Duality in finite dimensional complex space

C.H. Scott and T.R. Jefferson

The idea of duality is now a widely accepted and useful idea in
the analysis of optimization problems posed in real finite
dimensional vector spaces. Although similar ideas have filtered
over to the analysis of optimization problems in complex space,
these have mainly been concerned with problems of the linear and
quadratic programming variety. In this paper we present a
general duality theory for convex mathematical programs in finite
dimensional complex space, and, by means of an example, show that
this formulation captures all previous results in the area.

1.   Introduction

Over the last decade, considerable effort has been directed towards
optimization problems over a complex space. This work was initiated by
Levinson [7] who extended some basic theorems of linear inequalities to
complex space and derived a duality theory for complex linear programs.
Subsequently, these results were extended to polyhedral cones in complex
space by Ben-Israel [3] and Abrams and Ben-Israel [2]. Hanson and Mond
[5], Mond [8], and Jain and Saxena [6] have analysed complex programs
involving quadratic and square root functions. A general theory of
duality, in the sense of Wolfe [11], has been given by Abrams [1].

Another approach to duality in mathematical programming based on the
conjugate transform [4] has become established in the past few years. This
has been exhaustively pursued in finite dimensional real space by
Rockafellar [10] and Peterson [9]. Our purpose is to extend duality theory

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in the sense of Peterson to complex space. We obtain a completely
symmetric duality theory for convex programs in complex space. This
duality theory has a great advantage over that of Wolfe in that the primal
variables do not appear in the dual. As an example of the theory we
consider a convex program with an objective function containing both
differentiable and non differentiable parts subject to quadratic
constraints. All previously studied programs in complex space as listed
above are particular cases of this theory.

2.   Convexity

DEFINITION 1. $S \subset C^n$ is a convex set if for all x_1, $x_2 \in S$ and
for all λ with $0 \leq \lambda \leq 1$,

$$\lambda x_1 + (1-\lambda)x_2 \in S .$$

DEFINITION 2. $S \subset C^n$ is a convex cone if for all x_1, $x_2 \in S$ and
for all $\lambda_1 \geq 0$, $\lambda_2 \geq 0$,

$$\lambda_1 x_1 + \lambda_2 x_2 \in S .$$

DEFINITION 3. S^* , the polar of a convex cone S , is

$$\{y \in C^n \mid \mathrm{Re}\langle y, x \rangle \geq 0, \text{ for all } x \in S\} .$$

DEFINITION 4. A pair of function g and convex set $G \subset C^{2n}$, $[g, G]$,
is convex with respect to the closed convex cone S if for any z^1 and
z^2 , and $0 \leq \lambda \leq 1$,

$$\lambda g(z^1, \bar{z}^1) + (1-\lambda)g(z^2, \bar{z}^2) - g(\lambda z^1 + (1-\lambda)z^2, \lambda \bar{z}^1 + (1-\lambda)\bar{z}^2) \in S ,$$

where $G \subset Q = \{(w^1, w^2) \in C^{2n}\}$, $w^2 = \bar{w}^1$, $g : G \to C^m$, and $S \subset C^m$.

DEFINITION 5. The pair $[\mathrm{Re}\, g, G]$ is convex with respect to a closed
convex cone T in R^m if, for any z^1 and z^2 , and $0 \leq \lambda \leq 1$,

$$\lambda \,\mathrm{Re}\, g(z^1, \bar{z}^1) + (1-\lambda)\mathrm{Re}\, g(z^2, \bar{z}^2) - \mathrm{Re}\, g(\lambda z^1 + (1-\lambda)z^2, \lambda \bar{z}^1 + (1-\lambda)\bar{z}^2) \in T ,$$

where $G \subset Q \subset C^{2n}$ and $\mathrm{Re}\, g : G \to R^m$.

For the remainder of the paper we will consider g such that

$$\text{Re } g : G \to R \quad \text{and} \quad T = R_+ .$$

DEFINITION 6. The convex function $[g(z, \bar{z}), G]$ convex with respect to convex cone S is called closed if the set

$$\{(z, \bar{z}, \alpha) \mid \alpha = g(z, \bar{z}) + \beta, \beta \in S, (z, \bar{z}) \in G\}$$

is closed. We shall assume all functions are closed.

If g is convex with respect to S and $\text{Re } S = R_+$, then we can associate a set of dual points with each pair $(z_0, \bar{z}_0) \in G$ and convex function $[g : G \to C^1]$, called the subgradient.

