COVERINGS OF BIPARTITE GRAPHS

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1. Introduction and summary. For the purpose of analysing bipartite graphs (hereinafter called simply graphs) the concept of an exterior covering is introduced. In terms of this concept it is possible in a natural way to decompose any graph into two parts, an inadmissible part and a core. It is also possible to decompose the core into irreducible parts and thus obtain a canonical reduction of the graph. The concept of irreducibility is very easily and naturally expressed in terms of exterior coverings. The role of the inadmissible edges of a graph is to obstruct certain natural coverings of the graph.

Ore **(7)** has studied graphs using the notion of a set of maximal deficiency. For finite graphs a set of maximal deficiency in Ore's sense becomes the complement of a first member of a minimal exterior pair as defined by us. Because of this, a number of theorems obtained by us become equivalent to theorems of Ore when the graph is finite. For infinite graphs the situation is quite different since Ore's finiteness condition and ours can never be satisfied simultaneously.

Amongst the theorems obtained are generalizations of results due to König **(5)** which may be interpreted as theorems in distinct representatives of sets. In the sixth section inequalities are obtained connecting the dimension of a graph with certain simple parameters obtained from a matrix representation. These results are continuations of those obtained by the authors in **(2)**. Results of this type are of importance from the computational aspect and are connected with the theory of games through the optimal assignment problem as shown in von Neumann **(9)** and Dulmage and Halperin **(4)**.

2. Notation. Throughout this paper the following notation is used: S and T represent two arbitrary sets, and $S \times T$ their Cartesian product consisting of pairs (s, t) with $s \in S$, $t \in T$. Any subset K of $S \times T$ is called a graph, and its elements (s, t) are called edges. A, A_i, S_i, A^*, A_* are subsets of S and B, B_i, T_i, B^*, B_* are subsets of T. $A_i \cup A_j$, $A_i \cap A_j$ and $\bar{A}_i$ represent union, intersection, and complement (with respect to S). The null set is denoted by ϕ. If a set A_i contains a finite number n of elements, n is called the order of A_i and this is denoted by $\nu(A_i) = n$; otherwise, $\nu(A_i) = \infty$. If both S and T have a finite or countable number of elements ordered as $s_1, s_2, s_3, \ldots$, and $t_1, t_2, t_3, \ldots$, and K is any graph, a standard matrix representation for K is defined as follows: the entry $a_{ij} = 1$ if (s_i, t_j) is an edge of K, otherwise $a_{ij} = 0$. It will also be convenient to represent K by a more general matrix

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representation in which entries are non-negative real numbers having the property $a_{ij} > 0$ if $(s_i, t_j) \in K$, otherwise $a_{ij} = 0$.

3. The covering theorems. Let K be any graph. A pair of sets $[A, B]$, is an *exterior cover* (or simply cover) for K if for each $(s, t) \in K$, $s \in A$ or $t \in B$ (or both). Otherwise stated, $[A, B]$ is an exterior cover for K if

$$K \subseteq (A \times B) \cup (\bar{A} \times B) \cup (A \times \bar{B}) = (A \times T) \cup (S \times B).$$

Thus $K \cap (\bar{A} \times \bar{B}) = \phi$ if and only if $[A, B]$ covers K. The number $\nu(A) + \nu(B)$ is called the dimension of the covering and K is said to be of finite exterior dimension if there is a covering $[A, B]$ such that $\nu(A) + \nu(B)$ is finite; otherwise K is of infinite exterior dimension. A graph K consisting of an infinite number of edges may be of finite exterior dimension.

The *exterior dimension* $E(K)$ of K is defined as $E(K) = \min(\nu(A) + \nu(B))$, the minimum being taken over all exterior pairs $[A, B]$ which cover K. An exterior pair $[A, B]$ for which the minimum $E(K)$ is achieved is called a *minimal exterior pair*, abbreviated m.e.p.

Another concept of importance is that of a disjoint graph K^*. The graph K^* is said to be disjoint if for every two distinct edges (s_1, t_1), (s_2, t_2) of K^*, $s_1 \neq s_2$ and $t_1 \neq t_2$. It is obvious that a disjoint graph K^* of finite exterior dimension $E(K^*)$ contains exactly $E(K^*)$ edges, and conversely.

THEOREM 1. *If K is a graph of infinite exterior dimension, then K contains an infinite disjoint subgraph K^*.*

Proof. It is sufficient to show that to any disjoint subgraph K^* of exterior dimension n, there is at least one edge in K, which when added to K^* yields a disjoint subgraph of exterior dimension $n + 1$. Let (s_1, t_1), $(s_2, t_2), \ldots,$ (s_n, t_n) be the edges of K^*. Let $A = \{s_1, s_2, \ldots, s_n\}$ and $B = \{t_1, t_2, \ldots, t_n\}$. Since K is not of finite exterior dimension $[A, B]$ does not cover K. Hence, K contains an edge (s_{n+1}, t_{n+1}) which is contained in $\bar{A} \times \bar{B}$. This edge when added to K^* yields the required subgraph.

THEOREM 2. *If K is a graph of finite exterior dimension then K contains a disjoint subgraph K^* such that $E(K) = E(K^*)$.*

Proof. The proof is by induction on $E(K)$. If $E(K) = 1$, then K is non-null and any edge of K may be used as K^*.

To establish the theorem for any $E(K)$, we distinguish two cases.

In the first case, there exists an m.e.p. $[A, B]$ such that neither A nor B is null. Let $\nu(A) = u$, $\nu(B) = v$ so that $u + v = E(K)$. Put $K_1 = K \cap (A \times \bar{B})$. $[A, \phi]$ is an exterior pair for K_1 of dimension u. The pair $[A, \phi]$ is an m.e.p. for, if not, K_1 may be covered by a pair

$$[A_{K_1}, B_{K_1}], \text{ with } \nu(A_{K_1}) + \nu(B_{K_1}) = p < u.$$

But then

$$[A_{K1}, B \cup B_{K1}]$$

covers K, and its dimension is $p + v < E(K)$ which contradicts the fact that $[A, B]$ is an m.e.p. for K. By the induction hypothesis since $u < E(K)$ there exists a disjoint graph K^*_1 consisting of u edges of K_1 such $E(K^*_1) = E(K_1) = u$. Similarly, if $K_2 = K \cap (\bar{A} \times B)$, there exists a disjoint graph K^*_2 consisting of v edges of K_2 such that $E(K^*_2) = E(K_2) = v$. Putting $K^* = K^*_1 \cup K^*_2$ it follows that K^* is a disjoint subgraph of K such that $E(K^*) = E(K)$.

In the second case, if $[A, B]$ is an m.e.p. then $A = \phi$ or $B = \phi$. Suppose, for definiteness, that $B = \phi$. Then $K \subseteq A \times T$. Let (s, t) be any edge of K and let $L = K \cap ((A - s) \times (T - t))$. Since $[A, \phi]$ is an m.e.p. for $K, E(K) = \nu(A)$. Since $[A - s, \phi]$ covers L,

$$E(L) \leqslant \nu(A - s) + \nu(\phi) = E(K) - 1.$$

If $E(L) = E(K) - 1$, then by the inductive assumption there exists a disjoint subgraph L^* of L such that $E(L^*) = E(L) = E(K) - 1$. Putting $K^* = L^* \cup (s, t)$ it follows that K^* is a disjoint subgraph of K such that $E(K^*) = E(K)$. If $E(L) < E(K) - 1$, let $[A_L, B_L]$ be an m.e.p. for L. The pair $[A_L \cup s, B_L \cup t]$ is a covering for K of dimension $\leqslant E(K)$. Thus K has an m.e.p. in which neither set is null, which contradicts the hypothesis of the second case.

