Appendix B
Quantization in the temporal gauge

Gauge-field quantization in the temporal gauge in the continuum is often lacking in text books. Here follows a brief outline. Consider the action of $SU(n)$ gauge theory,

$$S = -\int d^4x\, \frac{1}{4g^2} G^p_{\mu\nu} G^{\mu\nu p}. \tag{B.1}$$

The stationary action principle leads to the equations of motion

$$D_\mu G^{\mu\nu p} = \partial_\mu G^{\mu\nu p} + f_{pqr} G^q_\mu G^{\mu\nu r} = 0. \tag{B.2}$$

where D_μ is the covariant derivative in the adjoint representation. Note that we are using a Minkowski-space metric with signature $(-1, 1, 1, 1)$, e.g. $G^{0np} = -G_0{}^{np} = -G^p_{0n}$. The Lagrangian is given by

$$L(G, \dot{G}) = \int d^3x \left(\frac{1}{2g^2} G^p_{0n} G^p_{0n} - \frac{1}{4g^2} G^p_{mn} G^p_{mn} \right), \tag{B.3}$$

where

$$G^p_{0n} = \dot{G}^p_n - \partial_n G^p_0 + f_{pqr} G^q_0 G^r_n, \tag{B.4}$$

and the canonical momenta are given by

$$\Pi^p_0 \equiv \frac{\delta L}{\delta \dot{G}^p_0} = 0, \tag{B.5}$$

$$\Pi^p_n \equiv \frac{\delta L}{\delta \dot{G}^p_n} = \frac{1}{g^2} G^p_{0n}. \tag{B.6}$$

The fact that L is independent of $\dot{G}^p_0$ and consequently the canonical momentum of G^p_0 vanishes is *in*compatible with the presumed canonical Poisson brackets $(G^p_0, \Pi^q_0) \stackrel{?}{=} \delta_{pq}\delta(\mathbf{x} - \mathbf{y})$, unless we eliminate G^p_0 as

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variable by a choice of gauge. This is the ‘temporal gauge’

$$G_0^p = 0. \tag{B.7}$$

The Hamiltonian in the temporal gauge is given by

$$\begin{aligned} H(G,\Pi) &= \int d^3x\, \Pi_m^p \dot{G}_m^p - L \\ &= \int d^3x \left(\frac{g^2}{2} \Pi_m^p \Pi_m^p + \frac{1}{4g^2} G_{mn}^p G_{mn}^p \right). \end{aligned} \tag{B.8}$$

However, one does not want to lose the time component ($\nu = 0$) of the equations of motion (B.2). In canonical variables this equation reads

$$\mathcal{T}^p \equiv \partial_m \Pi_m^p + f_{pqr} G_m^q \Pi_m^r = 0, \tag{B.9}$$

and we see that it does not contain a time derivative. It is a constraint equation for every space–time point. Imposing it at one time, the question of whether it is compatible with Hamilton’s equations arises.

Let us address this question directly in the quantized case, assuming the canonical commutation relations

$$[\hat{G}_m^p(\mathbf{x}), \hat{\Pi}_n^q(\mathbf{y})] = \delta_{pq}\delta(\mathbf{x}-\mathbf{y}), \quad [\hat{G}_m^p(\mathbf{x}), \hat{G}_n^q(\mathbf{y})] = 0 = [\hat{\Pi}_m^p(\mathbf{x}), \hat{\Pi}_n^q(\mathbf{y})]. \tag{B.10}$$

Now it is straightforward to check that the $\hat{\mathcal{T}}^p$ defined in (B.9) generate time-independent gauge transformations, e.g. $\hat{\Omega}^\dagger \hat{G}_m^p \hat{\Omega}$ = infinitesimally gauge-transformed $\hat{G}_m^p$, where $\hat{\Omega} = 1 + i\int d^3x\, \omega^p(\mathbf{x})\hat{\mathcal{T}}^p(\mathbf{x}) + O(\omega^2)$. The Hamiltonian is gauge invariant,

$$[\hat{\mathcal{T}}^p, \hat{H}] = 0, \tag{B.11}$$

and the constraints are compatible with the Heisenberg equations of motion. A formal Hilbert-space realization of the canonical commutation relations (B.10) is given by the coordinate representation

$$\langle G|\hat{G}_m^p(\mathbf{x})|\Psi\rangle = G_m^p(\mathbf{x})\langle G|\Psi\rangle, \tag{B.12}$$

$$\langle G|\hat{\Pi}_m^p(\mathbf{x})|\Psi\rangle = \frac{\delta}{i\delta G_m^p(\mathbf{x})}\langle G|\Psi\rangle, \tag{B.13}$$

with wave functionals $\Psi(G) = \langle G|\Psi\rangle$. Unlike quantization in other gauges, there are no negative norm states here, but physical states have to be gauge invariant,

$$\hat{\mathcal{T}}^p(\mathbf{x})\,|\Psi\rangle_{\rm phys} = 0. \tag{B.14}$$

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Such states can be formally written as a superposition of Wilson loops and this is useful for analytic calculations at strong coupling (on the lattice, of course, to make it well defined), but not at weak coupling.

Finally, the analogy with QED may be stressed in the notation by writing

$$E_k^p = \frac{1}{g} G^{0kp} = -g\Pi_m^p, \quad B_k^p = \frac{1}{2g}\epsilon_{klm} G_{lm}^p, \tag{B.15}$$

in terms of which

$$H = \int d^3x \left(\tfrac{1}{2}E^2 + \tfrac{1}{2}B^2\right). \tag{B.16}$$

In case other fields are present, there are additional contributions to $\mathcal{T}^p$ that act as generators for these fields, e.g. for QCD, $\rho_p = \psi^+\lambda_p\psi/2$, and (B.9) becomes the non-Abelian version of Gauss's law:

$$D_k E_k^p = g\rho_p. \tag{B.17}$$

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