DEFINITION 7. The subgradient set of $g(z, \bar{z})$ at $(z_0, \bar{z}_0)$ is defined by

$$\{(y, \bar{y}) \mid \text{Re } g(z_0, \bar{z}_0) - \text{Re}\langle y, z_0 - z \rangle \leq \text{Re } g(z, \bar{z}), \text{ for all } (z, \bar{z}) \in G\} .$$

We denote this by $\partial g(z_0, \bar{z}_0)$.

Moreover if g is convex with respect to S and $\text{Re } S = R_+$ we have the following definition of a convex conjugate transform for complex spaces. We will assume the condition that $\text{Re } S = R_+$ for the remainder of the paper.

DEFINITION 8. The conjugate transform of a function $g(z, \bar{z})$ defined on $(z, \bar{z}) \in G$, $[g(z, \bar{z}) : G]$ is $[h(y, \bar{y}) : H]$, where

$$\text{Re } h(y, \bar{y}) = \sup_{(z, \bar{z}) \in G} [\text{Re}\langle y, z \rangle - \text{Re } g(z, \bar{z})] .$$

$\text{Im } h(y, \bar{y})$ is the complex conjugate function of $\text{Re } h(y, \bar{y})$. Thus $h(y, \bar{y})$ is analytic in $(y, \bar{y})$. The complex conjugate is known up to a constant. For uniqueness choose some $(z, \bar{z}) \in G$ and $(y, \bar{y}) \in \partial g(z, \bar{z})$ and set

$$h(y, \bar{y}) = \langle y, z \rangle - g(z, \bar{z})$$

and

$$H = \{(y, \bar{y}) \mid \sup_{(z, \bar{z}) \in G} \text{Re}\langle y, z \rangle - \text{Re } g(z, \bar{z}) < \infty\} .$$

The above gives rise to the conjugate inequality

$$\text{Re } g(z, \bar{z}) + \text{Re } h(y, \bar{y}) \geq \text{Re}\langle y, z \rangle \quad \text{for} \quad (z, \bar{z}) \in G, \quad (y, \bar{y}) \in H.$$

This is satisfied at equality for

$$(y, \bar{y}) \in \partial g(z, \bar{z}),$$

or equivalently

$$(z, \bar{z}) \in \partial h(y, \bar{y}).$$

DEFINITION 9. The positive homogeneous extension of a convex function h is $[h^+(y, \bar{y}, \lambda), H^+]$ where

$$\text{Re } h^+(y, \bar{y}, \lambda) = \begin{cases} \sup_{(z,\bar{z}) \in G} \text{Re}\langle y, z \rangle & \text{if } \lambda = 0 \text{ and } \sup_{(z,\bar{z}) \in G} \text{Re}\langle y, z \rangle < \infty, \\ \text{Re } \lambda h(y/\lambda, \bar{y}/\bar{\lambda}) & \text{if } \lambda \in S^*\backslash\{0\} \text{ and } (y/\lambda, \bar{y}/\bar{\lambda}) \in H, \end{cases}$$

$$H^+ = \{(y, \bar{y}, \lambda) \mid \sup_{(z,\bar{z}) \in G} \text{Re}\langle y, z \rangle < \infty, \lambda = 0\}$$

$$\cup \{(y, \bar{y}, \lambda) \mid (y/\lambda, \bar{y}/\bar{\lambda}) \in H, \lambda \in S^*\backslash\{0\}\},$$

S^* is the polar of S, the set with respect to which g is convex.

The conjugate transform and its positive homogeneous extension are both closed convex functions by construction and the properties of analytic functions.

If $[\text{Re } g(z, \bar{z}), G]$ and $[\text{Re } h(y, \bar{y}), H]$ are closed conjugate functions, we have by construction that

$$\text{Re } g(z, \bar{z}) + \text{Re } h(y, \bar{y}) \geq \text{Re}\langle y, z \rangle$$

for $(z, \bar{z}) \in G$, $(y, \bar{y}) \in H$.

The following two lemmas are generalisations of Peterson's work [9] to complex space.

LEMMA 1. *The conjugate transform of* $[\text{Re } g^+(z, \bar{z}, \kappa) : \kappa G]$ *for fixed* $\kappa \in U^*\backslash\{0\}$ *is* $[\text{Re } \kappa h(y, \bar{y}) : H]$, *where* g *takes values over* U.

