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On a Yamabe Type Problem in Finsler Geometry

Bin Chen and Lili Zhao

Abstract. In this paper, a new notion of scalar curvature for a Finsler metric F is introduced, and two conformal invariants $Y(M,F)$ and $C(M,F)$ are defined. We prove that there exists a Finsler metric with constant scalar curvature in the conformal class of F if the Cartan torsion of F is sufficiently small and $Y(M,F)C(M,F) < Y(\mathbb{S}^n)$ where $Y(\mathbb{S}^n)$ is the Yamabe constant of the standard sphere.

1 Introduction

In 1960, H. Yamabe [14] attempted to find Riemannian metrics with constant scalar curvature by conformal deformations. Using the techniques of calculus of variations, he claimed the existence of the conformal deformation of a Riemannian metric to constant scalar curvature. In 1968, N. Trudinger [13] discovered an error in Yamabe's proof, which can be repaired by assuming an upper bound of the Yamabe invariant $Y(M^n,g) < \alpha_n$. In 1976, T. Aubin [3] showed that $\alpha_n = Y(\mathbb{S}^n)$ is the Yamabe constant of the standard n-sphere. Finally, T. Aubin [3] and R. Schoen [11] both showed that the strict inequality actually holds unless g is conformal to the standard sphere. J. Lee and T. Parker made a systematic study on this problem [10]. In this paper, we attempt to understand the Yamabe problem in Finsler geometry.

In the Finsler realm, two Finsler metrics $F(x,y)$ and $\overline{F}(x,y)$ on an n-manifold M are conformal if and only if there exists a smooth function $u = u(x)$ such that $\overline{F}(x,y) = e^{u(x)}F(x,y)$, where $y \in T_xM$. It seems natural to state the *Yamabe problem in Finselr geometry*. Unfortunately, there is no canonical definition of the scalar curvature for a Finsler metric. In 2014, X. Cheng and M. Yuan [8] studied the Yamabe problem for the scalar curvature defined by H. Akbar-Zadeh, and obtained a negative answer for Randers metrics. Intuitively, the Finsler scalar curvature should be constructed by the Riemannian curvature. However, once we considered the scalar curvature as the average of all the components of the curvature tensor, the expression of the Riemannian scalar curvature is a pure coincidence for its curvature tensor only has the Riemannian part. Since the Finsler curvature tensor has both Riemannian part and Landsberg part, the scalar curvature in Finsler realm should be an average in some sense of the flag curvature and the Landsberg curvature. In view of the calculus of variations, we introduce a new notion of *Finsler scalar curvature* $\mathrm{Scal}(x)$ (3.2) by

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adding a Landsberg term $j(x)$ to the Riemannian term $R(x)$, and prove that a Finsler metric with constant scalar curvature is a critical point of the total scalar curvature functional

$$\mathcal{S}(F) = \frac{1}{\mathrm{Vol}(M)^{1-\frac{2}{n}}} \int_M \mathrm{Scal}(x)\, d\mu_F$$

in its conformal class. In this paper, we consider the following Yamabe type problem.

On a compact Finsler manifold (M^n, F) *with* $n \geq 3$*, find* $\overline{F}$ *conformal to* F *with constant* $\mathrm{Scal}_{\overline{F}}$.

The Yamabe invariant is defined as $Y(M,F) = \inf_u \mathcal{S}(e^{u(x)}F)$. In order to have a lower bound of $\mathcal{S}$ in the conformal class $[F]$ of the metric F, we make an assumption on the Cartan torsion, and such F is said to be *C-convex* (see Definition 4.1) which is conformally invariant. For C-convex metrics, the PDE of the Yamabe problem is elliptic. This is the main reason for introducing C-convexity. Geometrically, C-convexity measures the distance from a Finsler metric to Riemannian metrics (for instance, the metric is C-convex when the Cartan torsion is small enough). Furthermore, we introduce another conformal invariant $C(M,F)$ (see (5.3)) which is again defined by the Cartan torsion. It is worth pointing out that any Riemannian metric g is indeed C-convex with $C(M,g) = 1$. The main result of this paper is the following theorem.

Theorem 1.1 *Let* (M^n, F) *be a compact C-convex Finsler manifold with* $n \geq 3$*. It holds* $Y(M,F)C(M,F) \leq Y(\mathbb{S}^n)$*. If* $Y(M,F)C(M,F) < Y(\mathbb{S}^n)$*, then there exists a metric* $\overline{F}$ *conformal to* F *such that* $\mathrm{Scal}_{\overline{F}}(x) = Y(M,\overline{F})$.

The contents of this paper are arranged as follows. In §2, we give a brief introduction of Finsler geometry. In §3, we define the Finsler scalar curvature and give the Euler–Lagrange equation of the total scalar curvature. In §4, we discuss the Finsler Yamabe invariant. In §5 and §6, we prove the main theorem.

2 Finsler Metrics

Throughout this paper, we are following reference [5] both for the concepts and the notions in Finsler geometry. Let M be an n-dimensional differentiable manifold with $n \geq 3$. The points in the tangent bundle TM are denoted by (x,y), where $x \in M$ and $y \in T_xM$. Let $(x^i; y^i)$ be the local coordinates of TM with $y = y^i\partial/\partial x^i$. A Finsler metric on M is a function $F\colon TM \to [0,+\infty)$ such that (i) F is smooth on the slit tangent bundle $TM^0 = \{(x,y) \in TM : y \neq 0\}$; (ii) $F(x,\lambda y) = \lambda F(x,y)$ for any $\lambda > 0$; and (iii) the fundamental quadratic form

$$g = g_{ik}(x,y)dx^i \otimes dx^k, \quad g_{ik} := \Big[\frac{1}{2}F^2\Big]_{y^i y^k}$$

is positively definite. Henceforth, the lower index x^i, y^i always means partial derivatives, $F_{y^i} := \frac{\partial F}{\partial y^i}$, $F_{x^i} := \frac{\partial F}{\partial x^i}$, $[F^2]_{y^i y^k} := \frac{\partial^2 F^2}{\partial y^i \partial y^k}$, etc. We shall use the convention that Latin indices range from 1 to n.