The concept of a disjoint graph has been given two interpretations in the literature in connection with the matrix representations of a graph. The authors have in **(2)** introduced the notion of a sub-permutation set of places in a matrix as a set of places which contains at most one place in any row or column of the matrix. It is clear that in any matrix representation of a disjoint graph the non-zero entries occupy a sub-permutation set of places. Ore **(7)** has defined the term rank ρ of a matrix A, to be the order of the greatest minor in A with a non-zero term in its determinant expansion. Theorem 2, then, states that $E(K)$ is equal to the term rank of any matrix representation of K.

An edge of a graph K is said to be *inadmissible* if it is not an edge of any disjoint subgraph K^* such that $E(K^*) = E(K)$; otherwise the edge is admissible. By the proof of Theorem 1 a graph K of infinite exterior dimension does not have inadmissible edges. It is clear that on removing any or all inadmissible edges from a graph leaves a new graph with the same admissible subset of edges. If K is any graph, the subset K_c consisting of all admissible edges of K is called the *core* of K.

THEOREM 3. *An edge of K is inadmissible if and only if it is in the union of all the sets $A \times B$ such that $[A, B]$ is an m.e.p. for K.*

Proof. Let $[A, B]$ be an m.e.p. for K and let $(s_1, t_1) \in K$ be an element of $A \times B$. Let $E(K) = q$ and let K' be any subgraph of K containing (s_1, t_1) and having exactly q edges. Explicitly let K' consist of the edges $(s_1, t_1), (s_2, t_2), \ldots, (s_q, t_q)$. Let $\nu(A) = u$, $\nu(B) = v$, where $u + v = q$. Since $[A, B]$ is an exterior cover for K', either $s_i \in A$ or $t_i \in B$ for $i = 2, 3, \ldots, q$. Also $s_1 \in A$ and $t_1 \in B$. Thus there are at least $q + 1$ elements belonging to A or B. Since the totality of elements belonging to A or B is q, either two of $s_1, s_2, \ldots, s_q$ or two of $t_1, t_2, \ldots, t_q$ are equal. If, for definiteness, two of $s_1, s_2, \ldots, s_q$ are equal and $A' = \{s_1, s_2, \ldots, s_q\}$, $\nu(A') \leqslant q - 1$ and $[A', \phi]$ covers K'. Hence $E(K') < q$ so that (s_1, t_1) is inadmissible.

Conversely, if $(s, t) \in K$ does not belong to $A \times B$ for any m.e.p. $[A, B]$ for K, it will be shown that (s, t) lies in a disjoint subgraph K^* for which $E(K^*) = E(K)$. Let $L = K \cap ((S - s) \times (T - t))$. Clearly $E(L) < E(K)$. If $E(L) = E(K) - 1$, let L^* be a disjoint subgraph of L such that $E(L^*) = E(L)$. Then $K^* = L^* \cup (s, t)$ is a disjoint subgraph of K, such that $E(K^*) = E(K)$. If $E(L) \leqslant E(K) - 2$, let $[A_L, B_L]$ be an m.e.p. for L. Then $[A_L \cup s, B_L \cup t]$ is an exterior cover for K of dimension $\leqslant E(K)$. Hence $[A_L \cup s,\ B_L \cup t]$ is an m.e.p. for K, and $(s, t) \in (A_L \cup s) \times (B_L \cup t)$. This gives the required contradiction.

Complementary to the concept of an exterior pair for a graph K is that of an *interior pair* for which the following is a definition. A pair $\{A, B\}$ where A and B are non-null subsets of S and T respectively is said to be an *interior pair* for a graph K if $(A \times B) \subseteq K$. From the definition it follows that if $[A, B]$ is an exterior cover for K such that $A \neq S$ and $B \neq T$ then $\{\bar{A}, \bar{B}\}$ is an interior pair for $\bar{K}$ (the complement of K in $S \times T$). Conversely, if $\{A, B\}$ is an interior pair for K, then $[\bar{A}, \bar{B}]$ is an exterior cover for $\bar{K}$. For any graph K an *interior dimension* $I(K)$ is defined by $I(K) = \max\ (\nu(A) + \nu(B))$ where the maximum is taken over all interior pairs $\{A, B\}$ for K. A pair $\{A, B\}$ for which the maximum value $I(K)$ is achieved is called a maximal interior pair. The number $\nu(A) + \nu(B)$ is called the dimension of the pair $\{A, B\}$ regardless of whether $\{A, B\}$ is a maximal interior pair. If a graph K has interior pairs $\{A, B\}$ of arbitrarily large dimension we put $I(K) = \infty$. Note that there is no necessary connection between the magnitudes of the dimensions $I(K)$ and $E(K)$. Either may be greater than, equal to, or less than the other and either may be infinite while the other remains finite. There is a duality theorem connecting the exterior dimension of a graph with the interior dimension of its complement (provided both are finite) which is now given.

THEOREM 4. *Let $\nu(S) = p$, $\nu(T) = q$, $p \leqslant q$. If K is a graph for which $E(K) < p$, then $E(K) + I(\bar{K}) = p + q$. If $E(K) = p$, then $E(K) + I(\bar{K}) \leqslant p + q$, the equality sign holding if and only if K has an m.e.p. $[A, B]$ for which $A \neq S$ and $B \neq T$.*

Proof. Suppose $E(K) < p$ and *let* $[A, B]$ be an m.e.p. for K. Then $A \neq S$ and $B \neq T$ so that $\{\bar{A}, \bar{B}\}$ is an interior pair for $\bar{K}$. Hence $I(\bar{K}) \geqslant \nu(\bar{A}) + \nu(\bar{B}) = p - \nu(A) + q - \nu(B) = p + q - E(K)$. Hence $I(\bar{K}) + E(K) \geqslant p + q$. Now let $\{A_1, B_1\}$ be a maximal interior pair for $\bar{K}$. Then $I(\bar{K}) = \nu(A_1) + \nu(B_1)$. Furthermore $[\bar{A}_1, \bar{B}_1]$ is an exterior cover for K so that

$$E(K) \leqslant \nu(\bar{A}_1) + \nu(\bar{B}_1) = p - \nu(A_1) + q - \nu(B_1) = p + q - I(\bar{K}).$$

Hence $E(K) + I(\bar{K}) \leqslant p + q$. This together with the previous inequality yields $E(K) + I(\bar{K}) = p + q$.

If now $E(K) = p$ and there is an m.e.p. $[A, B]$ such that $A \neq S$, $B \neq T$ then the above proof is valid and $E(K) \neq I(\bar{K}) = p + q$. On the other hand, if $[A, B]$ is an m.e.p. for K implies $A = S$ or $B = T$, thein either $\bar{K}$ has no interior pair in which case $E(K) = I(\bar{K}) = p < p + q$ or for any interior pair $\{A_1, B_1\}$ for $\bar{K}$, $[\bar{A}_1, \bar{B}_1]$ is an exterior cover for K with $\bar{A}_1 \neq S$ and $\bar{B}_1 \neq T$. Hence, $[\bar{A}_1, \bar{B}_1]$ is not an m.e.p. for K, which implies

$$E(K) < \nu(\bar{A}_1) + \nu(\bar{B}_1) = p - \nu(A_1) + q - \nu(B_1) = p + q - I(\bar{K}).$$

Hence $E(K) + I(\bar{K}) < p + q$.

Theorem 4 has the following interpretation for matrices. Let M be a p by q matrix of term rank ρ. If $\rho < p \leqslant q$ then M contains a u by v block of zeros and $\rho + u + v = p + q$. If $\rho = p \leqslant q$, then for any block of zeros of size u by v in M, $\rho + u + v \leqslant p + q$. In §4 we shall return to the matrix interpretation of the graphical theorems.

