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ON A THEOREM OF LICHNEROWICZ

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In his study on the structure of the complex Lie algebra of holomorphic vector fields on a compact Kähler manifold, Lichnerowicz ([3] Theorem 2, see also [1] and [4]) shows that if the first Chern class of the manifold is positive semi-definite, then to each harmonic (0.1)-form (i.e. anti-holomorphic 1-form) η, there exists a holomorphic vector field X such that the (0.1)-form $\iota(X)k$ is d''-cohomologous to η, where k is the Kähler form. The purpose of this note is to indicate that this result is a consequence of an existence theorem for solutions of a certain self-adjoint elliptic partial differential equation.

1. Let M be a connected compact Kähler manifold of complex dimension n. Let

$$g = \sum g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$$

and

$$k = \sum i g_{\alpha\bar{\beta}} dz^\alpha \Lambda d\bar{z}^\beta$$

be the fundamental tensor field and the Kähler form respectively. Let $\mathfrak{a}$ be the complex Lie algebra of holomorphic vector fields on M, and $\mathfrak{i}$ the ideal of $\mathfrak{a}$ consisting of holomorphic vector fields X such that for any holomorphic 1-form ω, $\omega(X) = 0$.

Take $X \in \mathfrak{a}$, then $\iota(X)k$ is a form of bidegree (0.1) and d''-closed. By a theorem of Hodge

$$\iota(X)k = H\iota(X)k + d''\lambda\ ,$$

where $H\iota(X)k$ is the harmonic component of $\iota(X)k$ and λ is a function on M. It is known that $X \in \mathfrak{i}$ if and only if $H\iota(X)k = 0$ (Lichnerowicz [3]). Thus, we have an injective linear map of $\mathfrak{a}/\mathfrak{i}$ into the complex

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vector space $\mathfrak{h}^{0,1}$ of anti-holomorphic 1-forms. The dimension of $\mathfrak{h}^{0,1}$ is the one half of the first Betti number $b_1(M)$.

THEOREM (*Lichnerowicz* [3]). *If the first Chern class of a connected compact Kähler manifold M is positive semi-definite, then there is a subalgebra $\mathfrak{b}$ of $\mathfrak{a}$ of dimension equal to $\frac{1}{2}b_1(M)$ such that*

$$\mathfrak{a} = \mathfrak{i} + \mathfrak{b}\,, \qquad \mathfrak{i} \cap \mathfrak{b} = 0$$

and that each vector in $\mathfrak{b}$ is nowhere vanishing.

2. Let us take an arbitrary volume element v on M, a positively oriented nowhere vanishing (n, n)-form. In terms of a local holomorphic coordinates (z^α),

$$v = (i)^{n^2} K dz^1 \wedge \cdots dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n\,.$$

Define a real closed (1.1)-form α_v on M by

$$\alpha_v = \frac{i}{2\pi} \sum \frac{\partial^2 \log K}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta\,.$$

Then, the cohomology class $[-\alpha_v]$ is the first Chern class of the manifold M.

Given a vector field X, let us denote by $\delta_v(X)$, the divergence of X, namely, $\theta(X)v = \delta_v(X)v$. The formula

$$\delta_v(fX) = f\delta_v(X) + Xf$$

for a function f is useful. From the above definition of α_v, it follows easily that if X is a holomorphic vector field and Y a vector field of bidegree (1.0),

$$\alpha_v(X, \bar{Y}) = -\bar{Y}\delta_v(X)$$

(Koszul [2]). Utilizing the Kähler connection ∇ on M and the property of the divergence above, we obtain the following formula for α_v valid for any vector field X of bidegree (1.0),

$$(1) \qquad 2\pi i \alpha_v(X, \bar{X}) = -\bar{X}\delta_v(X) + \delta_v(\nabla_{\bar{X}} X) - \rho(X)\,,$$

where, in terms of a holomorphic local coordinates (z^α),

$$\rho(X) = \sum_{\alpha,\beta} \frac{\partial \bar{\xi}^\beta}{\partial z^\alpha} \cdot \frac{\partial \xi^\alpha}{\partial \bar{z}^\beta}\,, \qquad \text{for } X = \sum \xi^\alpha \frac{\partial}{\partial z^\alpha}\,.$$

For later use, we examine $\rho(X)$ further and claim that if $d''\iota(X)k = 0$, then $\rho(X) \geqq 0$. Indeed, if this is the case, we see easily that

$$\rho(X) = g(\nabla''X, \overline{\nabla''X}) \geqq 0 . \tag{2}$$

By Stokes' Theorem, $\int_M \theta(\bar{X})(\delta_v(X)v) = 0$ and hence from (1) it follows that

$$\int_M 2\pi i\alpha_v(X, \bar{X})v = \int_M |\delta_v(X)|^2 v - \int_M \rho(X)v . \tag{3}$$