The canonical projection $\pi\colon TM^0 \to M$ gives rise to a covector bundle $\pi^* T^* M$, on which there exists the Hilbert form $\omega = F_{y^i}dx^i$, whose dual is the distinguished section of π^*TM: $\ell = \ell^i \frac{\partial}{\partial x^i}$, with $\ell^i := \frac{y^i}{F}$. The Cartan tensor (Cartan torsion) and the Cartan form are, respectively,

$$A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \quad A_{ijk} := \frac{F}{4}[F^2]_{y^i y^j y^k},$$
$$I = I_i dx^i, \quad I_i := A_{ijk} g^{jk}, (g^{jk}) = (g_{jk})^{-1}.$$

The metric is Riemannian if and only if $I = 0$ by Deicke's Theorem. The spray coefficients are given as $G^i = \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}$which determine the geodesic equation $\ddot{\sigma}^i + 2G^i(\sigma, \dot{\sigma}) = 0$. The nonlinear connection coefficients and the Berwald connection coefficients are given as $N^i_k = G^i_{y^k}$ and $B^i_{jk} = G^i_{y^j y^k}$, respectively. The horizontal-vertical decomposition of $T(TM^0)$ is given by the following notions of horizontal vectors and vertical covectors,

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}, \quad \delta y^i := dy^i + N^i_k dx^k.$$

The flag curvature tensor (Riemann curvature tensor) is given by

$$R^i{}_k = 2G^i_{x^k} - G^i_{x^j y^k} y^j + 2G^j G^i_{y^j y^k} - G^i_{y^j} G^j_{y^k},$$

while the Ricci curvature is defined as the trace

$$\mathrm{Ric}(x, y) := \frac{1}{F^2} R^i{}_i = \frac{y^j}{F^2}\Big[\frac{\delta N^i_j}{\delta x^i} - \frac{\delta N^i_i}{\delta x^j}\Big].$$

The most important non-Riemannian curvature in Finsler geometry is the Landsberg curvature, which is defined as the derivative of the Cartan torsion $L_{ijk} := A_{ijk:m}\ell^m$, where ":" is the horizontal covariant derivative with respect to the Berwald connection. All through this paper, the lower index "0" means taking contraction with the distinguished vector ℓ, *i.e.*, $T_{0k} = T_{ik}\ell^i$, $T_{00} = T_{0k}\ell^k$. Thus, we can express $A_{ijk:m}\ell^m$ as $A_{ijk:0}$. Another notation often used is $\dot{T}$, for example, $\dot{T}_{ij} := T_{ij:0}$ and $L_{ijk} = \dot{A}_{ijk}$. The mean Landsberg tensor is $J = J_k dx^k$, where $J_k := \dot{I}_k = g^{ij} L_{ijk}$.

On the slit tangent bundle TM^0, there is the Sasaki type metric

$$g_{ik} dx^i \otimes dx^k + g_{ik} \frac{\delta y^i}{F} \otimes \frac{\delta y^k}{F},$$

which induces a Riemannian metric on the projective sphere bundle SM:

$$\widehat{g} = g_{ik} dx^i \otimes dx^k + F[F]_{y^i y^k} \frac{\delta y^i}{F} \otimes \frac{\delta y^k}{F}.$$

Recall $SM = TM^0/_\sim$, where $(x, y) \sim (x', y')$ if and only if $x = x'$ and $y = \lambda y'$ for some positive number λ. The projective sphere bundle is obviously diffeomorphic to the unit sphere bundle $\{(x, y) \in TM^0 : F(x, y) = 1\}$. Assuming M orientable, the volume form of SM can be expressed as [4, 9]

$$d\mu_{SM} = \Omega d\eta \wedge dx, \quad \Omega := \det\Big(\frac{g_{ik}}{F}\Big),$$

where $d\eta := \sum(-1)^{i-1}y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n$ and $dx := dx^1 \wedge \cdots \wedge dx^n$. The volume form of M induced by SM can be defined by

$$d\mu_F = \sigma_F(x)dx, \quad \sigma_F(x) := \frac{1}{\omega_{n-1}} \int_{S_x M} \Omega\, d\eta, \tag{2.1}$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional standard sphere.

On the Riemannian manifold $(SM, \widehat{g})$, we have the following divergence formula.

Lemma 2.1 ([9]) *For any 1-form $\theta = \alpha_i dx^i + \beta_i \frac{\delta y^i}{F}$ $(\beta_i \ell^i = 0)$ on SM, its divergence is $\operatorname{div}_{\widehat{g}} \theta = g^{ik}(\alpha_{i:k} + \beta_{i,k})$, where ":" and "," denote the horizontal and vertical covariant derivatives of Berwald connection, respectively. Particularly, for any function f on SM, we have $\operatorname{div}_{\widehat{g}}(f\omega) = f_{:k}\ell^k = f_{:0}$, where ω is the Hilbert form.*

On the projective sphere fibre $S_x M$, we have the following Green-type formula.

Lemma 2.2 ([7,9]) *Let $\beta = \beta_i \frac{dy^i}{F}$ $(\beta_i \ell^i = 0)$ be a 1-form on $S_x M$, let ψ and ϕ be two smooth functions on $S_x M$. We have*

$$\int_{S_x M} g^{ij}\beta_{i,j}\Omega\, d\eta = 0, \qquad \int_{S_x M} \psi g^{ij}[F^2\phi]_{y^i y^j}\Omega\, d\eta = \int_{S_x M} \phi g^{ij}[F^2\psi]_{y^i y^j}\Omega\, d\eta.$$

Particularly, taking $\psi = 1$ and $\phi = \ell^r \ell^s$, we have $\int_{S_x M} \ell^r \ell^s \Omega\, d\eta = \frac{1}{n}\int_{S_x M} g^{rs}\Omega\, d\eta$.

3 Scalar Curvature

Henceforth, the manifold M is assumed to be connected, compact, and orientable. In order to find the variational meaning of Finsler–Einstein metrics, the following Einstein–Hilbert functional was considered in [1,7]

$$\mathcal{E}(F) = \frac{1}{\operatorname{Vol}(SM)^{1-2/n}} \int_{SM} \operatorname{Ric}\, d\mu_{SM}.$$

We note that $\mathcal{E}(\lambda F) = \mathcal{E}(F)$ for any positive number λ. In [7], the variation of this functional was calculated. Precisely, if $F(x, y; t) = e^{u(x,y)t}F(x, y)$, then

$$\frac{d\mathcal{E}(0)}{dt} = \frac{1}{\operatorname{Vol}(SM)^{1-2/n}} \int_{SM} u\big(g^{ij}[F^2 \operatorname{Ric}]_{y^i y^j} - (n+2)\operatorname{Ric} - (n-2)r + \iota\big)\, d\mu_{SM} \tag{3.1}$$

where

$$r = \frac{1}{\operatorname{Vol}(SM)} \int_{SM} \operatorname{Ric}\, d\mu_{SM}, \quad \iota = 2g^{ij}(J_{i:j} + \dot{J}_{i,j}).$$

In order to define the Finsler scalar curvature in view of calculus of variations, we define $R(x)$ as the average of Ric on each projective sphere

$$R(x) = \frac{n\int_{S_x M} \operatorname{Ric}\cdot\Omega\, d\eta}{\int_{S_x M}\Omega\, d\eta}.$$