4. The canonical decomposition of graphs. A graph K is said to be *irreducible* if for every m.e.p. $[A, B]$ for K, either $A = \phi$ or $B = \phi$; otherwise K is reducible. It is clear that an irreducible graph has no inadmissible edges. In this section the decompositions of reducible graphs of finite exterior dimension is considered.

THEOREM 5. *If $[A, B_1]$ and $[A_2, B_2]$ are m.e.p.'s for a graph K of finite exterior dimension then $[A_1 \cap A_2, B_1 \cup B_2]$ and $[A_1 \cup A_2, B_1 \cap B_2]$ are both m.e.p.'s for K.*

Proof. Let (s, t) be any edge of K. Then $s \in A_1$ or $t \in B_1$ and $s \in A_2$ or $t \in B_2$. If $s \bar{\in} (A_1 \cup A_2)$ then $s \bar{\in} A_1$ and $s \bar{\in} A_2$ so that $t \in B_1$ and $t \in B_2$. Hence $[A_1 \cup A_2, B_1 \cap B_2]$ is an exterior cover for K. Similarly, $[A_1 \cap A_2, B_1 \cup B_2]$ is an exterior cover for K. Now

$$E(K) = \nu(A_1) + \nu(B_1) = \nu(A_2) + \nu(B_2).$$

Since $[A_1 \cap A_2, B_1 \cup B_2]$ covers K,

$$\begin{aligned} E(K) &\leqslant \nu(A_1 \cap A_2) + \nu(B_1 \cup B_2) \\ &= \nu(A_1 \cap A_2) + \nu(B_1) + \nu(B_2) - \nu(B_1 \cap B_2). \end{aligned}$$

Since $[A_1 \cup A_2, B_1 \cap B_2]$ covers K, it follows that

$$E(K) \leqslant \nu(A_1 \cup A_2) + \nu(B_1 \cap B_2) = \nu(A_1) + \nu(A_2) - \nu(A_1 \cap A_2) + \nu(B_1 \cap B_2).$$

Both equalities must hold for, if not, we have

$$2E(K) < \nu(A_1) + \nu(B_1) + \nu(A_2) + \nu(B_2) = 2E(K),$$

a contradiction. Thus $[A_1 \cap A_2, B_1 \cup B_2]$ and $[A_1 \cup A_2, B_1 \cap B_2]$ are m.e.p.'s for K.

THEOREM 6. *If* $[A_1, B_1]$ *and* $[A_2, B_2]$ *are m.e.p.'s for* K *and if* $A_1 \subseteq A_2$ *then* $B_2 \subseteq B_1$.

Proof. Suppose $A_1 \subseteq A_2$ and $b_2 \in B_2$ but $b_2 \bar{\in} B_1$. There must exist an edge (a_2, b_2) of K such that $a_2 \bar{\in} A_2$; for if there is no such element then $[A_2, B_2 - b_2]$ is an exterior cover of smaller dimension than that of $[A_2, B_2]$. Since $a_2 \in A_2$ and $A_1 \subset A_2$, then $a_2 \bar{\in} A_1$. Since $b_2 \bar{\in} B_1$ the pair $[A_1, B_1]$ does not cover the edge (a_2, b_2), a contradiction.

COROLLARY. *If* $[A_1, B_1]$ *and* $[A_2, B_2]$ *are m.e.p.'s for* K *and if* A_1 *is a proper subset of* A_2 *or if* $A_1 = \phi$, *then* B_2 *is a proper subset of* B_1 *or* $B_2 = \phi$. *Also if* $[A, B_1]$ *and* $[A, B_2]$ *are m.e.p.'s for* K, *then* $B_1 = B_2$.

THEOREM 7. *For any graph* K *of finite exterior dimension there exist uniquely determined m.e.p.'s* $[A_*, B^*]$ *and* $[A^*, B_*]$ *such that if* $[A, B]$ *is any other m.e.p., then*

(i) A_* is a proper subset of A or $A_* = \phi$,
(ii) A is a proper subset of A^*,
(iii) B_* is a proper subset of B or $B_* = \phi$,
(iv) B is a proper subset of B^*.

Proof. If K has an m.e.p. $[A_*, B^*]$ where $A_* = \phi$, then (i) and (iv) hold for any m.e.p. $[A, B]$ and $[A_*, B^*]$ is the unique m.e.p. for K with this property. If there is no m.e.p. for K whose first member is null, let $[A_*, B^*]$ be an m.e.p. for K for which A_* contains the smallest number of elements of all first members of m.e.p.'s for K. A_* is uniquely determined, since if A_0 is the first member of an m.e.p. for K which contains the same number of elements as does A_*, then by Theorem 5, $A_0 \cap A_*$ is the first member of an m.e.p. for K. Hence if $A_0 \neq A_*$, the set $A_0 \cap A_*$ would have fewer members than A_*, a contradiction. Since A_* is uniquely determined, the corollary to Theorem 6 shows that B^* is also uniquely determined. Let $[A, B]$ be any other m.e.p. for K. $[A_* \cap A, B^* \cup B]$ is an m.e.p. so that $A_* \cap A = A_*$. This implies $B^* \cup B = B^*$. Hence $A_* \subseteq A$ and $B \subseteq B^*$. Both these inequalities are proper, otherwise $A_* = A$ and $B^* = B$ which contradicts the assumption that $[A, B]$ is different from $[A_*, B^*]$. Similarly, there is an m.e.p. $[A^*, B_*]$ for which (ii) and (iii) hold.

From the above proof it is seen that the sets A_*, A^*, B_*, B^* are definable as follows: $A_* = \bigcap A$, $A^* = \bigcup A$, $B_* = \bigcap B$, $B^* = \bigcup B$ where A ranges over all first members and B ranges over all second members of m.e.p.'s for K. The pairs $[A_*, B^*]$ and $[A^*, B_*]$ will be referred to as the *extreme* m.e.p.'s for K.

Let $[A_*, B^*]$ and $[A^*, B_*]$ be the extreme m.e.p.'s for a reducible graph K of finite exterior dimension $E(K)$. If $A^* = A_*$ the Cartesian product $S \times T$ is divided into three parts $R_1 = (A_* \times \bar{B}^*) \cup (\bar{A}_* \times B^*)$, $R_2 = A_* \times B^*$, and $R_3 = \bar{A}_* \times \bar{B}^*$. On the other hand, if $\nu(A^*) - \nu(A_*) > 0$, there is at least one non-null set A such that $A \cap A_* = \phi$, $A \cup A_*$ is the first member

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of an m.e.p. for K and such that $\nu(A)$ is minimal. Let $u_1 = \nu(A)$ and let S_1 be a particular set (possibly unique) amongst all such A. Put $A_1 = A_* \cup S_1$. Let B_1 be the uniquely determined second member such that $[A_1, B_1]$ is an m.e.p. for K and let $T_1 = B^* - B_1$. Now

$$\nu(T_1) = \nu(B^*) - \nu(B_1) = \{E(K) - \nu(A_*)\} - \{E(K) - \nu(A_1)\}$$
$$= \nu(A_1) - \nu(A_*) = \nu(S_1) = u_1.$$

Further, S_i and T_i are constructed inductively as follows. Provided $\nu(A^*) - \nu(A_* \cup S_1 \cup S_2, \ldots, \cup S_{i-1}) > 0$ there exists at least one non-null set A such that $\nu(A)$ is minimal, $A \cap (A_* \cup S_1 \ldots \cup S_{i-1}) = \phi$ and $A_* \cup S_1 \cup S_2 \ldots \cup S_{i-1} \cup A$ is the first member of an m.e.p. for K. Let S_i be any particular set which satisfies these requirements on A and put $\nu(S_i) = u_i$. Put $A_i = A_* \cup S_1 \cup S_2 \ldots \cup S_{i-1} \cup S_i$ and let B_i be the uniquely determined set such that $[A_i, B_i]$ is an m.e.p. for K. Let $T_i = B_{i-1} - B_i$. As before, $\nu(T_i) = \nu(S_i) = u_i$.

