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A RIGIDITY PROPERTY OF PLURIHARMONIC MAPS FROM PROJECTIVE MANIFOLDS

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Abstract

Suppose M is a complex projective manifold of dimension ≥ 2, V is the support of an ample divisor in M and U is an open set in M that intersects each irreducible component of V. We show that a pluriharmonic map $f : M \to N$ into a Kähler manifold N is holomorphic whenever $f|_{V\cap U}$ is holomorphic.

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Sampson [4] proved a unique continuation theorem for harmonic mappings of Riemannian manifolds. Siu [5] showed that a harmonic mapping of connected Kähler manifolds $f : M \to N$ is holomorphic whenever $f|_U$ is holomorphic for some nonempty open set $U \subset M$. It is not difficult to see that, in general, U cannot be replaced by an analytic hypersurface $V \subset M$. For example, take $f = \phi \times \psi$, where $\phi : X \to Y$ is a holomorphic mapping of connected Kähler manifolds and $\psi : S \to Y$ is a harmonic map from a Riemann surface S. If ψ is not holomorphic, then f is not holomorphic, but $f|_{X\times\{p\}}$ is holomorphic for every $p \in S$.

Siu's unique continuation theorem is a basic ingredient in the proof of the strong rigidity theorem for compact Kähler manifolds [5, Theorem 2]. Suppose $M \subset \mathbb{P}^k$ is a projective manifold of dimension ≥ 2 and V is a smooth hyperplane section of M. We shall see that if $f : M \to N$ is a pluriharmonic map into a Kähler manifold N such that $f|_V$ is holomorphic, then f is holomorphic. This result is motivated by the application of Toledo's theorem on plurisubharmonicity of the energy to rigidity theory [2, 6]. Note that we may weaken the assumption that $f|_V$ is holomorphic by applying Siu's unique continuation theorem.

The main purpose of this note is to establish the following result.

PROPOSITION 1. *Suppose $f : M \to N$ is a pluriharmonic map from a projective manifold M into a Kähler manifold N, with $\dim M \geq 2$. Suppose $D = \sum_{i=1}^{k} a_i V_i$ is an ample divisor in M, where $a_i \in \mathbb{Z}$, $a_i \neq 0$, and V_i is an irreducible hypersurface in M,* *for $i = 1, \dots, k$. Let V be the support of D and let U be an open set in M such that $V_i \cap U \neq \emptyset$ for each i. If $f|_{V \cap U}$ is holomorphic, then f is holomorphic.*

REMARK 2. Proposition 1 generalises the following result due to Gromov [2, Section 4.6] (see also [6, Theorem 5]). Suppose $f : M \to N$ is a pluriharmonic map from a smooth projective surface M into a quotient $N = \Omega / \Gamma$ of an irreducible bounded symmetric domain Ω. If $f|_V$ is holomorphic for some general element V of a Lefschetz pencil on M, then f is holomorphic.

PROOF OF PROPOSITION 1. Let ω be a Kähler form on M representing the first Chern class $c_1(\mathcal{O}(D))$ associated to D and let $\omega_N = \sqrt{-1}\, h_{\alpha\bar{\beta}}\, dw^\alpha \wedge \overline{dw^\beta}$ denote the Kähler form on N. If $\phi : X \to N$ is a smooth map from an n-dimensional complex submanifold $X \subset M$ into N, we let $\omega_X = \omega|_X$ and $\varepsilon''(\phi) = \sqrt{-1}\, h_{\alpha\bar{\beta}}\, \partial\overline{\phi^\beta} \wedge \bar{\partial}\phi^\alpha$, which are real $(1,1)$-forms on X such that

$$\varepsilon''(\phi) \wedge \omega_X^{n-1}/(n-1)! = |\bar{\partial}\phi|^2\, \omega_X^n/n! \quad \text{if } n \geq 1,$$

and we define $E''(\phi) = \int_X |\bar{\partial}\phi|^2\, dV_{\omega_X}$ provided that the integral is finite.