If F is a Riemannian metric, then Euler's Theorem and Lemma 2.2 lead to

$$R(x) = \frac{n\int_{S_xM} \mathrm{Ric}_{ij}\,\ell^i\ell^j \cdot \Omega d\eta}{\int_{S_xM}\Omega d\eta} = g^{ij}(x)\,\mathrm{Ric}_{ij}(x),$$

where $\mathrm{Ric}_{ij} = \frac{1}{2}[F^2\,\mathrm{Ric}]_{y^iy^j}$ is the Ricci tensor. Thus this $R(x)$ is compatible with Riemannian geometry. In the Finsler realm, Akbar-Zadeh's [1] scalar curvature is $H(x,y) = g^{ij}\,\mathrm{Ric}_{ij}$, which depends on the direction y. Cheng and Yuan considered the Yamabe problem for this H [8]. It can be deduced from Lemma 2.2 that $R(x)$ is the average of H on S_xM. For the variational meaning, let us consider

$$\mathcal{R}(F) = \frac{1}{\mathrm{Vol}(M)^{1-2/n}}\int_M R(x)\,d\mu_F.$$

It is easy to see $\mathcal{R}(F) = c_n\mathcal{E}(F)$ for a positive constant c_n depending only on n. Suppose $F(x,y;t) = e^{u(x)t}F(x,y)$ is a conformal deformation. Then the variation formula (3.1) becomes

$$\begin{aligned}\frac{d\mathcal{E}(0)}{dt} = \frac{1}{\mathrm{Vol}(SM)^{1-2/n}}\int_{SM}\Big\{ u(x)\big(g^{ij}[F^2\,\mathrm{Ric}]_{y^iy^j} - (n+2)\,\mathrm{Ric}\big) \\ - u(x)(n-2)r + \big(2u(x)g^{ij}J_{i:j} + 2g^{ij}(u(x)\dot{J}_i)_{,j}\big)\Big\}\,d\mu_{SM}.\end{aligned}$$

Noting that

$$\int_{SM} u(x)f(x,y)\,d\mu_{SM} = \int_M u(x)\int_{S_xM} f(x,y)\Omega\,d\eta\wedge dx,$$

by Lemmas 2.1 and 2.2 we obtain

$$\frac{d\mathcal{E}(0)}{dt} = \frac{1}{\mathrm{Vol}(SM)^{1-2/n}}\int_M u(x)\int_{S_xM}((n-2)(\mathrm{Ric}-r) + 2g^{ij}J_{i:j})\Omega\,d\eta\wedge dx.$$

Putting

$$j(x) = \frac{2n}{n-2}\frac{\int_{S_xM} g^{ij}J_{i:j}\cdot\Omega\,d\eta}{\int_{S_xM}\Omega\,d\eta},$$

we reach

$$\frac{d\mathcal{E}(0)}{dt} = \frac{\omega_{n-1}}{\mathrm{Vol}(SM)^{1-2/n}}\frac{n-2}{n}\int_M u(x)(R(x)+j(x)-nr)\,d\mu_F.$$

Being aware of $\mathrm{Vol}(SM) = \omega_{n-1}\,\mathrm{Vol}(M)$ and $\mathcal{R}(F) = c_n\mathcal{E}(F)$,

$$\frac{d\mathcal{R}(0)}{dt} = \frac{\bar{c}_n}{\mathrm{Vol}(M)^{1-2/n}}\int_M u(x)(R(x)+j(x)-nr)\,d\mu_F,$$

where $\bar{c}_n$ is a positive constant depending only on n. It is interesting that

$$\int_M j(x)\,d\mu_F = \frac{2n}{(n-2)\omega_{n-1}}\int_{SM} g^{ij}J_{i:j}\,d\mu_{SM} = 0$$

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by Lemma 2.1. Thus $\int_M R(x)\,d\mu_F = \int_M (R(x)+j(x))d\mu_F$. We now define the *Finsler scalar curvature* of F by

$$\operatorname{Scal}(x) = R(x) + j(x) \tag{3.2}$$

for dimension $n \geq 3$. This is a natural generalization of the scalar curvature for Riemannian metrics, since $R(x)$ is the scalar curvature if the metric is Riemannian and $j(x) = 0$ for Landsberg (in particular Berwald) manifolds. With this definition, we have the following result.

Theorem 3.1 *The critical points of the total scalar curvature functional*

$$\mathcal{S}(F) := \frac{1}{\operatorname{Vol}(M)^{1-2/n}} \int_M \operatorname{Scal}(x)\,d\mu_F$$

in a conformal class are the Finsler metrics with $\operatorname{Scal}(x) = \text{const}$.

Finsler Yamabe Problem *In the conformal class of F, is there a metric with* $\operatorname{Scal}(x) = \text{const}$*?*

4 Yamabe Invariant

Let us denote the conformal class of F by $[F] = \{e^u F : u \in C^\infty(M)\}$. We now define the *Finsler Yamabe invariant* of $[F]$ as

$$Y(M,[F]) = \inf_{u \in C^\infty(M)} \mathcal{S}(e^u F).$$

We may simply denote $Y(M,[F])$ by $Y(F)$. It is not obvious that $Y(F) > -\infty$ (we shall prove this in (4.5) below. For this purpose, we shall calculate the Ricci curvature of $\overline{F} = e^u F$. Thanks to [6], S. Bacso and X. Cheng obtained the Ricci curvature $\overline{\operatorname{Ric}}$ of $\overline{F}$ in terms of the curvatures of F,

$$\begin{aligned}\overline{\operatorname{Ric}} = e^{-2u}\Big\{ &\operatorname{Ric} + (n-2)(u_0^2 - u_{0:0} - u_i u_j g^{ij}) - g^{ij} u_{i:j} - 2u_i J^i - (u_i I^i)_{:0} \\ &- u_0 u_j I^j + 2u_i u_j A^{ij}_k I^k - u^i u^j I_{i,j} - u_i u_j A^{ir}_s A^{js}_r \Big\},\end{aligned} \tag{4.1}$$

where $u_i = u_{x^i}$, and the indices are lowered and raised by g_{ij} and g^{ij}, *e.g.*, $u^i = g^{ij}u_j$ and $A^{ij}_k = g^{ip}g^{jq}A_{pqk}$. One shall be aware that our I_i differs from Bacso and Cheng's

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I_i by a factor F, and $I_{i,j} = F[I_i]_{y^j}$. By $d\overline{\mu}_{SM} = e^{nu} d\mu_{SM}$ and Lemma 2.1–2.2, we have

$$\begin{aligned}
\int_{SM} e^{-2u} u_{0:0} d\overline{\mu}_{SM} &= \int_{SM} \big((e^{(n-2)u} u_0)_{:0} - (n-2) e^{(n-2)u} u_0^2 \big)\, d\mu_{SM} \\
&= -\int_{SM} (n-2) e^{(n-2)u} u_0^2 \, d\mu_{SM}, \\
\int_{SM} e^{-2u} u_i u_j g^{ij} d\overline{\mu}_{SM} &= \frac{1}{2} \int_{SM} e^{(n-2)u} g^{ij} [F^2 u_0^2]_{y^i y^j} \, d\mu_{SM} \\
&= \int_{SM} n e^{(n-2)u} u_0^2 \, d\mu_{SM}, \\
\int_{SM} e^{-2u} u_{i:j} g^{ij} d\overline{\mu}_{SM} &= \int_{SM} \big((e^{(n-2)u} u_i)_{:j} g^{ij} - (n-2) e^{(n-2)u} u_i u_j g^{ij} \big)\, d\mu_{SM} \\
&= -\int_{SM} n(n-2) e^{(n-2)u} u_0^2 \, d\mu_{SM}.
\end{aligned}$$

Then the integral of the Riemannian part of (4.1)

$$\mathrm{Rpart} := \int_{SM} e^{-2u} (\mathrm{Ric} + (n-2)(u_0^2 - u_{0:0} - u_i u_j g^{ij}) - g^{ij} u_{i:j})\, d\overline{\mu}_{SM}$$

turns to $\mathrm{Rpart} = \int_{SM} e^{(n-2)u} (\mathrm{Ric} + (n-1)(n-2) u_0^2) d\mu_{SM}$. For the rest of the terms, we note

$$g^{kj} (I_i u^i u_k - u_0 I_i u^i \ell_k)_{,j} = u^i u^j I_{i,j} - 2 I_i u_s A_j^{is} u^j - (n-1) u_0 u^i I_i.$$