The process stops when $A_* \cup S_1 \cup S_2 \ldots \cup S_k = A^*$. Thus $S = A_* \cup S \cup S_2 \ldots \cup S_k \cup \bar{A}^*$. This decomposition of S into $k + 2$ disjoint subsets is the *canonical decomposition* with respect to the reducible graph K of finite exterior dimension. $T = \bar{B} \cup T_1 \cup T_2 \ldots \cup T_k \cup B_*$ is the canonical decomposition of T. We have:

$$\begin{aligned}
&S_i \cap A_* = \phi \qquad \text{for } i = 1, 2, \ldots, k;\\
&S_i \cap S_j = \phi \qquad \text{for } \textit{all } i, j, i \neq j;\\
&T_i \cap B_* = \phi \qquad \text{for } i = 1, 2, \ldots, k;\\
&T_i \cap T_j = \phi \qquad \text{for all } i, j, i \neq j;\\
&\nu(S_i) = \nu(T_i) = u_i;\\
&E(K) = \nu(A_*) + \nu(B_*) + \sum_{i=1}^{k} u_i;\\
&[A_i, B_i] \text{ is an m.e.p. for } K, i = 1, 2, \ldots k,\\
&\text{where } A_i = A_* \cup S_1 \cup S_2 \ldots \cup S_i\\
&\text{and } B_i = T_{i+1} \cup T_{i+2} \ldots \cup T_k \cup B_*.
\end{aligned}$$

Let

$$R_1 = (A_* \times \bar{B}^*) \cup (S_1 \times T_1) \cup (S_2 \times T_2) \ldots \cup (S_k \times T_k) \cup (\bar{A}^* \times B_*);$$
$$R_2 = (A_* \times B^*) \cup (A^* \times B_*) \bigcup_{i<j} (S_i \times T_j);$$
$$R_3 = (\bar{A}_* \times \bar{B}^*) \cup (\bar{A}^* \times \bar{B}_*) \bigcup_{i>j} (S_i \times T_j):$$

R_1, R_2, R_3 are disjoint and $R_1 \cup R_2 \cup R_3 = S \times T$.

In the following figure, this decomposition is shown in the case where it is assumed that the elements of S are ordered so that the points of A_* come first, followed by those of $S_1, S_2, S_3, \ldots, S_k$ and finally $\bar{A}^*$, while those of T are ordered B_*, T_k, $T_{k-1}, \ldots T_1$, $\bar{B}^*$. In this representation R_2 appears in the upper left corner of the diagram, R_3 in the lower right corner, and R_1 separates R_2 from R_3.

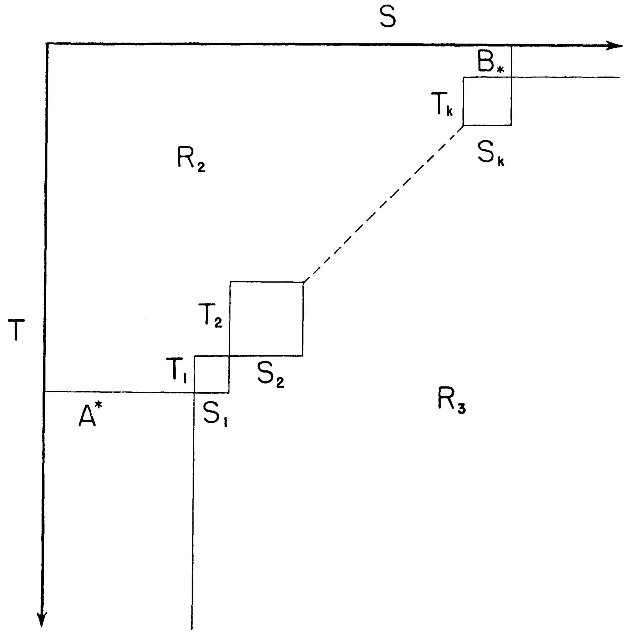

FIGURE 1

This decomposition of the Cartesian product $S \times T$ into R_1, R_2, and R_3 is the *canonical* decomposition of $S \times T$ with respect to the reducible graph K of finite exterior dimension.

THEOREM 8. *If R_1, R_2 and R_3 form the canonical decomposition of $S \times T$ with respect to a reducible graph K of finite exterior dimension, then* (i) *every element of $K \cap R_2$ is admissible and* (ii) $K \cap R_3 = \phi$.

Proof: Part (i) is implied by Theorem 3 for the following reasons. First, $[A_*, B^*]$ and $[A^*, B_*]$ are m.e.p.'s for K. Secondly, $[S_i, T_j]$ is an exterior cover for $S_i \times T_j$ and hence, if $i < j$, $[A_i, B_i]$ is an m.e.p. for K such that $(S_i \times T_j) \cap K \subset (A_i \times B_i)$.

To prove part (ii) we note that clearly no element of K is in $(\bar{A}_* \times \bar{B}^*)$ or in $(\bar{A}^* \times \bar{B}_*)$. Moreover $[A_{i-1}, B_{i-1}]$, which is an m.e.p. for K does not cover any edge of $S_i \times T_j$ when $i > j$.

COROLLARY. (1) *Corresponding to any edge (s, t) of R_2, there exists at least one m.e.p. $[A, B]$ for K such that (s, t) is in $A \times B$.*

(2) *Corresponding to any edge (s, t) of R_3, there exists at least one m.e.p. $[A, B]$ for K such that (s, t) is in $\bar{A} \times \bar{B}$.*

(3) *In both* (1) *and* (2) *the m.e.p. may be chosen from among the* $k + 1$ *m.e.p.'s* $[A_*, B^*]$, $[A_1, B_1]$, $[A_2, B_2]$, $\ldots$, $[A_{k-1}, B_{k-1}]$, $[A^*, B_*]$ *for* K.

K intersects R_1 in $k + 2$ disjoint irreducible subgraphs as indicated in the following theorem.

THEOREM 9. *If* $[A_*, B^*]$ *and* $[A^*, B_*]$ *are the extreme m.e.p.'s for a reducible graph* K *of finite exterior dimension, and if* $S = A_* \cup S_1 \cup S_2 \ldots \cup S_k \cup \bar{A}^*$ *and* $T = \bar{B}^* \cup T_1 \cup T_2 \ldots \cup T_k \cup B_*$ *are the canonical decompositions of* S *and* T, *then* (1) *the subgraphs* $K \cap (A_* \times \bar{B}^*)$ *and* $K \cap (\bar{A}^* \times B_*)$ *are irreducible and their only m.e.p.'s are* (A_*, ϕ) *and* (ϕ, B_*) *respectively, and* (2) *the subgraphs* $K \cap (S_i \times T_i)$ *are irreducible for* $i = 1, 2, 3, \ldots, k$.

