any two punctured disks Δ_1 and Δ_2, if both enclose the puncture a, must have an overlap. This shows that $\partial\Delta_1$ intersects $\partial\Delta_2$. On the other hand, by definition, curves in $\mathcal{C}$ are mutually disjoint. We see that there is exactly one curve c in $\mathcal{C}$ such that $\tilde{c}$ is trivial.

Now f is a self-map of S with $f(a) = a$, and the region $f(\Delta)$ must also be a twice punctured disk enclosing a. Then by the above argument, we obtain $\partial\Delta \cap \partial f(\Delta) \neq \emptyset$. Hence if $f(c) \neq c$, then $f(c) \notin \mathcal{C}$. So we must have $f(c) = c$.

Choose $[\omega] \in \mathrm{mod}(\tilde{S})$ so that $\varphi^*([\omega]) = \theta$ is represented by f. By Lemma 4.2, any representative ω_0 of $[\omega]$ fixes a parabolic fixed point x of G. On the other hand, we observe that $\varphi^*([\tau_\alpha^k g]) = t_\alpha^k \circ \varphi^*(g) = \theta$. We see that $[\omega] = [\tau_\alpha^k g] = [\tau_\alpha^k]g$, and thus

$$\omega|_{\mathbf{S}^1} = \tau_\alpha^k g|_{\mathbf{S}^1}. \tag{5.3}$$

From Lemma 5.1, we conclude that $\tau_\alpha^k g|_{\mathbf{S}^1}$ does not fix any parabolic fixed point of G. It follows from (5.3) that $\omega|_{\mathbf{S}^1}$ does not fix any parabolic fixed point of G, which leads to a contradiction. □

6. Proof of Theorem 1.1

By hypothesis, $c_g \cap \Sigma_\alpha \neq \emptyset$. Let $U \in \mathcal{U}_\alpha$ be the maximal element such that c_g intersects ∂U. Suppose that $\theta = \varphi^*([\tau_\alpha^k g])$ is reduced by (3.1). By Proposition 5.2, the curve system $\mathcal{C}$ does not contain any curve c with $\tilde{c}$ trivial on $\tilde{S}$. Choose $c \in \mathcal{C}$. Then Lemma 3.2 leads to $\theta^2(c) = c$. By Lemma 3.3, we can assume that $c \in \Lambda_1$. Thus, by

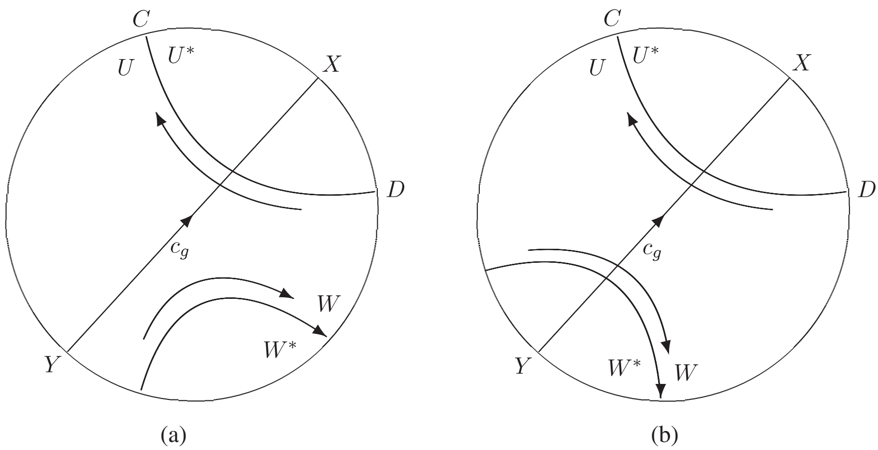

FIGURE 2. Two of the cases in the proof of Theorem 1.1.

Lemma 3.5, there exists a maximal element $W \in \mathcal{U}_c$ such that either $\partial U \cap \partial W = \emptyset$, $U \cap W \neq \emptyset$, and $U \cup W = \mathbf{H}$, or both U and U_0 are contained in W, where U_0 is as in Lemma 3.5. If the former possibility occurs, we let Y denote the intersection $c_g \cap (U \cap \mathbf{S}^1)$ and X the intersection $c_g \cap ((\mathbf{H} - U) \cap \mathbf{S}^1)$. There are four cases in total:

(i) Y is the repelling fixed point of g and ∂W is disjoint from c_g;
(ii) Y is the repelling fixed point of g and ∂W intersects c_g;
(iii) Y is the attracting fixed point of g and ∂W is disjoint from c_g; and
(iv) Y is the attracting fixed point of g and ∂W intersects c_g.