Therefore, Lemma 2.1 gives

$$\begin{aligned}
\mathrm{NRpart} := \int_{SM} & e^{(n-2)u} \Big\{ -2u_i J^i - (u_i I^i)_{:0} - u_0 u_j I^j \\
& + 2u_i u_j A_k^{ij} I^k - u^i u^j I_{i,j} - u_i u_j A_s^{ir} A_r^{js} \Big\}\, d\mu_{SM} \\
= \int_{SM} & e^{(n-2)u} \Big\{ -2u_i J^i + (n-2) u_0 u_j I^j - u_0 u_j I^j \\
& - (n-1) u_0 u_j I^j - u_i u_j A_s^{ir} A_r^{js} \Big\}\, d\mu_{SM} \\
= \int_{SM} & e^{(n-2)u} \big\{ -2u_i J^i - 2u_0 u_j I^j - u_i u_j A_s^{ir} A_r^{js} \big\}\, d\mu_{SM}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{SM} \overline{\mathrm{Ric}}\, d\overline{\mu}_{SM} &= \mathrm{Rpart} + \mathrm{NRpart} \\
&= \int_{SM} e^{(n-2)u} \big\{ \mathrm{Ric} + (n-2)(n-1) u_0 u_0 \big\}\, d\mu_{SM} \\
&\quad + \int_{SM} e^{(n-2)u} \big\{ -2u_i J^i - 2u_0 u_j I^j - u_i u_j A_s^{ir} A_r^{js} \big\}\, d\mu_{SM}.
\end{aligned}$$

Setting $e^{(n-2)u} = \phi^2$ where $\phi > 0$, we reach

$$\begin{aligned}
\int_{SM} \overline{\mathrm{Ric}}\, d\overline{\mu}_{SM} = \int_{SM} & \Big\{ \phi^2 Ric - \frac{2}{n-2} (\phi^2)_i J^i + \frac{4(n-1)}{(n-2)} \phi_0 \phi_0 \Big\}\, d\mu_{SM} \\
& + \frac{4}{(n-2)^2} \int_{SM} \Big\{ -2\phi_0 \phi_j I^j - \phi_i \phi_j A_s^{ir} A_r^{js} \Big\}\, d\mu_{SM}.
\end{aligned}$$

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Now, applying $g^{ik}(\phi^2 J_k)_{:i} = (\phi^2)_i J^i + \phi^2 g^{ik} J_{k:i}$, one can obtain

$$\begin{aligned}(4.2)\quad \int_{SM} \overline{\mathrm{Ric}}\, d\overline{\mu}_{SM} &= \int_{SM} \Big(\phi^2 \Big\{ \mathrm{Ric} + \frac{2}{n-2} g^{ij} J_{i:j} \Big\} + \frac{4(n-1)}{(n-2)} \phi_0 \phi_0 \Big)\, d\mu_{SM} \\ &\quad + \int_{SM} \frac{4}{(n-2)^2} \{ -2\phi_0 \phi_j I^j - \phi_i \phi_j A_s^{ir} A_r^{js} \}\, d\mu_{SM} \\ &= \frac{\omega_{n-1}}{n} \int_M (\phi^2 \mathrm{Scal} + a \phi_i \phi_j h^{ij})\, d\mu_F,\end{aligned}$$

where $a = \frac{4(n-1)}{(n-2)}$ and

$$h^{ij} = \frac{1}{\int_{S_x M} \Omega\, d\eta} \int_{S_x M} \Big\{ g^{ij} - \frac{n}{(n-1)(n-2)} (\ell^i I^j + \ell^j I^i + A_s^{ir} A_r^{js}) \Big\}\, \Omega\, d\eta.$$

For a Riemannian metric, $h^{ij} = g^{ij}$ is just the metric tensor since $I^j = A_s^{ir} = 0$.

Definition 4.1 We say F is C-convex if the matrix $(h^{ij})_n$ is positively definite. In this case, $h = h_{ij} dx^i \otimes dx^j$ becomes a Riemannian metric on M where $(h_{ij}) = (h^{ij})^{-1}$.

One may observe that $\overline{h}^{ij} = e^{-2u} h^{ij}$ given $\overline{F} = e^u F$. Hence C-convexity is a conformal invariant, and we can say that the conformal class $[F]$ is C-convex. It is clear that F shall be C-convex if the Cartan tensor is sufficiently small.

Proposition 4.2 *If the Cartan tensor satisfies $\|A\|^2 = A_{ijk} A^{ijk} < \frac{1}{36} n$, then F is C-convex.*

Proof Note that $\|A\|$ is a conformal invariant. We will prove that the matrix

$$\Big(g^{ij} - \frac{n}{(n-1)(n-2)} (\ell^i I^j + \ell^j I^i + A_s^{ir} A_r^{js}) \Big)_{n\times n}$$

is positive definite. For any nonzero $\theta_i dx^i$, by $\ell \perp_g I$, we have

$$|\theta_i \theta_j (\ell^i I^j + \ell^j I^i + A_s^{ir} A_r^{js})| \le \|\theta\|^2 \|\ell\| \|I\| + \|\theta\|^2 \|A\|^2.$$

Since $\|\ell\| = 1$ and $\|I\|^2 \le n\|A\|^2$, one can easily get $|\theta_i \theta_j (\ell^i I^j + \ell^j I^i + A_s^{ir} A_r^{js})| < \frac{(n-1)(n-2)}{n} \|\theta\|^2$. ∎

By this proposition, one can easily construct C-convex metrics of Randers type. Henceforth, let us assume that $[F]$ is C-convex. According to (4.2), we can express the total scalar curvature in terms of the conformal factor.

Theorem 4.3 *Suppose $\overline{F} = \phi^{\frac{2}{n-2}} F$, where $\phi = \phi(x)$ is a positive smooth function. If F is C-convex, then*

$$(4.3)\qquad \int_M \overline{\mathrm{Scal}}\, d\mu_{\overline{F}} = \int_M (a \|d\phi\|_h^2 + \phi^2 \cdot \mathrm{Scal}) d\mu_F,$$

$$(4.4)\qquad \mathcal{S}(\phi^{\frac{2}{n-2}} F) = \frac{\int_M (a \|d\phi\|_h^2 + \phi^2 \cdot \mathrm{Scal}_F)\, d\mu_F}{\big(\int_M \phi^p d\mu_F\big)^{2/p}}.$$

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where $p = \frac{2n}{n-2}$ *and* $a = \frac{4(n-1)}{(n-2)}$.

