STRONG BANDS OF GROUPS OF LEFT QUOTIENTS

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1. Introduction and preliminaries. An interesting concept of semigroups (and also rings) of (left) quotients, based on the notion of *group inverse* in a semigroup, was developed by J. B. Fountain, V. Gould and M. Petrich, in a series of papers (see [**5**]–[**12**]). Among the most interesting are semigroups having a semigroup of (left) quotients that is a union of groups. Such semigroups have been widely studied. Recall from [**3**] that a semigroup has a group of left quotients if and only if it is right reversible and cancellative. A more general result was obtained by V. Gould [**10**]. She proved that a semigroup has a semilattice of groups as its semigroup of left quotients if and only if it is a semilattice of right reversible, cancellative semigroups. This result has been since generalized by A. El-Qallali [**4**]. He proved that a semigroup has a left regular band of groups as its semigroup of left quotients if and only if it is a left regular band of right reversible, cancellative semigroups. Moreover, he proved that such semigroups can be also characterised as punched spined products of a left regular band and a semilattice of right reversible, cancellative semigroups. If we consider the proofs of their theorems, we will observe that the principal problem treated there can be formulated in the following way: Given a semigroup S that is a band B of right reversible, cancellative semigroups S_i, $i \in B$, to each S_i we can associate its group of left quotients G_i. When is it possible to define a multiplication of $Q = \bigcup_{i \in B} G_i$ such that Q becomes a semigroup having S as its left order, and especially, that Q becomes a band B of groups G_i, $i \in B$? Applying the methods developed in [**1**] (see also [**2**]), in the present paper we show how this problem can be solved for Q to become a strong band of groups (that is in fact a band of groups whose idempotents form a subsemigroup, by [**16**, Theorem 2]. Moreover, we show how Gould's and El-Quallali's constructions of semigroups of left quotients of a semilattice and a left regular band of right reversible, cancellative semigroups, can be simplified.

Throughout this paper, for a semilattice $Y, S = (Y; S_\alpha)$ will mean that a semigroup S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$. Especially, for a band $B, B = (Y; B_\alpha)$ will mean that B is a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$ (i.e. Y is the greatest semilattice homomorphic image of B). For a congruence ρ, $\rho^\natural$ will denote its natural homomorphism.

Let B be a band. By $\leq$ we will denote the natural partial order on B, i.e. a relation on B defined by: $j \leq i \Leftrightarrow ij = ji = j$ $(i, j \in B)$, and $\preccurlyeq$ will denote a quasi-order on B defined by: $j \preccurlyeq i \Leftrightarrow j = jij$ $(i, j \in B)$. Clearly, $\leq$ and $\preccurlyeq$ coincide if and only if B is a semilattice. Further, for $i \in B$, $[i]$ will denote the class of i with respect to the smallest semilattice congruence on B. It is easy to verify that $j \preccurlyeq i \Leftrightarrow [j] \leq [i]$, for all $i, j \in B$.

Let B be a band. To each $i \in B$ we associate a semigroup S_i and an oversemigroup D_i of S_i such that $D_i \cap D_j = \emptyset$, if $i \neq j$. For $i, j \in B$, $i \succcurlyeq j$, let $\phi_{i,j}$ be a mapping of S_i into D_j and suppose that the family of $\phi_{i,j}$ satisfies the following conditions:

(1) $\phi_{i,i}$ is the identity mapping on S_i, for each $i \in B$;

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(2) $(S_i\phi_{i,ij})(S_j\phi_{j,ij}) \subseteq S_{ij}$, for all $i, j \in B$;

(3) $[(a\phi_{i,ij})(b\phi_{j,ij})]\phi_{ij,k} = (a\phi_{i,k})(b\phi_{j,k})$, for $a \in S_i$, $b \in S_j$, $ij \geqslant k$, $i, j, k \in B$.

Define a multiplication $*$ on $S = \bigcup_{i \in B} S_i$ by:

(4) $a * b = (a\phi_{i,ij})(b\phi_{j,ij}) \quad (a \in S_i, b \in S_j)$.

Then S is a band B of semigroups S_i, $i \in B$, in notation $S = (B; S_i, \phi_{i,j}, D_i)$ [**1**]. The symbol "$*$" will be further omitted. If we assume $i = j$ in (3), then we obtain that $\phi_{i,k}$ is a homomorphism, for all $i, k \in B$, $i \geqslant k$.