Proof. By definition, the conjugate transform of $[\text{Re } g^+(z, \bar{z}, \kappa) : \kappa G]$ for fixed $\kappa \in U^*\backslash\{0\}$ is given by

$$(1) \quad \sup_{(z,\bar{z}) \in \kappa G} [\text{Re}\langle y, z \rangle - \text{Re } g^+(z, \bar{z}, \kappa)]$$

$$= \sup_{(z/\kappa,\bar{z}/\bar{\kappa}) \in G} [\text{Re}\langle y, z \rangle - \text{Re } \kappa g(z/\kappa, \bar{z}/\bar{\kappa})], \text{ since } \kappa \in U^*\backslash\{0\}.$$

Substituting for the inner product and using Definition 8, equation (1) gives

$$\text{Re } \kappa h(y, \, \bar{y}) \quad \text{with} \quad (y, \, \bar{y}) \, \in H \, .$$

LEMMA 2. *The conjugate transform of* $[\text{Re } g^+(z, \, \bar{z}, \, \kappa) : \dot{G}^+]$ *is the identically zero function on*

$$\{(y, \, \bar{y}) \mid (y, \, \bar{y}) \, \in H \text{ and } h(y, \, \bar{y})+\nu \, \in U\} \, .$$

Proof. The conjugate transform of $[\text{Re } g^+(z, \, \bar{z}, \, \kappa) : G^+]$ is given by

$$(2) \quad \sup_{(z,\bar{z},\kappa)\in G^+} \{\text{Re}\langle\nu, \, \kappa\rangle+\text{Re}\langle y, \, z\rangle-\text{Re } g^+(z, \, \bar{z}, \, \kappa)\}$$

$$= \sup_{\kappa\in U^*} \left[\text{Re}\langle\nu, \, \kappa\rangle + \sup_{(z,\bar{z})\in\kappa G} \{\text{Re}\langle y, \, z\rangle-\text{Re } g^+(z, \, \bar{z}, \, \kappa)\}\right] \, .$$

Explicit evaluation of equation (2) using Definitions 8 and 9, and Lemma 1, gives a finite value of zero if

$$h(y, \, \bar{y}) + \nu \, \in -U \, .$$

The primal formulation of a convex mathematical program in complex space is given below as Program A.

PROGRAM A. $\Phi = \inf \text{Re } g_0(z_0, \, \bar{z}_0) + \sum_j \text{Re } g_j^+(z_j, \, \bar{z}_j, \, \kappa_j)$ subject to the explicit constraints

$$g_i(z_i, \, \bar{z}_i) \, \in -W_i \, , \quad i \, \in I \, ,$$

the implicit constraints

$$(z_0, \, \bar{z}_0) \, \in G_0 \, ,$$

$$(z_i, \, \bar{z}_k) \, \in G_i \, , \quad i \, \in I \, ,$$

$$(z_j, \, \bar{z}_j, \, \kappa_j) \, \in G_j^+ \, , \quad j \, \in J \, ,$$

and the cone condition

$$(z, \, \bar{z}) \, \in \chi \, ,$$

where

$$z = z_0 \underset{i \in I}{\times} z_i \underset{j \in J}{\times} z_j \ ,$$

$$G_j^+ = \{(z_j, \bar{z}_j, \kappa_j) \mid \kappa_j = 0 \quad \text{and} \quad \underset{y_j \in H_j}{\sup} \ \text{Re} \langle y_j, z_j \rangle < \infty \}$$

$$\cup \{(z_j, \bar{z}_j, \kappa_j) \mid \kappa_j \in U_j^* \backslash \{0\} \text{ and } (z_j/\kappa_j, \bar{z}_j/\bar{\kappa}_j) \in G_j \} \ ,$$

and

$$\text{Re } g_j(z_j, \bar{z}_j, \kappa_j) = \begin{cases} \underset{(y_i, \bar{y}_j) \in H_j}{\sup} \ \text{Re} \langle y_j, z_j \rangle \quad \text{if} \quad \kappa_j = 0 \quad \text{and} \\ \\ \qquad\qquad\qquad\qquad \underset{(y_j, \bar{y}_j) \in H_j}{\sup} \ \text{Re} \langle y_j, z_j \rangle < \infty \ , \\ \\ \text{Re } \kappa_j g_j(z_j/\kappa_j, \bar{z}_j/\bar{\kappa}_j) \quad \text{if} \quad \kappa_j \in U_j^* \backslash \{0\} \quad \text{and} \\ \qquad\qquad\qquad\qquad\qquad (z_j/\kappa_j, \bar{z}_j/\bar{\kappa}_j) \in G_j \ . \end{cases}$$