Proof. If there exists an m.e.p. $[A', B']$ for $K \cap (A_* \times \bar{B}^*)$ such that A' is a proper subset of A_*, we have $\nu(A') + \nu(B') \leqslant \nu(A_*)$. Since $K \cap R_3$ is null, $[A', B' \cup B^*]$ is an exterior pair for K and since its dimension is

$$\nu(A') + \nu(B') + \nu(B^*) \leqslant \nu(A_*) + \nu(B^*) = E(K)$$

it is a minimal pair. Since A' is a proper subset of A_*, this contradicts the fact that $[A_*, B^*]$ is an extreme m.e.p. for K. Thus $K \cap (A_* \times \bar{B}^*)$ is irreducible and its only m.e.p. is $[A_*, \phi]$. Similarly $K \cap (\bar{A}^* \times B_*)$ is irreducible and its only m.e.p. is (ϕ, B_*).

If there exists an m.e.p. $[A', B']$ for $K \cap (S_i \times T_i)$ such that neither A' nor B' is null, then $\nu(A') + \nu(B') \leqslant \nu(S_i) = \nu(T_i)$. Since $[A_{i-1}, B_i]$ is an exterior cover for $K \cap R_2$, and since $K \cap R_3$ is null, the pair $[A, B] = [A_{i-1} \cup A', B_i \cup B']$ is an exterior cover for K. This cover is minimal, since its dimension is

$$\nu(A_{i-1}) + \nu(A') + \nu(B') + \nu(B_i) \leqslant \nu(A_{i-1}) + \nu(S_i) + \nu(B_i) = E(K).$$

Since $\nu(A_{i-1}) < \nu(A) = \nu(A_{i-1}) + \nu(S') < \nu(A_{i-1}) + \nu(S_i) = \nu(A_i)$, this contradicts the minimality assumption in the definition of S_i. Thus $K \cap (S_i \times T_i)$ is irreducible for $i = 1, 2, \ldots, k$.

THEOREM 10. *If* K *is a reducible graph of finite exterior dimension with a corresponding canonical decomposition of* S *and* T, *and if* α *denotes the collection of* $k + 1$ m.e.p.'s $[A_*, B^*]$, $[A_1, B_1]$, $[A_2, B_2]$, $[A_3, B_3], \ldots, [A_{k-1}, B_{k-1}]$, $[A^*, B_*]$ *for* K, *and if* β *denotes the collection of all m.e.p.'s* $[A, B]$ *for* K, *and if* γ *denotes the collection of* 2^k *pairs* $[A, B]$ *defined by*

$$A = \Big(\bigcup_{i \in \Lambda} S_i\Big) \cup A_*, \quad B = \Big(\bigcup_{i \in \Pi} T_i\Big) \cup B_*$$

in which Λ *and* Π *are complementary subsets of* $1, 2, 3, \ldots, k$, *then*

(i) $\alpha \subseteq \beta \subseteq \gamma$,
(ii) *the admissible subset of* K *is* $K_c = K \cap R_1$,
(iii) *the inadmissible subset of* K *is* $K_I = K \cap R_2$, *and*
(iv) *an exterior pair* $[A, B]$ *is an m.e.p. for* K_c *if and only if* $[A, B] \in \gamma$.

Proof. Let $K_1 = K \cap R_1$. We show, first, that $[A, B]$ is an m.e.p. for K_1 if and only if $[A, B]$ belongs to γ. By Theorem 8, no element of $K - K_1$ is admissible and hence by Theorem 2, $E(K_1) = E(K)$. If $[A, B]$ belongs to γ it covers K_1, and

$$\nu(A) + \nu(B) = \nu(A_*) + \nu(B_*) + \sum_{i=1}^{k} u_i = E(K) = E(K_1)$$

so that any $[A, B] \in \gamma$ is an m.e.p. for K_1.

If $[A, B]$ is any m.e.p. for K_1 then, to show that it belongs to γ it is sufficient to show that $A_* \subseteq A \subseteq A^*$, $B_* \subseteq B \subseteq B^*$, and that, for $i = 1, 2, \ldots, k$, either $A \cap S_i = S_i$ and $B \cap T_i = \phi$ or $A \cap S_i = \phi$ and $B \cap T_i = T_i$. Since $[A, B]$ and $[A_*, B^*]$ are m.e.p.'s for K_1, by Theorem 5, $[A \cup A_*, B \cap B^*]$ is an m.e.p. for K_1 and its dimension is

$$\nu(A) + \nu(A_*) - \nu(A \cap A_*) + \nu(B \cap B^*) = E(K_1) = E(K) = \nu(A) + \nu(B).$$

Thus $\nu(A_* \cap A) + \nu(B) - \nu(B \cap B^*) = \nu(A_*)$ or $\nu(A_* \cap A) + \nu(B \cap \bar{B}^*) = \nu(A_*)$. Thus, by Theorem 9, the exterior covering $[A_* \cap A, B \cap \bar{B}^*]$ for $K \cap (A_* \times \bar{B}^*)$ is minimal and $A_* \cap A = A_*$ and $B \cap \bar{B}^* = \phi$. Thus $A_* \subseteq A$ and $B \subseteq B^*$. Similarly, since $[A \cap A_*, B \cup B^*]$ is an m.e.p. for K_1, $A \subseteq A^*$ and $B_* \subseteq B$. Since $[A, B]$ and $[A_* \cup S_i, B^* - T_i]$ are m.e.p.'s for K_1,

$$[A \cup (A_* \cup S_i), B \cap (B^* - T_i)] = [A \cup S_i, B - B \cap T_i]$$

is an m.e.p. for K_1 by Theorem 5. Its dimension is

$$\nu(A) + \nu(S_i) - \nu(A \cap S_i) + \nu(B) - \nu(B \cap T_i) = E(K_1) = E(K) = \nu(A) + \nu(B).$$

Hence $\nu(A \cap S_i) + \nu(B \cap T_i) = \nu(S_i)$. Thus, by Theorem 9, the exterior covering $[A \cap S_i, B \cap T_i]$ of $K \cap (S_i \times T_i)$ is minimal and either $A \cap S_i = \phi$ and $B \cap T_i = T_i$ or $A \cap S_i = S_i$ and $B \cap T_i = \phi$.

Since every m.e.p. for K is an m.e.p. for K_1, $\beta \subseteq \gamma$ and hence $\alpha \subseteq \beta \subseteq \gamma$.

Since $\beta \subseteq \gamma$, if $[A, B] \in \beta$ and if (s, t) is any edge of $A \times B$ then either $s \in A_*$ and $t \in B^*$ or $s \in A_i$ and $t \in B_j$ with $i \neq j$, or $s \in A^*$ and $t \in B_*$. Thus (s, t) is not in R_1. By Theorems 3 and 8 the inadmissible subset K_I of K is $K \cap R_2$. Hence the admissible subset is $K_c = K \cap R_1$. Since $K_c = K_1$, (iv) follows.

This completes the proof of Theorem 10.

5. Some further properties of the canonical decomposition. Any graph K of finite exterior dimension decomposes the Cartesian product $S \times T$ into three regions R_1, R_2, R_3. In this section the stability of this decomposition under alterations of the graph K is discussed as is also the role played by the inadmissible edges in obstructing some of the m.e.p.'s for K.

Property 1. If the graph K is altered by the addition or removal of edges from R_2, the resulting graph has the same core as does K and the regions R_1, R_2, R_3 are unaltered. The proof is obvious.

Property 2. If edges in R_1 are added to K, the resulting graph produces the same decomposition of $S \times T$ as does K and hence each added element is admissible. The proof again is immediate.

Property 3. Edges may be removed from $K \cap R_1$ without changing the decomposition of $S \times T$ provided the following condition holds. If K_0 is the resulting graph, then for each i the subgraph $K_0 \cap (S_i \times T_i)$ has exterior dimension u_i while $(S_i \times T_i) - K_0 \cap (S_i \times T_i)$ has interior dimension less than u_i in the space $(S_i \times T_i)$. A similar statement must hold for the "tails" $(A_* \times \bar{B}^*)$ and $\bar{A}^* \times B_*$. Again the proof is omitted.