Further, if $D_i = S_i$, for each $i \in B$, then we write $S = (B; S_i, \phi_{i,j})$. Here the condition (2) can be omitted. If $S = (B; S_i, \phi_{i,j})$ and if $\{\phi_{i,j} \mid i, j \in B, i \geqslant j\}$ is a *transitive system of homomorphisms*, i.e. if $\phi_{i,j}\phi_{j,k} = \phi_{i,k}$, for $i \geqslant j \geqslant k$, then we will write $S = [B; S_i, \phi_{i,j}]$, and we will say that S is a *strong band* B of semigroups S_i. In the case when B is a semilattice, we obtain a *strong semilattice* of semigroups.

If P and Q are two semigroups with a common homomorphic image Y, then a *spined product of* P *and* Q *with respect to* Y is $S = \{(a, b) \in P \times Q \mid a\varphi = b\psi\}$, where $\varphi : P \to Y$ and $\psi : Q \to Y$ are homomorphisms onto Y. If $P_\alpha = \alpha\varphi^{-1}$, $Q_\alpha = \alpha\psi^{-1}$, $\alpha \in Y$, then $S = \bigcup_{\alpha \in Y} P_\alpha \times Q_\alpha$. Clearly, S is a subdirect product of P and Q. A *punched spined product of* P *and* Q *with respect to* Y is any semigroup isomorphic to some subdirect product of P and Q contained in their spined product with respect to Y [**4**].

An element a of a semigroup S is *completely regular* if there exists $x \in S$ such that $a = axa$ and $ax = xa$. It is well known that a is completely regular if and only if it lies in some subgroup of S, so completely regular elements will be also called *group elements*. If a is completely regular, then there exists a unique $x \in S$ such that $a = axa$, $x = xax$ and $ax = xa$, which is the inverse of a in the maximal subgroup of S containing it, so such an element will be called a *group inverse* of a and it will be denoted by a^{-1}.

An element a of a semigroup S is *square-cancellable* if, for all $x, y \in S^1$,

$$a^2x = a^2y \text{ implies } ax = ay \quad \text{and} \quad xa^2 = ya^2 \text{ implies } xa = ya.$$

Let S be a subsemigroup of a semigroup Q. Recall from [**9**] that S is a *left order* in Q or that Q is a *semigroup of left quotients* of S if

(i) every square-cancellable element of S lies in a subgroup of Q;

(ii) every element q of Q can be written as $q = a^{-1}b$, for some elements $a, b \in S$.

Clearly, if Q is a union of groups, then the condition (i) can be omitted.

A semigroup S is *right reversible* if $Sa \cap Sb \neq \varnothing$, for all $a, b \in S$.

For undefined notions and notation we refer to [**3**], [**13**] and [**15**].

2. The main results.

First we will prove the following lemma.

Lemma 1. *Let* $S = (B; S_i, \phi_{i,j}, G_i)$, *and for each* $i \in B$, *let* S_i *be a right reversible, cancellative semigroup with* G_i *as its group of left quotients. Then, for all* $i, j \in B$, $i \geqslant j$, $\phi_{i,j}$ *can be extended to a homomorphism* $\varphi_{i,j}$ *of* G_i *into* G_j *such that there exists a* $Q = [B; G_i, \varphi_{i,j}]$.

Proof. Let $\circ$ denote the multiplications in groups G_i, $i \in B$. For $i, j \in B$, $i \geqslant j$, $\phi_{i,j}$ can be (uniquely) extended to a homomorphism $\varphi_{i,j}$ of G_i into G_j and then for $a, b \in S_i$ we