In Section 3, we show that the dual problem, Program B, corresponding to Program A is given by

PROGRAM B. $\Psi = \inf \text{Re } h_0(y_0, \bar{y}_0) + \sum_I \text{Re } h_i^+(y_i, \bar{y}_i, \lambda_i)$ subject to

the explicit constraints

$$h_j(y_j, \bar{y}_j) \in -U_j \ , \text{ for all } \ j \in J \ ,$$

the implicit constraints

$$(y_0, \bar{y}_0) \in H_0 \ ,$$

$$(y_i, \bar{y}_i, \lambda_i) \in H_i^+ \ , \quad i \in I \ ,$$

$$(y_j, \bar{y}_j) \in H_j \ , \quad j \in J \ ,$$

and the cone condition

$$(y, \bar{y}) \in \chi^* \ ,$$

where

$$H_i^+ = \{(y_i, \bar{y}_i, \lambda_i) \mid \lambda_i = 0 \quad \text{and} \quad \underset{(z_i, \bar{z}_i) \in G_i}{\sup} \text{Re} \langle y_i, z_i \rangle < \infty \}$$

$$\cup \{(y_i, \bar{y}_i, \lambda_i) \mid \lambda_i \in W_i^* \backslash \{0\} \text{ and } (y_i/\lambda_i, \bar{y}_i/\bar{\lambda}_i) \in H_i \} \ ,$$

and

$$
\mathrm{Re}\ h_i^+\!\left(y_i,\ \bar{y}_i,\ \lambda_i\right)\ =\ \begin{cases} \displaystyle\sup_{\left(z_i,\bar{z}_i\right)\in G_i}\ \mathrm{Re}\langle y_i,\ z_i\rangle & \text{if}\ \ \lambda_i\,=\,0 \\[4pt] \qquad\qquad\qquad\quad \text{and}\ \ \displaystyle\sup_{\left(z_i,\bar{z}_i\right)\in G_i}\ \mathrm{Re}\langle y_i,\ z_i\rangle\,<\,\infty\ , \\[16pt] \mathrm{Re}\ \lambda_i h_i\!\left(y_i/\lambda_i,\ \bar{y}_i/\bar{\lambda}_i\right) & \text{if}\ \ \lambda_i\,\in\,W_i^*\backslash\{0\} \\[4pt] \qquad\qquad\qquad\quad \text{and}\ \ \left(y_i/\lambda_i,\ \bar{y}_i/\bar{\lambda}_i\right)\,\in\,H_i\ , \end{cases}
$$

and χ^* is the polar cone of χ .

We note that Programs A and B are completely symmetric.

The primal and dual solutions are related by

$$
\left(z_0,\ \bar{z}_0\right)\,\in\,\partial h_0\!\left(y_0,\ \bar{y}_0\right)\ ,
$$

$$
\left(z_i,\ \bar{z}_i\right)\,\in\,\partial h_i\!\left(y_i/\lambda_i,\ \bar{y}_i/\bar{\lambda}_i\right)\ ,\ \ \lambda_i\,\in\,W_i^*\ ,
$$

and

$$
\left(z_j/\kappa_j,\ \bar{z}_j/\bar{\kappa}_j\right)\,\in\,\partial h_j\!\left(y_j,\ \bar{y}_j\right)\ ,\ \ \kappa_j\,\in\,U_j^*\ .
$$

Alternatively we have

$$
\left(y_0,\ \bar{y}_0\right)\,\in\,\partial g_0\!\left(z_0,\ \bar{z}_0\right)\ ,
$$

$$
\left(y_i/\lambda_i,\ \bar{y}_i/\bar{\lambda}_i\right)\,\in\,\partial g_i\!\left(z_i,\ \bar{z}_i\right)\ ,\ \ \lambda_i\,\in\,W_i^*\ ,
$$

$$
\left(y_j,\ \bar{y}_j\right)\,\in\,\partial g_j\!\left(\bar{z}_j/\bar{\kappa}_j\right)\ ,\ \ \kappa_j\,\in\,U_j^*\ .
$$

In addition, the primal and dual objective functions sum to zero (see Section 3 for the proof). Hence

$$
\Phi\,+\,\Psi\,=\,0\ .
$$

These optimality conditions combined with feasibility conditions on the primal variables provide us with sufficient information to determine a primal optimal point from a dual optimal point.