If the condition given in property 3 is violated the following may occur. If the exterior dimension of each of the blocks $K_0 \cap (A_* \times \bar{B}^*)$, $K_0 \cap (S_i \times T_i)$, $K_0 \cap (\bar{A}^* \times B_*)$ is the same as that of the corresponding block with K_0 replaced by K, then in the decomposition of $S \times T$ with respect to K_0, some of the blocks $A_* \times \bar{B}^*$, $S_i \times T_i$, $\bar{A}^* \times B_*$ may break down into smaller irreducible sub-blocks, the remaining parts of the blocks going into R_2 and R_3. If the exterior dimension of any of the blocks $K_0 \cap (A_* \times \bar{B}^*)$, $K_0 \cap (S_i \times T_i)$, $K_0 \cap (\bar{A}^* \times B_*)$ is less than that of the corresponding blocks with K_0 replaced by K, the whole nature of the decomposition may be destroyed. Certain edges originally in R_2 may become admissible and some edges originally in R_1 may become inadmissible.

If K is altered by adding edges from the region R_3, the new graph may produce an entirely different decomposition of $S \times T$. As an example of the effect of adding a single edge of R_3 to K, consider the following: Let $S = \{a_1, a_2, \ldots, a_k\}$, $T = \{b_1, b_2, \ldots, b_k\}$, $K =$ the set of all (a_i, b_j) with $j \geqslant i$. Here $A_* = B_* = \phi$, and the admissible edges are (a_i, b_i) $(i = 1, 2, \ldots, k)$. The irreducible block $S_i \times T_i$ consists of the single edge (a_i, b_i). R_2 consists of all edges (a_i, b_j) with $j > i$ and R_3 all edges (a_i, b_j) with $j < i$. If the edge (a_k, b_1) is added to K, the resulting block is irreducible and hence all points become admissible. If instead of (a_k, b_1) another edge (a_i, b_j) with $j < i$ is added to K then in the new graph some but not all of the edges which were inadmissible in K become admissible in the augmented graph.

The role of the inadmissible elements of K as obstructions to m.e.p.'s is now considered. The core K_c of K has the 2^k m.e.p.'s $[A, B]$,

$$A = \left(\bigcup_{i \in \Lambda} S_i \right) \cup A_*, B = \left(\bigcup_{i \in \Pi} T_i \right) \cup B_*,$$

where Λ and Π are complementary subsets of $1, 2, \ldots, k$. Because of the occurrence of inadmissible elements in K, some of the 2^k m.e.p.'s of K_c may not be m.e.p.'s for K. The following theorem shows that in the extreme case the number of m.e.p.'s may be reduced from 2^k to $k + 1$.

THEOREM 11. *An m.e.p.* $[A, B]$,

$$A = \left(\bigcup_{i \in \Lambda} S_i \right) \cup A_* \quad B = \left(\bigcup_{i \in \Pi} T_i \right) \cup B_*,$$

Λ *and* Π *as above, for the core of* K *is an m.e.p. for* K, *if and only if* $\bigcup (S_j \times T_k)$, *taken over all pairs* j, k, *in which* $j < k$, $j \in \Pi$, $k \in \Lambda$, *contains no edge of* K.

Proof. Let (s, t) be an edge of K. It is immediate that (s, t) is in some $(S_j \times T_k)$, $j \cup k$, $j \in \Pi$, $k \in \Lambda$ if and only if $s \bar{\in} A$ and $t \bar{\in} B$.

COROLLARY. *If every set $S_j \times T_k$ in which $j < k$ contains at least one edge of K then K has exactly the $k + 1$ m.e.p.'s of the collection α, namely $[A_*, B^*]$, $[A_1, B_1], \ldots, [A_{k-1}, B_{k-1}]$, $[A^*, B_*]$.*

6. Other decompositions of $S \times T$. We have already considered the decompositions R_1, R_2, R_3 of $S \times T$. Although R_1 is intrinsic, depending only on S, T, and K, there are cases in which the sets A_i, B_i are not uniquely determined and, in such cases, R_2 and R_3 are not uniquely defined.

We now present two completely intrinsic decompositions of $S \times T$.

We use β to denote the collection of m.e.p.'s for K. By

$$\bigcap_{\beta} \quad \text{or} \quad \bigcup_{\beta}$$

we mean the intersection or union taken over the collection β.

We define

$$V_1 = W_1 = \bigcap_{\beta} ((\bar{A} \times B) \cup (A \times \bar{B})),$$
$$V_2 = \bigcap_{\beta} ((A \times B) \cup (\bar{A} \times B) \cup (A \times \bar{B})) - V_1,$$
$$V_3 = \bigcup_{\beta} (\bar{A} \times \bar{B}),$$
$$W_2 = \bigcup_{\beta} (A \times B),$$
$$W_3 = \bigcap_{\beta} ((\bar{A} \times \bar{B}) \cup (\bar{A} \times B) \cup (A \times \bar{B})) - W_1.$$

(Note that W_1 may be obtained from V_1, W_3 from V_2, and W_2 from V_3, by replacing all A's and B's by their complements.) Since, for every $[A, B]$, $(A \times B) \cup (\bar{A} \times B) \cup (A \times \bar{B})$ and $\bar{A} \times \bar{B}$ are complementary subsets of $S \times T$, V_1, V_2, and V_3 are disjoint and have $S \times T$ as their union. W_1, W_2, and W_3 have the same property.

THEOREM 12. *If K_c and K_I are the admissible and inadmissible subsets of a reducible graph K of finite exterior dimension then*

(1) $K_c \subseteq R_1 = V_1 = W_1$,
(2) $K_I \subseteq V_2 \subseteq R_2 \subseteq W_2$,
(3) $W_3 \subseteq R_3 \subseteq V_3$.

Proof (1) We need only prove $R_1 = V_1$. If (s, t) is an edge of R_1, if $[A, B]$ is any m.e.p., then, by Theorem 10 (since $\beta \subseteq \gamma$) $s \in A$ or $t \in B$ but not both. Thus $(s, t) \in ((\bar{A} \times B) \cup (A \times \bar{B}))$ for every m.e.p. $[A, B]$. Hence $R_1 \subseteq V_1$.

By Corollary (1) to Theorem 8, if (s, t) is in R_2, then (s, t) is in $A \times B$ for some m.e.p. and by Theorem 8, Corollary (2), if (s, t) is in R_3, then (s, t) is in $\bar{A} \times \bar{B}$ for some m.e.p. Thus $R_2 \cap V_1 = R_3 \cap V_1 = \phi$, and hence $V_1 \subseteq R_1$.

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(2) V_2 consists of all the edges of $S \times T$ which are in every cover of K but not in V_1 (that is, not in R_1). Thus $K_I \subseteq V_2$. By Corollary (2) to Theorem 8, any edge of R_3 is in V_3 so that $V_2 \subseteq R_2$. By Corollary (1) to Theorem 8, $R_2 \subseteq W_2$. (3) follows from (1) and (2).

The following examples illustrate these decompositions. If $S = \{a_1, a_2, \ldots, a_k\}$, $T = \{b_1, b_2, \ldots, b_k\}$ and if K is the set of all (a_i, b_j) with $j \geqslant i$, then $K_c = R_1 = V_1 = W_1$, $K_I = R_2 = V_2 = W_2$ and $R_3 = V_3 = W_3$.