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have

$$(a^{-1}\circ b)\varphi_{i,j}=(a\phi_{i,j})^{-1}\circ(b\phi_{i,j}).$$

Let us prove that $\{\varphi_{i,j} \mid i,j\in B, i\geqslant j\}$ is a transitive system of homomorphisms. Since G_i is the group of left quotients of S_i, for any $i\in B$, it is enough to show that $a\varphi_{i,j}\varphi_{j,k}=a\varphi_{i,k}$, for all $i,j,k\in B$ such that $i\geqslant j\geqslant k$ and any $a\in S_i$. Assume $x,y\in S_j$ such that $a\varphi_{i,j}=x^{-1}\circ y$, i.e. $x\circ(a\varphi_{i,j})=y$. Then $yx=y\circ x=x\circ(a\varphi_{i,j})\circ x=x\circ(a\phi_{i,j})\circ x$. By (3) and (4) it follows that $xax=x\circ(a\phi_{i,j})\circ x$, and hence $yx=xax$. Again by (3) and (4) we obtain

$$(y\phi_{j,k})\circ(x\phi_{j,k})=(yx)\phi_{j,k}=(xax)\phi_{j,k}=(x\phi_{j,k})\circ(a\phi_{i,k})\circ(x\phi_{j,k}),$$

whence $y\phi_{j,k}=(x\phi_{j,k})\circ(a\phi_{i,k})$, by the cancellativity in G_k. Hence,

$$a\varphi_{i,k}=a\phi_{i,k}=(x\phi_{j,k})^{-1}\circ(y\phi_{j,k})=(x^{-1}\circ y)\varphi_{j,k}=a\varphi_{i,j}\varphi_{j,k},$$

which was to be proved.

Now we go to the main theorem of this paper.

THEOREM 1. *The following conditions on a semigroup S are equivalent*:

(i) *S is a left order in a strong band of groups*;

(ii) *$S=(B;S_i,\phi_{i,j},G_i)$, where, for each $i\in B$, S_i is a right reversible, cancellative semigroup with G_i as its group of left quotients*;

(iii) *S is a punched spined product of a band $B=(Y;B_\alpha)$ and a semigroup $T=(Y;T_\alpha)$, with respect to a semilattice Y, where, for each $\alpha\in Y$, T_α is a right reversible, cancellative semigroup.*

Proof. (i)⇒(ii). This follows immediately by Propositions 2 and 4 of [**11**].

(ii)⇒(i). This follows by Lemma 1.

(ii)⇒(iii). Let $B=(Y;B_\alpha)$. By [**1**, Theorem 2], S is a semilattice Y of semigroups $(B_\alpha;S_i,\phi_{i,j},G_i)$, a relation ρ on S defined by: $a\ \rho\ b$ if and only if $a\in S_i$, $b\in S_j$, $i,j\in B$, $[i]=[j]$ and $a\phi_{i,k}=b\phi_{j,k}$, for each $k\in B$ such that $i,j\geqslant k$, is a congruence, $T=S/\rho$ is a semilattice Y of semigroups $T_\alpha=S_\alpha\rho^\natural$, $\alpha\in Y$, and S is a punched spined product of B and T with respect to Y. It remains to prove that for each $\alpha\in Y$, T_α is cancellative and right reversible.

Let $\alpha\in Y$. Assume $u,v,w\in T_\alpha$ such that $uw=vw$. Then $u=a\rho^\natural$, $v=b\rho^\natural$ and $w=c\rho^\natural$, for some $a,b,c\in S_\alpha$. Let $a\in S_i$, $b\in S_j$, $c\in S_k$, for some $i,j,k\in B_\alpha$. Assume $l\in B$ such that $i,j\geqslant l$. Then k, ik, $jk\geqslant l$ and

$$\begin{aligned}(a\phi_{i,l})(c\phi_{k,l})&=[(a\phi_{i,ik})(c\phi_{k,ik})]\phi_{ik,l}=(ac)\phi_{ik,l}\\&=(bc)\phi_{jk,l}=[(b\phi_{j,jk})(c\phi_{k,jk})]\phi_{jk,l}=(b\phi_{j,l})(c\phi_{k,l}),\end{aligned}$$

since $ac\ \rho\ bc$, i.e. $uw=vw$. Now, by the cancellativity in G_l, $a\phi_{i,l}=b\phi_{j,l}$. Thus, $a\ \rho\ b$, i.e. $u=v$. Hence, T_α is right cancellative. Similarly we prove left cancellativity in T_α.