By (4.4)((4.2) is enough in fact), one can reprove Theorem 3.1 by putting $\phi(x,t) = 1 + t\varphi(x)$. For simplicity, let us set $\mathcal{S}_F(\phi) = \mathcal{S}(\phi^{\frac{2}{n-2}}F)$ and define the energy

$$E(\phi) := \int_M (a\|d\phi\|_h^2 + \phi^2 \cdot \mathrm{Scal}_F)\, d\mu_F,$$

and the L^q norm $\|\phi\|_q := (\int_M |\phi|^q\, d\mu_F)^{1/q}$. Then

$$\mathcal{S}_F(\phi) = \frac{E(\phi)}{\|\phi\|_p^2} \quad \text{and} \quad Y(F) = \inf_{\phi>0} \mathcal{S}_F(\phi).$$

Hölder's inequality tells us that $\mathcal{S}_F(\phi) \geq -(\int_M |\mathrm{Scal}_F(x)|^{\frac{n}{2}}\, d\mu_F)^{\frac{2}{n}}$. Thus, the Yamabe invariant of a C-convex conformal class $[F]$ is bounded from below by

$$Y(F) \geq -\|\,\mathrm{Scal}_F\,\|_{\frac{n}{2}}. \tag{4.5}$$

5 The Variational Approach

Following Aubin, Trudinger, and Yamabe, we solve the Finsler–Yamabe problem through the variational approach (we follow Lee and Parker [10]). Consider $\mathcal{S}_F$ as a functional on the Sobolev space $W^{1,2}(M)$, where the $W^{1,2}$-norm is given by

$$\|\phi\|_{1,2} = \Big(\int_M (\|d\phi\|_h^2 + \phi^2)\, d\mu_F \Big)^{1/2}.$$

In general, the $W^{k,q}$-norm is

$$\|\phi\|_{k,q} = (\sum_{i=0}^{k} \int_M \|\nabla^i \phi\|_h^q d\mu_F)^{1/q},$$

where "∇" is the covariant derivative with respect to h. Particularly, $W^{0,q}(M)$ is just the Lebesgue space $L^q(M)$. Since M is compact, different metrics induce equivalent $W^{k,q}$-norms.

Since $C^\infty(M)$ is dense in $W^{k,q}(M)$ and $\mathcal{S}_F(|\phi|) \leq \mathcal{S}_F(\phi)$, we see

$$Y(F) = \inf_{\|\phi\|_p=1} \mathcal{S}_F(\phi) = \inf_{\|\phi\|_p=1} E(\phi).$$

The Euler–Lagrange equation of $\mathcal{S}_F$ on $W^{1,2}(M)$ is

$$L\phi = a\Delta_h\phi + a\langle d\phi, d\tau\rangle_h - \phi \cdot \mathrm{Scal} = -\frac{E(\phi)}{\|\phi\|_p^p}\phi^{p-1},$$

where Δ_h is the Laplacian of the induced Riemannian metric h, and

$$\tau = \frac{\sigma_F(x)}{\sqrt{\det(h_{ij}(x))}},$$

where $\sigma_F(x)$ was defined in (2.1). Since $\mathcal{S}_F(\lambda\phi) = \mathcal{S}_F(\phi)$ for any $\lambda \neq 0$, we may assume the critical function ϕ is normalized as $\|\phi\|_p = 1$. Thus, a nonnegative minimizing

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function ϕ with $\|\phi\|_p = 1$ satisfies

$$L\phi = -Y(F)\phi^{p-1}. \tag{5.1}$$

In order to seek the minimizer, since $\mathcal{S}_F(\phi)$ has a finite lower bound $Y(F)$, we can pick a sequence $\phi_i \in W^{1,2}$ such that $\lim_{i\to\infty} \mathcal{S}_F(\phi_i) = Y(F)$, for $\|\phi_i\|_p = 1$. Then by Hölder's inequality, the $W^{1,2}$-norm of $\{\phi_i\}$ areis uniformly bounded,

$$\|\phi_i\|_{1,2}^2 = \int_M (\|d\phi_i\|_h^2 + \phi_i^2)\, d\mu_F \le \frac{1}{a}\mathcal{S}_F(\phi_i) + C_1\|\phi_i\|_p^2 \le \frac{1}{a}Y(F) + C_2,$$

where C_1, C_2 are constants depending on F. Since $W^{1,2}$ is a Hilbert space, we see $\phi_i \to \phi$ weakly in $W^{1,2}$. But since the inclusion $W^{1,2} \subset L^p$ is not compact, the L^p-norm is not preserved and the limit ϕ may be identically zero. Following Yamabe, let us consider the *disturbed functional*

$$\mathcal{S}_F^t(\phi) := \frac{E(\phi)}{\|\phi\|_t^2}, \quad 2 \le t \le p = \frac{2n}{n-2}.$$

For each t, set $Y_t := \inf_{\|\phi\|_t=1} \mathcal{S}_F^t(\phi)$. One may observe that each $Y_t \ge -\|\operatorname{Scal}_F\|_{\frac{t}{t-2}}$ and $Y_p = Y(F)$. A nonnegative minimizer of $\mathcal{S}_F^t$ with $\|\phi\|_t = 1$ satisfies

$$L\phi = -Y_t\phi^{t-1}. \tag{5.2}$$

If we can find a minimizer ϕ_t of $\mathcal{S}_F^t$ for each t, and (5.2) converges to (5.1) in a suitable sense when $t \to p$, then we can find a minimizer of $\mathcal{S}_F^p(\phi) = \mathcal{S}_F(\phi)$. We first show the existence of ϕ_t.

Lemma 5.1 *For $2 \le t < p$, there exists a nonnegative function $\phi_t \in W^{1,2}$ such that $\mathcal{S}_F^t(\phi_t) = Y_t$ and $\|\phi_t\|_t = 1$.*

Proof Let $\{\phi_i\}$ be a minimizing sequence of $\mathcal{S}_F^t$ in $W^{1,2}$ such that

$$\lim_{i\to\infty} \mathcal{S}_F^t(\phi_i) = Y_t, \quad \|\phi_i\|_t = 1.$$

Since $\mathcal{S}_F^t(|\phi|) \le \mathcal{S}_F^t(\phi)$, we may assume $\phi_i \ge 0$. The Hölder's inequality gives

$$\|\phi_i\|_{1,2}^2 = \frac{1}{a}\mathcal{S}_F^t(\phi_i) + \int_M \phi_i^2\Big(1 - \frac{1}{a}\operatorname{Scal}\Big)\, d\mu_F \le \frac{1}{a}Y_t + C_3 + C_3\|\phi_i\|_t^2,$$

and we may assume $\phi_i \to \phi_t$ weakly in $W^{1,2}$. By the compact embedding $W^{1,2} \subset L^s(2 \le s < p = \frac{2n}{n-2})$, we obtain $\phi_i \to \phi_t$ strongly in L^t and $\|\phi_t\|_t = 1$. The strong convergence gives

$$\Big|\int_M \operatorname{Scal}(\phi_i^2 - \phi_t^2)\, d\mu_F\Big| \le \|\operatorname{Scal}(\phi_i + \phi_t)\|_2 \|\phi_i - \phi_t\|_2 \le C_4\|\phi_i - \phi_t\|_t \to 0.$$

And the weak convergence gives

$$\int_M \langle d\phi_t, d\phi_t\rangle_h\, d\mu_F = \lim_{i\to\infty} \int_M \langle d\phi_t, d\phi_i\rangle_h\, d\mu_F \le \limsup_{i\to\infty} \|d\phi_i\|_2 \|d\phi_t\|_2.$$

Hence, $Y_t \le \mathcal{S}_F^t(\phi_t) = \int_M (a\|d\phi\|_h^2 + \phi^2 \cdot \operatorname{Scal}_F)\, d\mu_F \le \lim_{i\to\infty} \mathcal{S}_F^t(\phi_i) = Y_t$. So $Y_t = \mathcal{S}_F^t(\phi_t)$ and ϕ_t is a minimizer of $\mathcal{S}_F^t$. ■

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At this point, we need the following regularity lemma to ensure that ϕ_t can be a smooth conformal factor.