On the other hand, if $S = \{a_1, a_2, \ldots, a_k\}$, $T = \{b_1, b_2, \ldots, b_k\}$ and K is the set of all (a_i, b_i) then $K_c = K = R_1 = V_1 = W_1$ and $K_I = \phi = V_2 = W_3$.

R_2 depends on the manner in which the construction of $A_1, A_2, \ldots$ is effected, but, in any case, it consists of $\frac{1}{2}k(k-1)$ edges. For example, if (a_i, b_i) is $S_i \times T_i$, then R_2 consists of all (a_i, b_j) with $i < j$.

W_2 consists of all (a_i, b_j) with $i \neq j$.

7. Application to matrices and computation. In this section some properties of the matrix representation of a graph are studied. Throughout this section the following notation is used. C is a p by q matrix with non-negative entries of term rank ρ. It is assumed that $p \leqslant q$. S represents the sum of all the entries in C, M the maximum sum of the entries in any row or column of C, m the minimum sum of the entries in any row or column of C. Also to be used is the null dimension n of the matrix C, defined as the maximum value of $u + v$ where C contains a u by v block of zeros. Theorem 4 states that $\rho + n = p + q$ unless $\rho = p$ in which case $n \leqslant q$. A problem of some interest is to estimate ρ for a given matrix C. For a large-sized matrix this is a problem of considerable difficulty. A systematic computing machine programme for the exact determination of ρ would involve a search through $q!(q-p)!$ terms which appear in the p by p minors of C. In what follows estimates of ρ in terms of p, q, S, M, m are obtained. Furthermore, transformations in the matrix are introduced which lead to improved estimates of ρ. The results obtained can be applied to the problem of distinct representations of sets **(6)** and to variants of the optimal assignment problem in the theory of games. It is not to be expected that exact values of ρ can be obtained using the above-mentioned parameters only. In fact, in a recent paper, Ryser **(8)** has shown that for standard matrices (those in which the non-zero entries are 1) and using the transformation of replacing

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ by } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

that ρ may be varied between two values ρ_1 and ρ_2 while each of the parameters S, M, m are held constant.

The following result has been found recently by the authors in **(2)**: If $p - r < S/M \leqslant p - r + 1$, then $\rho \geqslant p - r + 1$. It will soon be shown how to modify the matrix C in such a way that ρ is held constant but that S/M is increased. This could lead to a better lower bound for ρ.

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THEOREM 13. *For any matrix C with non-negative entries, $\rho \geqslant p + q - S/m$, unless $\rho = p$.*

Proof. If $\rho \neq p$, $\rho + n = p + q$. Suppose C has a u by v block of zeros with $u + v = n$. By adding the entries in the u rows and v columns of C which contain this block of zeros, it follows that $um + vm \leqslant S$. Hence $nm \leqslant S$, or $(p + q - \rho)m \leqslant S$. Hence $\rho \geqslant p + q - S/m$.

COROLLARY. *If $q \geqslant S/m$, then $\rho = p$.*

THEOREM 14. *For any standard matrix C,*

$$\rho \geqslant \frac{S - m^2 + m(q - p)}{q - m},$$

unless $\rho = p$.

Proof. Suppose $\rho \neq p$, and C has a u by v block of zeros with $u + v = n$. $u \leqslant p - m$, $v \leqslant q - m$. The smallest possible value for the number of zeros in such a block occurs when $|v - u|$ is a maximum and this occurs when $v = q - m$ and $u = n - q + m = p + m - \rho$. The maximum number of 1's in C occurs when all places except the u by v block of 0's are occupied with 1's.

Hence $qp - (q - m)(p + m - \rho) \geqslant S$, which proves the required inequality.

THEOREM 15. *For any standard matrix C, $\rho \geqslant 2m - m^2/q$ unless $\rho = p$.*

Proof. From the inequalities obtained in Theorems 13 and 14,

$$qp - (q - m)(p + m - \rho) \geqslant S \geqslant (p + q - \rho)m.$$

The inequality of the extreme terms reduces to that of the theorem.

Remark. The inequality for ρ given by Theorem 15 is not necessarily weaker than those given in Theorems 13 and 14. Also, in general the inequalities connecting ρ with p, q, S, m give a better lower bound for ρ than the inequalities connecting with p, S, M (when $p = q$) quoted previously.

The estimate $\rho \geqslant p + q - S/m$ will be improved if the matrix C can be replaced by another having the same ρ but a smaller S/m. In what follows we may always assume that $m \neq 0$ since this occurs only if some rows or columns of C have only zero entries. On deleting these rows and columns the new matrix has a value of $m \neq 0$.

A matrix C^* is said to be graph equivalent to C if it is obtained from C by a finite sequence of the following types of operation:

(1) Interchange of two rows.

(2) Interchange of two columns.

(3) Replacement of a non-zero entry by any positive number.

For the next two theorems a systematic method will be given for replacing C by a graph equivalent matrix C^* for which S/M is increased and S/m is decreased. The method is easily adaptable for machine computation.

Let C be any p by q matrix with non-negative entries. Let M be the maximum value of any row or column sum in C and M^* the next largest row or column sum in C. Let C^* be the matrix obtained from C as follows: If the entry c_{ij} of C does not occur in a row or column at which the maximum sum M is attained, put $c^*_{ij} = c_{ij}$. If c_{ij} appears in a row or column at which the maximum sum M is attained, put

$$c^*_{ij} = \frac{M^*}{M} c_{ij}.$$

THEOREM 16. *If S^* is the sum of all entries in C^* then*

$$\frac{S^*}{M^*} \geqslant \frac{S}{M}.$$

Proof. Rearranging rows and columns in C will not change the values of the parameters S, M, S^*, M^*. Suppose the first u rows and first v columns of C have the sum M, all other rows and columns having sums $< M$. Partition C into four blocks as follows:

Let A be the matrix c_{ij}, $i \leqslant u$, $j \leqslant v$,
B be the matrix c_{ij}, $i \leqslant u$, $j > v$,
D be the matrix c_{ij}, $i > u$, $j \leqslant v$,
E be the matrix c_{ij}, $i > u$, $j > v$.

Let a, b, d, e be the sums of the entries in A, B, D, E respectively; let A^*, B^*, D^*, E^* be the corresponding submatrices of C^* and let a^*, b^*, d^*, e^* be the corresponding entry sums. Then M^* is the maximum row or column of C^* and

$$a^* = \frac{M^*}{M} a,\ b^* = \frac{M^*}{M} b,\ d^* = \frac{M^*}{M} d,\ e^* = e.$$

Also

$$S^* = a^* + b^* + d^* + e^* = \frac{M^*}{M} a + \frac{M^*}{M} b + \frac{M^*}{M} d + e.$$

Hence

$$\frac{S^*}{M^*} = \frac{a+b+d}{M} + \frac{e}{M^*} = \frac{S-e}{M} + \frac{e}{M^*}$$

or

$$\frac{S^*}{M^*} - \frac{S}{M} = e\left(\frac{1}{M^*} - \frac{1}{M}\right) \geqslant 0.$$

COROLLARY. If the M^* does not exist, then $p = q$ and the matrix C is doubly stochastic so that (see **(2)**) $\rho = p$. If $S^*/M^* = S/M$ then $e = 0$. But then C contains a block of zeros of size $p - u$ by $q - v$ so that, from Theorem 4, $\rho \leqslant u + v$. This supplies an upper bound for ρ.

From the computational point of view this corollary is not as trivial as might first appear. For a large matrix the problem of locating a u by v block of zeros might require a search of prohibitive length.