3. Now let us assume that the first Chern class of M is positive semi-definite. This means that we can choose a volume element v so that $2\pi i\alpha_v(X, \bar{X}) \geqq 0$ for any vector field X of bidegree (1.0). First, we prove that the map of $\mathfrak{a}/\mathfrak{i}$ into $\mathfrak{h}^{0.1}$ in **1** is onto, in other words, given an anti-holomorphic 1-form η, there is a holomorphic vector field X such that

$$H\iota(X)k = \eta .$$

For this purpose, we show that given an anti-holomorphic 1-form η, we can choose a function λ on M so that a vector field X of bidegree (1.0) determined by

$$\iota(X)k = \eta + d''\lambda \tag{4}$$

is of zero divergence, and hence is holomorphic on account of (2) and (3).

From (4),

$$\theta(X)k = dd''\lambda$$

and

$$\theta(X)k^n = (\tfrac{1}{2}\Delta\lambda)k^n ,$$

where Δ denotes the Laplacian associated to the Kähler metric g. Put $v = e^f k^n$ with a real valued function f on M.

$$\delta_v(X)v = \theta(X)v = \left(Xf + \frac{i}{2}\Delta\lambda\right)v .$$

Define vector fields Y and Z of bidegree (1.0) by

$$\iota(Y)k = \eta \quad \text{and} \quad \iota(Z)k = d''\lambda .$$

Then $X = Y + Z$, $Z = -\sum ig^{\alpha\bar{\beta}} \dfrac{\partial\lambda}{\partial\bar{Z}^{\beta}} \dfrac{\partial}{\partial Z^{\alpha}}$ and

$$Xf = Yf - ig(d''\lambda, d'f) .$$

Therefore, for the vector field X defined by (4), $\delta_v(X) = 0$ if and only if λ satisfies the equation

$$\Delta\lambda - 2g(d''\lambda, d'f) = -Yf \tag{5}$$

where f and Yf are given. Moreover

$$\int_M (-Yf)v = 0 . \tag{6}$$

In order to see the above equality, we remark that η is harmonic. Thus

$$\theta(Y)k = d\iota(Y)k + \iota(Y)dk = d\eta = 0 ,$$

and

$$0 = \int_M \theta(Y)v = \int_M (Yf)v + \int_M e^f\theta(Y)k^n = \int_M (Yf)v .$$

Put

$$D\lambda = \Delta\lambda - 2g(d''\lambda, d'f) .$$

Then, D is an elliptic differential operator of degree 2. By a straight forward computation, we see that D is self-adjoint with respect to the inner product

$$\langle\lambda, \mu\rangle = \int_M \lambda\bar{\mu}v .$$

The condition (6) means that the function $-Yf$ is orthogonal to the eigen space of D belonging to the eigen value 0, which consists of constant functions on M. Therefore, the equation (5) has a unique smooth solution up to an additive constant ([5] Theorem, p. 43).

We have seen that for each $\eta \in \mathfrak{H}^{0,1}$, there is a unique holomorphic vector field X such that $H\iota(X)k = \eta$ and that $\delta_v(X) = 0$.

4. Let $\mathfrak{b}$ be the set of all holomorphic vector fields X on M whose divergence $\delta_v(X) = 0$. Let us show that $\mathfrak{b} \cap \mathfrak{i} = (0)$, which is valid under no assumption on the first Chern class of M.

If $X \in \mathfrak{i}$, then $\iota(X)k = d''\lambda$ and $k(X, \bar{X}) = \bar{X}\lambda$. By Stokes' theorem, $\int_M \theta(\bar{X})(\lambda v) = 0$. Hence

$$\int_M k(X\bar{X})v = -\int_M \lambda\delta_v(\bar{X})v = 0 . \tag{7}$$

If $X \in \mathfrak{b} \cap \mathfrak{i}$, then from (7) it follows that $X = 0$.

In our case where the first Chern class is positive semi-definite, $\dim \mathfrak{b} = \frac{1}{2}b_1(M)$. We have finished the proof of the theorem.

Remark. Another theorem of Lichnerowicz ([2] Theorem 1) asserts that if the first Chern class of M is negative semi-definite, namely if $2\pi i\alpha_v(X, \bar{X}) \leq 0$ for a certain volume element v, then $\mathfrak{i} = (0)$. This fact follows immediately from (3) and (7), both being valid without any assumption on the first Chern class. Indeed, if $X \in \mathfrak{i}$, $\delta_v(X) = 0$ on account of (3) and hence $X = 0$ by (7).

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