3.   Duality theory

From the construction of the conjugate transform and the lemmas of the previous section we have

$$
\mathrm{Re}\ g_0\!\left(z_0,\ \bar{z}_0\right)\,+\,\mathrm{Re}\ h_0\!\left(y_0,\ \bar{y}_0\right)\,\geq\,\mathrm{Re}\langle z_0,\ y_0\rangle
$$

for feasible $\left(z_0, \bar{z}_0\right)$ and $\left(y_0, \bar{y}_0\right)$;

$$\text{Re } g_j^+\left(z_j, \bar{z}_j, \kappa_j\right) + 0 \geq \text{Re}\langle z_j, y_j\rangle$$

for feasible $\left(z_j, \bar{z}_j\right)$, $\left(y_j, \bar{y}_j\right)$, and for $j \in J$;

$$0 + \text{Re } h_i^+\left(y_i, \bar{y}_i, \lambda_i\right) \geq \text{Re}\langle z_i, y_i\rangle$$

for feasible $\left(z_i, \bar{z}_i\right)$, $\left(y_i, \bar{y}_i\right)$, and for $i \in I$.

Summing the equalities we get

(3) $\text{Re } g_0\left(z_0, \bar{z}_0\right) + \sum_{j \in J} \text{Re } g_j^+\left(z_j, \bar{z}_j, \kappa_j\right)$

$$+ \text{Re } h_0\left(y_0, \bar{y}_0\right) + \sum_{i \in I} \text{Re } h_i^+\left(y_i, \bar{y}_i, \lambda_i\right) \geq \text{Re}\langle z, y\rangle = 0 \ ,$$

since $(z, \bar{z}) \in \chi$ and $(y, \bar{y}) \in \chi^{\perp}$.

THEOREM. *Given Programs* A *and* B *are both consistent and the functions are closed convex, then*

$\text{Re } g_0\left(z_0, \bar{z}_0\right) + \sum_{j \in J} \text{Re } g_j^+\left(z_j, \bar{z}_j, \kappa_j\right)$

$$+ \text{Re } h_0\left(y_0, \bar{y}_0\right) + \sum_{i \in I} \text{Re } h_i\left(y_i, \bar{y}_i, \lambda_i\right) = 0 \ ,$$

for optimal (z, κ) *and* (y, λ) .

Proof. By construction $\text{Re } g_0\left(z_0, \bar{z}_0\right) + \sum_{j \in J} \text{Re } g_j^+\left(z_j, \bar{z}_j, \kappa_j\right)$ is

convex in $(z, \bar{z}, \kappa)$ and $\text{Re } h_0\left(y_0, \bar{y}_0\right) + \sum_{i \in I} \text{Re } h_i^+\left(y_i, \bar{y}_i, \lambda_i\right)$ is convex

in $(y, \bar{y}, \lambda)$. Suppose Program B attains its minimum at a point $(y', \bar{y}', \lambda')$.

Consider the subgradient of $\text{Re } h_0\left(y_0, \bar{y}_0\right) + \sum_{i \in I} \text{Re } h_i\left(y_i, \bar{y}_i, \lambda_i\right)$ at

the point $(y', \bar{y}', \lambda')$ and call it $\partial h(y', \bar{y}', \lambda')$. Now
$\partial h(y', \bar{y}', \lambda') \subset G_0 \underset{i \in I}{\times} G_i \underset{j \in J}{\times} G_j^+$. If there exists

$(z, \bar{z}, \kappa) \in \partial h(y', \bar{y}', \lambda')$ which is feasible $\left((z, \bar{z}) \in \chi\right)$, then the left hand side of (3) is equal to zero.

Suppose there does not exist $(z, \bar{z}, \kappa) \in \partial h(y', \bar{y}', \lambda')$ such that $(z, \bar{z}) \in \chi$. Let $P(y', \bar{y}', \lambda')$ be the closed convex cone of feasible directions at $(y', \bar{y}', \lambda')$. Now $\partial h(y', \bar{y}', \lambda') \cap P(y', \bar{y}', \lambda')$ is a compact convex set and χ is a subspace. By the Hahn-Banach Theorem, there exists a function $\hat{y}$ such that $\mathrm{Re}\langle z, \hat{y}\rangle = 0$ for $(z, \bar{z}) \in \chi$ and $\mathrm{Re}\langle z, \hat{y}\rangle < 0$ for $(z, \bar{z}, \kappa) \in \partial h(y', \bar{y}', \lambda') \cap P(y', \bar{y}', \lambda')$.