Lemma 5.2 *The function ϕ_t is positive and smooth.*

Proof The minimizer ϕ_t is a weak solution of $a\Delta_h\phi + a\langle d\phi, d\tau\rangle_h = f$, where

$$f = \{\phi_t \cdot \mathrm{Scal} - Y_t\phi_t^{t-1}\} \in L^{t'=\frac{t}{t-1}}.$$

By the elliptic regularity, $\phi_t \in W^{2,t'}$ and thus $\phi_t \in L^{t_1=\frac{nt'}{n-2t'}}$ by the Sobolev embedding. Note that $t_1 - t > 0$. Now $f \in L^{t_1'=\frac{t_1}{t-1}}$, $\phi_t \in W^{2,t_1'}$, and thus

$$\phi_t \in L^{t_2=\frac{nt_1'}{n-2t_1'}}.$$

Inductively, if $t_i' < \frac{n}{2}$, we can set the next pair as

$$t_{i+1} = \frac{nt_i'}{n-2t_i'} = \frac{nt_i}{n(t-1)-2t_i}, \qquad t_{i+1}' = \frac{t_{i+1}}{t-1}.$$

Since $t_1 > \frac{n(t-2)}{2}$, we have $t_2 > t_1$ and $t_{i+1} > t_i > \frac{n(t-2)}{2}$. Thus $\{t_i\}$ is increasing. If all $t_i' < \frac{n}{2}$, then $\{t_i\}$ is bounded and converges to $\frac{n(t-2)}{2}$ which contradicts to $t_1 > \frac{n(t-2)}{2}$. Assume t_{i_0}' is the first one such that $t_{i_0}' \geq \frac{n}{2}$. If $t_{i_0}' > \frac{n}{2}$, then Sobolev embedding gives $\phi_t \in C^\alpha$. If $t_{i_0}' = \frac{n}{2}$, then the embedding gives $\phi_t \in L^q$ for any $q > 1$, and thus $\phi_t \in W^{2,q}$ for large q, and again we have $\phi_t \in C^\alpha$. Moreover we have $f \in C^\alpha$, so $\phi_t \in C^{2,\alpha}$ by the Schauder estimates.

Now putting $m = \max|Y_t\phi_t^{t-2} - \mathrm{Scal}|$, by $\phi_t \geq 0$ we have

$$a\Delta_h\phi_t + a\langle d\phi_t, d\tau\rangle_h - m\phi_t \leq 0.$$

Thus, if $\min\phi_t = 0$, then $\phi_t = 0$ by the strong maximum principle. By $\|\phi_t\|_t = 1$, we see that $\min\phi_t > 0$. Therefore, ϕ_t^{t-1} is $C^{2,\alpha}$ and then C^∞ by iterating. ■

Now we are forced to show the convergence of $L\phi_t = -Y_t\phi_t^{t-1}$ when $t \to p$. Henceforth, we assume the initial metric F has unit volume $\mathrm{Vol}_F(M) = 1$.

Lemma 5.3 *If $\mathrm{Vol}_F(M) = 1$, then $|Y_t|$ is nonincreasing for $2 \leq t \leq p$. If $Y_p = Y(F) \geq 0$, then $\lim_{t\to p-} Y_t = Y_p$. If $Y_p < 0$, then $\limsup_{t\to p-} Y_t \leq Y_p$.*

Proof Given $t' < t$, we have

$$\|\phi\|_{t'} = \Big(\int_M \phi^{t'}\cdot 1\Big)^{1/t'} \leq \Big(\int_M \phi^t\Big)^{1/t} (\mathrm{Vol}_F(M))^{\frac{1}{t'}-\frac{1}{t}} = \|\phi\|_t.$$

One can get $|Y_t| \leq |Y_{t'}|$ immediately from

$$\mathcal{S}_F^t(\phi) = \frac{\|\phi\|_{t'}^2}{\|\phi\|_t^2}\cdot \mathcal{S}_F^{t'}(\phi).$$

Moreover, if $Y_{t_0} < 0$ for some t_0, then there exists ϕ_0 such that

$$Y_t \leq \mathcal{S}_F^t(\phi_0) = \frac{\|\phi_0\|_{t_0}^2}{\|\phi_0\|_t^2}\cdot \mathcal{S}_F^{t_0}(\phi_0) < 0$$

for all t. So if $Y_p \geq 0$, then $Y_t \geq 0$, $Y_p \leq Y_t$, and $Y_p \leq \liminf_{t\to p-} Y_t$. For any $\epsilon > 0$, pick u such that $\mathcal{S}_F^p(u) \leq Y_p + \epsilon$. For the fixed u, $\mathcal{S}_F^t(u)$ is continuous with respect to t. Hence $Y_p \leq \liminf_{t\to p-} Y_t \leq \limsup_{t\to p-} Y_t \leq \lim_{t\to p-} \mathcal{S}_F^t(u) = \mathcal{S}_F^p(u) \leq Y_p + \epsilon$, which ends the proof. It is easy to get the case $Y_p < 0$. ∎

At present, let us define a new quantity of F by

$$C(M,F) = \sup_{x\in M} \tau(x)^{-2/n} = \sup_{x\in M}\Bigg[\frac{\sqrt{\det(h_{ij}(x))}}{\sigma_F(x)}\Bigg]^{2/n}. \tag{5.3}$$

A moment's thought gives the conformal invariance of $C(M,F)$. Thus we may write it as $C(M,[F])$. Providing F Riemannian, $C(M,[F]) = 1$.

Lemma 5.4 *Let (M,F) be a compact C-convex Finsler manifold. Then for any $\epsilon > 0$, there exists C_ϵ such that*

$$\|w\|_p^2 \leq (1+\epsilon)\frac{C(M,F)}{Y(\mathbb{S}^n)}\int_M a\|dw\|_h^2 d\mu_F + C_\epsilon \int_M w^2\, d\mu_F,$$

where $Y(\mathbb{S}^n)$ is the Yamabe constant of the standard n-sphere.

Proof Putting $\widetilde{h}_{ij} = \tau^{2/n}h_{ij}$, it proves to be the case that $d\mu_{\widetilde{h}} = d\mu_F$. It is well known [2, 3, 10] that $\|w\|_p^2 \leq (1+\epsilon)\frac{a}{Y(\mathbb{S}^n)}\int_M \|dw\|_{\widetilde{h}}^2 d\mu_{\widetilde{h}} + C_\epsilon \int_M w^2 d\mu_{\widetilde{h}}$. Then the lemma follows from $\|dw\|_{\widetilde{h}}^2 = \tau^{-2/n}\|dw\|_h^2 \leq C(M,F)\|dw\|_h^2$. ∎

With the help of the above lemma, one can obtain the following uniform L^{p_0} estimate for some $p_0 > p$.