We only discuss the first two cases, which are drawn in Figures 2(a) and (b). The other two cases can be treated by considering the inverse of $\tau_\alpha^k g$.

Case (i). The geodesic boundary ∂W is disjoint from c_g. In this case, the complement W^* of W is in U (Figure 2(a)). Now W^* is disjoint from c_g to $W \in \mathcal{U}_c$ is maximal. Since $\partial W = \partial W^*$ projects to $\tilde{c}$ which is nontrivial on $\tilde{S}$, we see that $g(W^*) \cap W^* = \emptyset$. From Lemma 3.1, the geodesic $\tilde{c}$ is disjoint from $\tilde{\alpha}$. Since ∂U projects to $\tilde{\alpha}$, either $g(W^*) \subset U$ or $g(W^*) \subset U^*$. If $g(W^*) \subset U^*$, then since τ_α^k keeps U^* invariant, we see that $\tau_\alpha^k g(W^*) \subset U^*$ and that $(\tau_\alpha^k g)^2(W^*) \subset U^*$. Hence $(\tau_\alpha^k g)^2(W^*) \cap W^* = \emptyset$. By Lemma 4.3, we conclude that $(\tau_\alpha^k g)^2(W)$ is not a maximal element of $\mathcal{U}_c$.

Assume that $g(W^*) \subset U$. Note that the Euclidean diameter of W^* is positive and that ∂W^* projects to $\tilde{c}$, and all boundaries of elements of $\mathcal{U}_\alpha$ project to $\tilde{\alpha}$. Since $\tilde{c}$ is disjoint from $\tilde{\alpha}$, all boundaries of elements of $\mathcal{U}_\alpha$ are disjoint from ∂W^*. We see that there are only finitely many elements

$$U_0 = U, U_1, \ldots, U_r$$

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of $\mathcal{U}_\alpha$ so that

$$W^* \subset U_r \subset \cdots \subset U_1 \subset U. \tag{6.1}$$

Let $\epsilon_1(\tau_\alpha, W^*)$ denote the number of elements of $\mathcal{U}_\alpha$ that cover W^*. By definition of τ_α, we know that for $i = 0, \ldots, r$, the half-planes $\tau_\alpha^{-k}(U_i)$ are elements of $\mathcal{U}_\alpha$ and cover $\tau_\alpha^{-k}(W^*)$. It follows that

$$\epsilon_1(\tau_\alpha, \tau_\alpha^{-k}(W^*)) \geq \epsilon_1(\tau_\alpha, W^*). \tag{6.2}$$

Notice that U and all $g^{-1}\tau_\alpha^{-k}(U_i)$, for $i = 0, \ldots, r$, are elements of $\mathcal{U}_\alpha$. Since g is a Möbius transformation, from (6.1) we obtain

$$g^{-1}\tau_\alpha^{-k}(W^*) \subset g^{-1}\tau_\alpha^{-k}(U_r) \subset \cdots \subset g^{-1}\tau_\alpha^{-k}(U_0) \subset U. \tag{6.3}$$

From (6.2) along with (6.3), we assert that

$$\epsilon_1(\tau_\alpha, g^{-1}\tau_\alpha^{-k}(W^*)) > \epsilon_1(\tau_\alpha, \tau_\alpha^{-k}(W^*)) \geq \epsilon_1(\tau_\alpha, W^*). \tag{6.4}$$

In particular, (6.4) yields that $\epsilon_1(\tau_\alpha, g^{-1}\tau_\alpha^{-k}(W^*) \neq \epsilon_1(\tau_\alpha, W^*))$. Thus we must have $g^{-1}\tau_\alpha^{-k}(W^*) \neq W^*$. A similar argument yields that

$$\epsilon_1(\tau_\alpha, (g^{-1}\tau_\alpha^{-k})^2(W^*)) > \epsilon_1(\tau_\alpha, W^*). \tag{6.5}$$

Thus $(g^{-1}\tau_\alpha^{-k})^2(W^*) \neq W^*$.