This implies that

$$(y'+\alpha y, \overline{y'+\alpha\hat{y}}) \in \chi^{\perp} .$$

Continuity tells us that there exists a small positive α such that

$$\mathrm{Re}\, h_0\big(y_0, \bar{y}_0\big) + \sum_{i \in I} \mathrm{Re}\, h_i^+\big(y_i, \bar{y}_i, \lambda_i\big)$$

at $(y'+\alpha y, \overline{y'+\alpha\hat{y}}, \lambda'+\alpha\hat{\lambda})$ is less than at $(y', \bar{y}', \lambda')$, which contradicts the optimality of $(y', \bar{y}', \lambda')$. The result follows.

4. Example

Consider the following problem:

$$\text{minimize} \quad \mathrm{Re}\left\{\tfrac{1}{2}\bar{z}H_0 z + h_0 z + b_0 + (\bar{z}Az)^{\frac{1}{2}}\right\}$$
$$\text{subject to} \quad \tfrac{1}{2}\bar{z}H_i z + h_i z + b_i \in -W_i , \text{ for all } i \in I ,$$

where H_i , $i \in I \cup \{0\}$, and A are positive semidefinite hermitian $m \times m$ matrices, h_i , $i \in I \cup \{0\}$, and b_i , $i \in I \cup \{0\}$ are fixed column vectors in C_m and scalars respectively, and W_i , $i \in I$, are convex polyhedral cones. Hence we are minimizing an objective which is part differentiable and part nondifferentiable subject to a finite number of quadratic constraints.

Since H_i , $i \in I \cup \{0\}$, and A are positive semidefinite hermitian matrices, they may be decomposed into the product form

$$H_i = \bar{D}_i D_i , \quad i \in I \cup \{0\} ,$$

and

$$A = \bar{B}B .$$

With this decomposition, our problem may be cast in the separable form of Program A as

minimize $\mathrm{Re}\left[\frac{1}{2}\bar{z}_{01}z_{01}+z_{02}+b_0+\left(\bar{z}_{03}z_{03}\right)^{\frac{1}{2}}\right]$

subject to $\frac{1}{2}\bar{z}_{i1}z_{i1} + z_{i2} + b_i \in -W_i$, for all $i \in I$ and $(z, \bar{z}) \in \chi$,

where $z = \left(z_{01},\ z_{02},\ z_{03},\ z_{11},\ \ldots,\ z_{12},\ \ldots\right)^T$ and z belongs to the column space of the $n \times m$ matrix $\left(D_0 h_0 B D_1 h_1 D_2 h_2 \ldots\right)^T$, where T denotes the transpose.

For the dual program, Program B, we require the conjugate transform of quadratic functions, linear functions, and the nondifferentiable form $(\bar{z}z)^{\frac{1}{2}}$. We also require the dual space χ^* of χ .

Since at optimality we have $\langle y, z \rangle = 0$, the polar cone is

$$\chi^* = \left\{ (y,\ \bar{y}) \ \middle|\ D_0 y_{01}+h_0 y_{02}+B y_{03} + \sum_{i \in I} \left(D_i y_{i1}+h_i y_{i2}\right) = 0 \right\} \ .$$

We can now determine the dual program from the detailed prescription given in Program B. This is found to be:

minimize $\mathrm{Re}\left(\frac{1}{2}\bar{y}_{01}y_{01}-b_0\right) + \sum_{i \in I} \mathrm{Re}\left(\frac{1}{2}\bar{\lambda}\bar{y}_{i1}y_{i1}+b_i\lambda_i\right)$

subject to $\lambda_i \in W_i^*$ for all i ,

$$D_0 y_{01} + h_0 + B y_{03} + \sum_{i \in I} \left(D_i y_{i1}+h_i y_{i2}\right) = 0 \ ,$$

$$\bar{y}_{03}y_{03} \leq 1 \ ,$$

$$y_{i2} = \lambda_i e \ , \text{ for all } i \ ,$$

where e is a vector with components of unity.

It is now straightforward to generate the results of previous researchers by appropriate specification of given matrices H_i , A vectors h_i , and scalars b_i , $i \in I \cup \{0\}$.

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School of Mechanical and Industrial Engineering,
University of New South Wales,
Kensington, New South Wales,