Lemma 5.5 *If $Y(M,F)\cdot C(M,F) < Y(\mathbb{S}^n)$, then there exists $t_0 < p < p_0$ such that $\|\phi_t\|_{p_0} \leq C_8$, $t_0 \leq t < p$,, where C_8 is independent of t.*

Proof Following Trudinger and Aubin, for $\delta > 0$ we have

$$\phi_t^{1+2\delta}(\Delta_h\phi_t + \langle d\phi_t, d\tau\rangle_h - \phi_t\cdot \mathrm{Scal}) = -Y_t\phi_t^{t+2\delta}$$

and

$$\int_M ((1+2\delta)\phi_t^{2\delta}\|d\phi_t\|_h^2 + \phi_t^{2+2\delta}\cdot\mathrm{Scal})\, d\mu_F = Y_t\int_M \phi_t^{t+2\delta}\, d\mu_F.$$

Put $w = \phi_t^{1+\delta}$. It proves to be the case that

$$\begin{aligned}\int_M \frac{(1+2\delta)}{(1+\delta)^2}\|dw\|_h^2\, d\mu_F &= Y_t\int_M \phi_t^{t-2}w^2 d\mu_F - \int_M w^2\cdot\mathrm{Scal}\, d\mu_F\\ &\leq Y_t\|w\|_p^2\|\phi_t\|_{n(t-2)/2}^{t-2} + C_5\|w\|_2^2.\end{aligned}$$

Recalling $\mathrm{Vol}(M) = 1$, one can see that $\|\phi_t\|_{n(t-2)/2} \leq \|\phi_t\|_t = 1$ by $t < p$. Therefore,

$$\|w\|_p^2 \leq (1+\epsilon)\frac{(1+\delta)^2}{(1+2\delta)}\frac{C(M,F)\cdot Y_t}{Y(\mathbb{S}^n)}\|w\|_p^2 + C_\epsilon\|w\|_2^2.$$

If $Y_p < 0$, then $Y_t < 0$ and $\|w\|_p^2 \le C_\epsilon \|w\|_2^2$. If $Y_p \ge 0$, then $Y_t \ge 0$. Since $\lim_{t\to p-} Y_t = Y_p < \frac{Y(\mathbb{S}^n)}{C(M,F)}$, there exists t_0 such that for $t_0 \le t < p$ we have

$$C(M,F)\cdot Y_t \le C(M,F)\cdot Y_{t_0} < Y(\mathbb{S}^n).$$

Hence, we can choose small ϵ and δ, which are independent of t, such that

$$(1+\epsilon)\frac{(1+\delta)^2}{(1+2\delta)}\frac{C(M,F)\cdot Y_t}{Y(\mathbb{S}^n)} \le (1+\epsilon)\frac{(1+\delta)^2}{(1+2\delta)}\frac{C(M,F)\cdot Y_{t_0}}{Y(\mathbb{S}^n)} < 1.$$

Thus, again we have $\|w\|_p^2 \le C_6\|w\|_2^2$, where C is independent of t. Finally, setting $2+2\delta < t_0 \le t$ and $p_0 = (1+\delta)p$, we have

$$\|\phi_t\|_{p_0}^{1+\delta} = \|w\|_p^2 \le C_7\|w\|_2^2 = C_7\|\phi_t\|_{2(1+\delta)}^{1+\delta} \le C_7\|\phi_t\|_t^{1+\delta} = C_7.$$

Therefore, $\phi_t(t_0 \le t < p)$ are uniformly bounded in L^{p_0}. ∎

Theorem 5.6 *If $Y(M,F)\cdot C(M,F) < Y(\mathbb{S}^n)$, then there exists a smooth positive function ϕ such that $\mathcal{S}_F(\phi) = Y(M,F)$.*

Proof By $Y_t \ge -\|\operatorname{Scal}_F\|_{\frac{t}{t-2}}$ and $\limsup_{t\to p-} Y_t \le Y_p$, we see that $\{Y_t\}$ are bounded for $t_0 \le t < p$. Since $L\phi_t = -Y_t\phi_t^{t-1}$ and $\|\phi_t\|_{p_0} \le C_8$, the regularity theory (similar to Lemma 5.2) shows $\{\phi_t\}$ are uniformly bounded in $C^{2,\alpha}(M)$. Then $\phi_{t_i} \to \phi$ in $C^2(M)$ for some $t_i \to p$. Then the limit gives $-L\phi \le Y(F)\phi^{p-1}$ and $S_F(\phi) \le Y(F)$. Then $S_F(\phi) = Y(F)$ by the definition of $Y(F)$. Moreover, $-L\phi = Y(F)\phi^{p-1}$ and then ϕ is smooth and positive. ∎

Therefore, any Finsler metric satisfying $Y(M,F)C(M,F) < Y(\mathbb{S}^n)$ is conformally deformable to one with constant scalar curvature.

6 A Bound of the Yamabe Invariant

We shall prove $Y(M,F)\cdot C(M,F) \le Y(\mathbb{S}^n)$ in this section by modifying [3,10]. Let us recall some well-known facts [12]. On the Euclidean space $\mathbb{R}^n$, a family of radial functions achieving the best Sobolev constant is

$$u_\epsilon := \Big(\frac{\epsilon}{\epsilon^2+r^2}\Big)^{\frac{n-2}{2}}, \quad \epsilon > 0,$$

where $r = |x|$. A direct computation gives $\partial_r u_\epsilon = -(n-2)\frac{r}{\epsilon^2+r^2}u_\epsilon$ and

$$\Delta_{\mathbb{R}^n} u_\epsilon = \partial_r^2 u_\epsilon + \frac{n-1}{r}\partial_r u_\epsilon = -n(n-2)u_\epsilon^{p-1},$$

which implies

$$\int_{B(R)-B(\rho)} |du_\epsilon|^2\,dx = n(n-2)\int_{B(R)-B(\rho)} u_\epsilon^p\,dx + (2-n)\omega_{n-1}\epsilon^{n-2}\Big[\frac{R^n}{(\epsilon^2+R^2)^{n-1}} - \frac{\rho^n}{(\epsilon^2+\rho^2)^{n-1}}\Big] \tag{6.1}$$

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by the divergence theorem, where $B(R) = \{x : |x| < R\}$. The Yamabe constant of $\mathbb{S}^n$ is then

$$Y(\mathbb{S}^n) = \frac{a \int_{\mathbb{R}^n} |du_\epsilon|^2 \, dx}{(\int_{\mathbb{R}^n} u_\epsilon^p dx)^{\frac{2}{p}}} = an(n-2)\Big(\int_{\mathbb{R}^n} u_\epsilon^p dx\Big)^{\frac{2}{n}}.$$

Thus, we have

$$\text{(6.2)} \qquad \int_{B(\rho)} |du_\epsilon|^2 dx < n(n-2) \int_{B(\rho)} u_\epsilon^p \, dx < \frac{Y(\mathbb{S}^n)}{a} \Big(\int_{B(\rho)} u_\epsilon^p dx\Big)^{\frac{2}{p}}.$$