By iteration of the process with $S^*/M^* = S_1/M_1$ a sequence of values

$$\frac{S}{M} \leqslant \frac{S_1}{M_1} \leqslant \frac{S_2}{S_2} \cdots$$

is obtained. Either for some i,

$$\frac{S_i}{M_i} = \frac{S_{i-1}}{M_{i-1}},$$

in which case the corollary to Theorem 14, together with the results in **(2)** quoted previously, give upper and lower bounds for ρ, or else the sequence

$$\frac{S}{M} < \frac{S_1}{M_1} < \frac{S_2}{M_2} < \frac{S_3}{M_3} < \ldots$$

is an infinite properly increasing sequence. This sequence is bounded above since for all i,

$$\frac{S_i}{M_i} \leqslant p.$$

In this case the terms approach a limit, and the result quoted previously together with an approximation to this limit gives a lower bound for ρ.

Let C be any p by q matrix with non-negative entries. Let $m \neq 0$ be the minimum value of any row or column sum in C. Let C^* be the matrix obtained from C as follows: If the entry c_{ij} of C does not occur in a row or column at which the minimum sum m is attained, put $c^*_{ij} = c_{ij}$. If c_{ij} appears in a row or column at which the minimum sum m is attained put

$$c^*_{ij} = \frac{m^*}{m} c_{ij}.$$

THEOREM 17. The sum S^* of the entries in C^* satisfies

$$\frac{S^*}{m^*} \leqslant \frac{S}{m}.$$

Proof. The proof is identical with that given for Theorem 16.

The corollaries to Theorem 16 and the remarks concerning the iteration of the transformation have immediate analogues in Theorem 17.

8. Concluding remarks. In the previous sections have we avoided the language of lattice theory in describing our results. In this language some of the results take on an interesting form and it is possible that the lattice formulation might lead to further ramification of the theory. We also note in what follows that our notion of an interior pair can be used to reformulate the map colouring problem.

The m.e.p.'s of a graph K may be partially ordered in a natural manner as follows: $[A, B] \subseteq [C, D]$ if and only if $A \subseteq C$ and $B \supseteq D$ by set inclusion. In this ordering the lattice-theoretic join and meet are given by

$$[A, B] \cup [C, D] = (A \cup C, B \cap D]$$

and

$$[A, B] \cap [C, D] = [A \cap C, B \cup D],$$

respectively. By Theorem 5, if $[A, B]$ and $[C, D]$ are m.e.p.'s for a graph K, then $[A, B] \cup [C, D]$ and $[A, B] \cap [C, D]$ are also m.e.p.'s for K. Also, since the definitions have been given in terms of set inclusion, the resulting lattice is distributive. Hence we have the following theorem:

THEOREM 18. *If K is any graph, the set of all* m.e.p.'s *for K form a distributive lattice.*

Using the notation of §3, we can define complements of an m.e.p. as follows. The complement of $[A, B]$ is taken as $[(A^* - A) \cup A_*, (B^* - B) \cup B_*]$. The complement of $[A, B]$ is not necessarily an m.e.p. for K. In fact the remark preceding Theorem 11, together with the proof used in Theorem 11, yields the following theorem.

THEOREM 19. *The lattice of all m.e.p.'s for a graph K is complemented if and only if K contains no inadmissible elements.*

We remark here without going into details that the region R_2 can be subdivided into subregions such that for each of these subregions the presence of elements of K obstructs the complements of certain m.e.p.'s for K from being m.e.p.'s for K.

In another direction it may be worth while to examine the polarity construction given by Garrett Birkhoff (**1**, p. 54). Let K be any graph in $S \times T$ and $\bar{K}$ its complement. To any subset A of S we associate the subset $B = \psi_K(A)$ of T defined as follows: $b \in \psi_K(A)$ if and only if $(a, b) \in K$ for all a in A. Similarly with any subset B of T we associate the subset $A = \phi_k(B)$ of S defined as follows: $a \in \phi_k(B)$ if and only if $(a, b) \in K$ for all $b \in B$. The following properties of this construction are given in (**1**). For any A and any B,

$$\phi_k\psi_k(A) \supseteq A,\ \psi_k\phi_k(B) \supseteq B.\ \psi_k\phi_k\psi_k(A) = \psi_k(A),\ \phi_k\psi_k\phi_k(B) = \phi_k(B).$$

If $A = \phi_k\psi_k(A)$, A is said to be closed. Similarly, B is closed if $B = \psi_k\phi_k(B)$. A pair (A, B) is called a *polar pair* with respect to K if $B = \psi_k(A)$ and $A = \phi_k(B)$. If (A, B) is a polar pair with respect to K, A and B are both necessarily closed sets. To establish a connection between the concept of a polar pair and the concepts of interior and exterior pairs for a graph we make the following further definitions. An exterior cover $[A, B]$ of K is said to be uncontractable if for any other exterior cover $[A_1, B_1]$ such that $A_1 \subseteq A$ and $B_1 \subseteq B$ then $A_1 = A$ and $B_1 = B$. By this definition an m.e.p. $[A, B]$ is an uncontractable cover of minimum dimension. More generally we may say that the cover $[A, B]$ is uncontractable with respect to A, if for any other cover $[A_1, B]$ we have $A \subseteq A_1$. Uncontractability with respect to B is defined in a similar way. An interior pair $\{A, B\}$ for K is said to be inextensible if for any interior pair $\{A_1, B_1\}$ such that $A \subseteq A_1$ and $B \subseteq B_1$ then $A = A_1$ and $B = B_1$. A maximal interior pair $\{A, B\}$ is then simply an inextensible interior pair of maximum dimension. We can now state the following theorem.

THEOREM 20. *Let K be any graph and $\bar{K}$ its complement, and let $[A, B]$ be an exterior pair for K. Then*

(1) $\psi_{\bar{k}}(\bar{A}) \supseteq \bar{B}$, $\phi_{\bar{k}}(\bar{B}) \supseteq \bar{A}$.

(2) $\psi_{\bar{k}}(\bar{A}) = \bar{B}$ *if and only if* $[A, B]$ *is uncontractable with respect to* B; $\phi_{\bar{k}}(\bar{B}) = \bar{A}$ *if and only if* $[A, B]$ *is uncontractable with respect to* A.

(3) $[A, B]$ *is an uncontractable cover for* K *if and only if* $(\bar{A}, \bar{B})$ *is a polar pair with respect to* $\bar{K}$.

The proof of Theorem 20 is immediate and is not given here. Another theorem whose proof we omit is the following:

THEOREM 21. *With respect to any graph K an interior pair $\{A, B\}$ is inextensible if and only if (A, B) is a polar pair.*

Whether the concepts of uncontractable and inextensible pairs can lead to important properties of graphs is a matter of speculation. It may be worth mentioning here that the set of all uncontractable pairs for a graph K do not form a lattice in any natural way as do the m.e.p.'s for K.

Finally, it might be worth while investigating whether the concepts of cover and interior pair can lead to interesting results in connection with symmetric graphs or with dominance matrices. While we have done no work on these problems, it turns out that many of the interesting problems can be formulated in terms of concepts introduced here. We give one example—that of colouring a map in λ colours. Let M be a map of r regions $a_1, a_2, a_3, \ldots, a_r$. Put $S = T = \{a_1, a_2, \ldots, a_r\}$. The graph K corresponding to the map M is the set of all (a_i, a_j) such that the regions a_i and a_j are contiguous. A colouring of M using λ colours consists of decomposing S into λ mutually exclusive sets $S_1, S_2, S_3, \ldots, S_\lambda$ in such a way that the pairs $\{S_1, S_1\}$, $\{S_2, S_2\}, \ldots$ $\{S_\lambda, S_\lambda\}$ are interior to the complement of K.

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