If $(g^{-1}\tau_\alpha^{-k})^2(W^*) \cap W^* = \emptyset$, by Lemma 4.3, the half-plane $(g^{-1}\tau_\alpha^{-k})^2(W)$ is not a maximal element of $\mathcal{U}_\alpha$. If $(g^{-1}\tau_\alpha^{-k})^2(W^*) \supset W^*$, then (6.5) is impossible. If $(g^{-1}\tau_\alpha^{-k})^2(W^*) \subset W^*$, then $(g^{-1}\tau_\alpha^{-k})^2(W) \supset W$. This says that if $(g^{-1}\tau_\alpha^{-k})^2(W)$ were a maximal element of $\mathcal{U}_c$, then W would not be a maximal element of $\mathcal{U}_c$. It follows that $(g^{-1}\tau_\alpha^{-k})^2(W)$ is not maximal. But this contradicts Lemma 4.1.

Case (ii). The geodesic boundary ∂W intersects c_g. In this case, $W^* \subset U$; see Figure 2(b). Since Y is the attracting fixed point of g that is covered by W^*, we have $W^* \subset g(W^*)$.

If $U \subseteq g(W^*)$, then $g(W) \subseteq U^* \subset W$. Since τ_α^k keeps U^* invariant, we have $\tau_\alpha^k g(W) \subseteq U^* \subset W$. Since X is the attracting fixed point of g that is covered by U^*, we must have $g\tau_\alpha^k g(W) \subseteq U^* \subset W$. Hence, $\tau_\alpha^k g \tau_\alpha^k g(W) = (\tau_\alpha^k g)^2(W) \subseteq U^* \subset W$, which says that $(\tau_\alpha^k g)^2(W)$ is not a maximal element of $\mathcal{U}_c$.

If $g(W^*) \subset U$, then since U is a maximal element of $\mathcal{U}_\alpha$, the map τ_α^k keeps U invariant. Hence $\tau_\alpha^k g(W^*) \subset U$. Clearly, $\tau_\alpha^k g(W^*) \cap g(W^*) = \emptyset$. Now we can also easily check that $(\tau_\alpha^k g)^2(W^*) \cap g(W^*) = \emptyset$. By Lemma 4.3, we see that $(\tau_\alpha^k g)^2(W)$ is not a maximal element of $\mathcal{U}_c$.

Finally, if W contains both U and U_0, then $c_g \subset W$ and W^* is disjoint from c_g and U. In this case, we use the same argument as in Case (i) above to conclude that $(\tau_\alpha^k g)^2(W)$ is not a maximal element of $\mathcal{U}_c$.

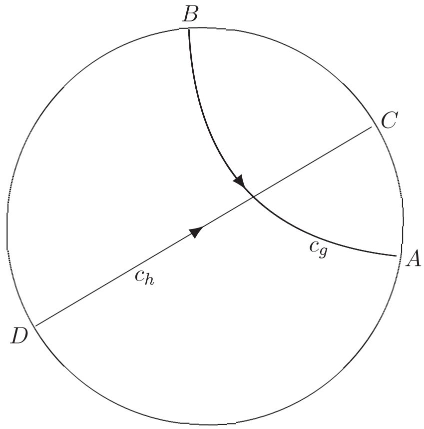

FIGURE 3. Hyperbolic elements whose axes intersect.

7. Proof of Corollary 1.2

We first prove the following result.

LEMMA 7.1. *Let $g, h \in G$ be hyperbolic elements. Assume that their axes c_g and c_h intersect. Then for all integers r and s, the elements $h^r g^s \in G$ are also hyperbolic, and their axes intersect both c_h and c_g.*

PROOF. Let $\{A, B\}$ and $\{C, D\}$ denote the fixed points of g and h, respectively, where A and C are the attracting fixed points and B and D are the repelling fixed points; see Figure 3.

We assume that both r and s are positive. Consider the motion of $\xi = h^r g^s$ on $\mathbf{S}^1$. Notice that A and C are the attracting fixed points of g^s and h^r respectively, and B and D are the repelling fixed points of g^s and h^r respectively. We also observe that the motion ξ at A is toward C in the counterclockwise direction, and at C is toward A in the clockwise direction. Similarly, the motion ξ at B is toward C in the clockwise direction, and at D is toward A in the counterclockwise direction. Therefore, by calculus there is an attracting fixed point X for the motion ξ in the arc (AC) and a repelling fixed point Y for the motion ξ in the arc (BD) (not shown in Figure 3). Since ξ is a Möbius transformation, it has at most two fixed points on $\mathbf{S}^1$. It follows that ξ has exactly two fixed points X and Y. We conclude that ξ is hyperbolic and its axis c_ξ is the geodesic connecting X and Y.