Since $c_1(\mathcal{O}(D)) = [\omega]$ is the Poincaré dual of the fundamental homology class $[D]$ of D,

$$E''(f) = \int_M \varepsilon''(f) \wedge \frac{\omega^{m-1}}{(m-1)!} = \sum_{i=1}^k a_i \cdot \int_{D_i} \varepsilon''(f|_{D_i}) \wedge \frac{\omega_{D_i}^{m-2}}{(m-2)!} = \sum_{i=1}^k a_i E''(f|_{D_i}), \tag{1}$$

where $m = \dim M$ and D_i is the smooth locus of V_i for $1 \leq i \leq k$. Note that $\varepsilon''(f|_{D_i}) = \varepsilon''(f)|_{D_i}$ for each i, and that $\varepsilon''(f)$ is closed because f is pluriharmonic [3, Lemma 3.10]. Since $f|_{V \cap U}$ is holomorphic, $f|_{D_i}$ is holomorphic for each i (see [5, pages 88–89]) and we conclude from (1) that f is holomorphic. □

We shall see that the analogue of (1) holds for the Dirichlet energy $E(f)$. Suppose that $f : M \to N$ is a pluriharmonic map from a projective manifold M of dimension $m \geq 2$ into a Riemannian manifold $(N, h = h_{ij}\, dy^i \otimes dy^j)$. We fix a Kähler form ω on M with integral cohomology class and choose a divisor $D = \sum_{i=1}^k a_i V_i$ such that $c_1(\mathcal{O}(D)) = [\omega]$, where a_i is a nonzero integer and V_i is an irreducible hypersurface in M with smooth locus D_i for each i.

Recall that if $\phi : X \to N$ is a smooth map from a Kähler submanifold $X \subset M$, with Kähler form $\omega_X = \omega|_X$, then the energy density $e(\phi)$ of ϕ is given by $e(\phi) = \frac{1}{2}|d\phi|^2 = \langle \varepsilon(\phi), \omega_X \rangle$, where $\varepsilon(\phi) = \sqrt{-1}\, h_{ij}\, \partial\phi^i \wedge \bar{\partial}\phi^j$ is a real (1,1)-form on X, and the energy $E(\phi)$ of ϕ is given by $E(\phi) = \int_X e(\phi)\, dV_{\omega_X}$.

Since f is pluriharmonic, the $(1,1)$-form $\varepsilon(f)$ is closed (see, for example, [3, Lemma 2.2]). For $i = 1, \dots, k$, $\varepsilon(f)|_{D_i} = \varepsilon(f|_{D_i})$. Since $c_1(\mathcal{O}(D)) = [\omega]$ is the Poincaré dual of $[D]$, it follows that

$$E(f) = \int_M \varepsilon(f) \wedge \frac{\omega^{m-1}}{(m-1)!} = \sum_{i=1}^k a_i \cdot \int_{D_i} \varepsilon(f|_{D_i}) \wedge \frac{\omega_{D_i}^{m-2}}{(m-2)!} = \sum_{i=1}^k a_i E(f|_{D_i}). \tag{2}$$

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EXAMPLE 3. Suppose S is a compact Riemann surface of genus $g \geq 2$. The Jacobian $J(S)$ of S is an Abelian variety with a principal polarisation $[\omega] = c_1(\mathcal{O}(\Theta))$, where Θ is a theta divisor in $J(S)$, that is, a translate of the zero locus of the Riemann theta function. Put $M = J(S)$ and let $f : M \to N$ be a pluriharmonic map into a Kähler manifold N. Then Proposition 1 shows that if $f|_G$ is holomorphic for some nonempty open subset G of Θ, then f is holomorphic.

Let $W = \mu(S)$ be the image of S under the Abel–Jacobi map $\mu : S \to J(S)$. We have

$$E''(f) = \int_M \varepsilon''(f) \wedge \frac{\omega^{g-1}}{(g-1)!} = \int_W \varepsilon''(f|_W) = E''(f|_W),$$

since $[(1/(g-1)!)\,\omega^{g-1}]$ is the Poincaré dual of $[W]$ (see, for example, [1, page 350]). Similarly, $E(f) = E(f|_W)$. Using (1) and (2), we conclude that

$$E''(f) = E''(f|_{\Theta^*}) = E''(f|_W) \quad \text{and} \quad E(f) = E(f|_{\Theta^*}) = E(f|_W),$$

where Θ^* denotes the smooth locus of Θ. As in the proof of Proposition 1, it follows that if $f|_G$ is holomorphic for some nonempty open subset G of W, then f is holomorphic.

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