We give one more estimate

$$\text{(6.3)} \qquad \int_{B(\rho)} u_\epsilon^p \, dx = \omega_{n-1} \int_0^\rho \Big(\frac{\epsilon}{\epsilon^2 + r^2}\Big)^n r^{n-1} dr = \omega_{n-1} \int_0^{\rho/\epsilon} \frac{t^{n-1}}{(1+t^2)^n} dt.$$

Let $\eta = \eta(r)$ be a radial cutoff function on $\mathbb{R}^n$ such that $0 \le \eta \le 1$, $\eta|_{B(1)} = 1$, $\eta|_{\mathbb{R}^n - B(2)} = 0$, and $|d\eta| = |\partial_r \eta| \le 2$. Putting $\eta_\rho := \eta(\frac{r}{\rho})$ for $\rho > 0$, then $0 \le \eta_\rho \le 1$, $\eta|_{B(\rho)} = 1$, $\eta|_{\mathbb{R}^n - B(2\rho)} = 0$, and $|d\eta_\rho| = |\partial_r \eta_\rho| \le \frac{2}{\rho}$. Consider the test function $\varphi := \eta_\rho u_\epsilon$ for $\epsilon \ll \rho$.

Now, for the C-convex Finsler metric F, as we did in Lemma 5.4, putting $\widetilde{h}_{ij} = \tau^{2/n} h_{ij}$, it proves to be the case that

$$\text{(6.4)} \quad E(\phi) = \int_M (a\|d\phi\|_h^2 + \phi^2 \cdot \mathrm{Scal}) d\mu_F \le a \int_M \tau^{2/n} \|d\phi\|_{\widetilde{h}}^2 \, d\mu_{\widetilde{h}} + c_1 \int_M \phi^2 \, d\mu_{\widetilde{h}}.$$

Pick a point $x_0 \in M$ such that $C(M, F) = \sup_{x \in M} \tau^{-2/n}(x) = \tau^{-2/n}(x_0)$. Take a normal coordinate system of $\widetilde{h}$ around x_0. We use the same notation $B(\rho)$ to denote the geodesic balls centered at x_0. By the continuity, we have

$$\tau^{2/n}(x) \le \frac{1}{C(M,F)} + \delta(\rho), \quad x \in B(2\rho),$$

where $\delta(\rho) \to 0$ when $\rho \to 0$.

Suppose 2ρ is less than the injectivity radius of x_0 with respect to $\widetilde{h}$. The test function $\varphi = \eta_\rho u_\epsilon$ can be considered as a smooth function on M. Let us estimate $\mathcal{S}_F(\varphi)$. Assume $(1 - c_2 r) dx \le d\mu_{\widetilde{h}} \le (1 + c_2 r) dx$ in the normal coordinate system. By the Hölder inequality and (6.3), one gets the estimate of the second term of (6.4)

$$\int_M \varphi^2 \, d\mu_{\widetilde{h}} \le (1 + 2c_2\rho) \int_{B(2\rho)} u_\epsilon^2 \, dx \le c_3 \Big(\int_{B(2\rho)} u_\epsilon^p \, dx\Big)^{\frac{2}{p}} \rho^2 \le c_4 \rho^2.$$

For the first term, since

$$a \int_{B(2\rho)} \tau^{2/n} \|d\varphi\|_{\widetilde{h}}^2 \, d\mu_{\widetilde{h}} \le \Big(\frac{1}{C(M,F)} + \delta(\rho)\Big) a \int_{B(2\rho)} \|d\varphi\|_{\widetilde{h}}^2 \, d\mu_{\widetilde{h}},$$

we need to estimate

$$\begin{aligned} \int_M a\|d\varphi\|_{\widetilde{h}}^2 \, d\mu_{\widetilde{h}} &\le (1 + 2c_2\rho) \int_{B(2\rho)} a|\partial_r \varphi|^2 \, dx \\ &= (1 + 2c_2\rho) \Big[\int_{B(\rho)} a|\partial_r u_\epsilon|^2 \, dx + \int_{B(2\rho) - B(\rho)} a|\partial_r(\eta_\rho u_\epsilon)|^2 \, dx \Big]. \end{aligned}$$

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The first term is just (6.2). For the second term, we see from (6.1) that

$$\begin{aligned}\int_{B(2\rho)-B(\rho)} |\partial_r(\eta_\rho u_\epsilon)|^2\,dx &\le \frac{8}{\rho^2}\int_{B(2\rho)-B(\rho)} u_\epsilon^2\,dx + 2\int_{B(2\rho)-B(\rho)} |\partial_r u_\epsilon|^2\,dx\\ &\le c_5\Big(\int_{B(2\rho)-B(\rho)} u_\epsilon^p\,dx\Big)^{\frac{2}{p}} + c_5\int_{B(2\rho)-B(\rho)} u_\epsilon^p\,dx\\ &\quad + c_5\rho^{2-n}\epsilon^{n-2}.\end{aligned}$$

Being aware of (6.3), we see that

$$(1+2c_2\rho)a\int_{B(2\rho)-B(\rho)} |\partial_r(\eta_\rho u_\epsilon)|^2 dx \le \frac{c_6\epsilon^{n-2}}{\rho^{n-2}}.$$

On the other hand, for any $\epsilon < \rho < \frac{1}{2c_2}$,

$$\Big(\int_M \varphi^p\,d\mu_F\Big)^{\frac{2}{p}} = \Big(\int_M \varphi^p\,d\mu_{\widetilde{h}}\Big)^{\frac{2}{p}} \ge (1-c_2\rho)^{\frac{2}{p}}\Big(\int_{B(\rho)} u_\epsilon^p dx\Big)^{\frac{2}{p}} \ge c_7. \tag{6.5}$$

Together with (6.2)–(6.5), we reach

$$\mathcal{S}_F(\varphi) \le \Big(\frac{1}{C(M,F)} + \delta(\rho)\Big)\Big[\frac{(1+2c_2\rho)}{(1-c_2\rho)^{\frac{2}{p}}}Y(\mathbb{S}^n) + \frac{c_6\epsilon^{n-2}}{c_7\rho^{n-2}}\Big] + \frac{c_1c_4}{c_7}\rho^2.$$

By letting $\epsilon \to 0$ and $\rho \to 0$, we see $Y(M,F) \le \frac{1}{C(M,F)}Y(\mathbb{S}^n)$.

Theorem 6.1 *For any C-convex Finsler metric F on a compact n-manifold M, we have $Y(M,F)\cdot C(M,F) \le Y(\mathbb{S}^n)$.*

Based on Schoen's result [11], it is natural to ask whether the strict inequality holds for the manifolds that are non diffeomorphic to the spheres. Noting that $Y(M,F)$ and $C(M,F)$ depend on the derivatives of F up to order six, it should be true if F is close enough to a Riemannian metric in the sense of C^6. However we have not found a simple quantity to describe such an approximation.

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Department of Mathematics, Tongji University, Shanghai, China, 200092
e-mail: chenbin@tongji.edu.cn

Department of Mathematics, Shanghai Jiao Tong University, Shanghai, China, 200240
e-mail: zhaolili@sjtu.edu.cn