Since X and Y lie on different sides of c_h, we see that c_ξ intersects c_h. Similarly, we note that X and Y also lie on different sides of c_g. Thus c_ξ also intersects c_g. Therefore c_ξ intersects both c_h and c_g. □

From Lemma 7.1, we conclude that the axis c_{hg} of hg intersects c_h. In particular, this implies that $c_{hg} \cap \Sigma_\alpha \neq \emptyset$ and $c_{hg} \cap \Sigma_{\alpha_0} \neq \emptyset$. Hence by Theorem 1.1 we see that

$t_\alpha \circ \varphi^*(hg)$ is pseudo-Anosov. But since $\varphi^*(h) = t_\alpha^{-1} \circ t_{\alpha_0}$, we obtain

$$t_\alpha \circ \varphi^*(hg) = t_\alpha \circ (t_\alpha^{-1} \circ t_{\alpha_0}) \circ \varphi^*(g) = t_{\alpha_0} \circ \varphi^*(g).$$

Hence $t_{\alpha_0} \circ \varphi^*(g)$ is pseudo-Anosov.

To prove that $t_\alpha^{-1} \circ \varphi^*(g)$ pseudo-Anosov, we use Theorem 1.1 once again. By assumption, $c_{hg} \cap \Sigma_0 \neq \emptyset$. So Theorem 1.1 asserts that $t_{\alpha_0}^{-1} \circ \varphi^*(hg)$ is pseudo-Anosov. A computation shows that

$$t_{\alpha_0}^{-1} \circ \varphi^*(hg) = t_\alpha^{-1} \circ \varphi^*(g).$$

Hence $t_\alpha^{-1} \circ \varphi^*(g)$ is pseudo-Anosov. This proves Corollary 1.2.

8. Examples

In this section we give an example to show that Theorem 1.1 is no longer true if we drop the assumption that $c_g \cap \Sigma_\alpha \neq \emptyset$. We take a simple closed geodesic α on S that is also nontrivial on $\tilde{S}$. Let $f \in \mathcal{F}(S, a)$ be an arbitrary element. Then it is well known (see Masur and Minsky [10], for example) that for a sufficiently large integer k, the pair $\{\alpha, f^k(\alpha)\}$ fills S. Denote by β the geodesic homotopic to $f^k(\alpha)$. From Thurston [14], for any positive integer i, the mapping class θ_i induced by

$$t_\alpha^{-i} \circ t_\beta^i \tag{8.1}$$

is pseudo-Anosov. Since it also projects to the identity on $\tilde{S}$, by [7, Theorem 2], there is an essential hyperbolic element $g_i \in G$ such that $\varphi^*(g_i) = \theta_i$ that is represented by (8.1).

Let $[\tau_\alpha]$, $[\tau_\beta] \in \mathrm{mod}(\tilde{S})$ be such that $\varphi^*([\tau_\alpha]) = t_\alpha$ and $\varphi^*([\tau_\beta]) = t_\beta$. By the same argument as Lemma 3.4, there exist a maximal element U of $\mathcal{U}_\alpha$ and a maximal element V of $\mathcal{U}_\beta$ such that $U \cap V \neq \emptyset$, $\partial U \cap \partial V = \emptyset$, and $U \cup V = \mathbf{H}$. In particular, it follows that the region

$$\Sigma_0 = \mathbf{H} - \{\text{all maximal elements of } \tau_\alpha \text{ and } \tau_\beta\}$$

is empty. Now from [17, Theorem 1.2], the axis c_i of g_i stays in the region $U \cap V$. Since $U \cup V = \mathbf{H}$, the axis c_i does not cross Σ_α and Σ_β (defined as in (1.2)). This implies that $c_i \cap \Sigma_\alpha = \emptyset$ and $c_i \cap \Sigma_\beta = \emptyset$ (certainly, the geodesic c_i intersects some nonmaximal elements of τ_α and τ_β since $\tilde{c}_i$ is a filling closed geodesic that intersects $\tilde{\alpha} = \tilde{\beta}$). Now if we choose $k = i$ and consider the mapping class $t_\alpha^k \circ \varphi^*(g_i)$, then from (8.1), we obtain

$$t_\alpha^i \circ \varphi^*(g_i) = t_\alpha^i \circ (t_\alpha^{-i} \circ t_\beta^i) = t_\beta^i.$$

So for any integer i, the mapping class $t_\alpha^i \circ \varphi^*(g_i)$ is not pseudo-Anosov.

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C. ZHANG, Department of Mathematics, Morehouse College,
Atlanta, GA 30314, USA
e-mail: czhang@morehouse.edu