Let $u,v\in T_\alpha$. Then $u=a\rho^\natural$, $v=b\rho^\natural$, for some $a,b\in S_\alpha$, and $a\in S_i$, $b\in S_j$, for some $i,j\in B_\alpha$. By Lemma 1, for all $i,j\in B$, $i\geqslant j$, $\phi_{i,j}$ can be extended to a homomorphism $\varphi_{i,j}$ of G_i into G_j such that there exists a $Q=[B;G_i,\varphi_{i,j}]$. Now $(a\phi_{i,ij})(b\phi_{j,ij})^{-1}\in G_{ij}$, so $(a\phi_{i,ij})(b\phi_{j,ij})^{-1}=x^{-1}y$, for some $x,y\in S_{ij}$, i.e. $x(a\phi_{i,ij})=y(b\phi_{j,ij})$, whence

$$yb=y(b\phi_{j,ij})=x(a\phi_{i,ij})=(xa)\phi_{iji,ij}.$$

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Assume $k \in B$ such that $ij, iji \geq k$. Then

$$(yb)\phi_{ij,k} = (xa)\phi_{iji,ij}\phi_{ij,k} = (xa)\varphi_{iji,ij}\varphi_{ij,k} = (xa)\varphi_{iji,k} = (xa)\phi_{iji,k}.$$

Therefore, $yb \ \rho \ xa$, whence $(x\rho^{\natural})u = (y\rho^{\natural})v$, so T_α is right reversible.

(iii)$\Rightarrow$(i). Without loss of generality we can assume that $S \subseteq B \times T$, i.e. $S \subseteq \bigcup_{\alpha \in Y} B_\alpha \times T_\alpha$. By [**10**, Theorem 3.1], T is a left order in a semigroup Q, where $Q = (Y; G_\alpha)$ and for each $\alpha \in Y$, G_α is a group, and also, for each $\alpha \in Y$, G_α is a group of left quotients of T_α. Let P be the spined product of B and Q with respect to Y, i.e. let $P = \bigcup_{\alpha \in Y} B_\alpha \times G_\alpha$. By [**16**, Theorem 4] (see also [**14**, Theorem 3.2]), P is a strong band of groups. It remains to prove that P is a semigroup of left quotients of S. Assume an arbitrary $(i, a) \in P$. Then $i \in B_\alpha$, $a \in G_\alpha$, for some $\alpha \in Y$. Since S is a subdirect product of B and T, there exists $b \in T$ such that $(i, b) \in S$, and hence $b \in T_\alpha$. Thus, $ba^{-1} \in G_\alpha$, so $ba^{-1} = x^{-1}y$, for some $x, y \in T_\alpha$, and further, there exists $j, k \in B$ such that (j, x), $(k, y) \in S$. Now $j, k \in B_\alpha$, so $(i, bxb) = (i, b)(j, x)(i, b) \in S$, $(ik, by) = (i, b)(k, y) \in S$, and $(by)^{-1}bxb = y^{-1}b^{-1}bxb = y^{-1}xb = (ba^{-1})^{-1}b = ab^{-1}b = a$, whence

$$(i, a) = (i, (by)^{-1}bxb) = (ik, by)^{-1}(i, bxb).$$

Therefore, P is a semigroup of left quotients of S.

Semigroups having a rectangular group of (left) quotients have been considered by several authors. By Theorem 1 we obtain the following corollary.

Corollary 1. *The following conditions on a semigroup S are equivalent*:

(i) *S is a left order in a rectangular group*;

(ii) *$S = (B; S_i, \phi_{i,j}, G_i)$, where B is a rectangular band and for each $i \in B$, S_i is a right reversible, cancellative semigroup with G_i as its group of left quotients*;

(iii) *S is a subdirect product of a rectangular band and a right reversible, cancellative semigroup.*

Finally, the next theorem, together with Lemma 1, shows how Gould's and El-Qallali's constructions of semigroups of left quotients of a semilattice and a left regular band of right reversible, cancellative semigroups, can be simplified.

Theorem 2. *Let S be a left regular band B of right reversible, cancellative semigroups S_i, $i \in B$, and for each $i \in B$, let G_i be the group of left quotients of S_i. Then $S = (B; S_i, \phi_{i,j}, G_i)$.*

Proof. Let $\circ$ denote the multiplications in groups G_i, $i \in B$, and let $\{u_i \mid i \in B\} \subseteq S$ such that $u_i \in S_i$, for each $i \in B$. For $i, j \in B$, $i \geq j$, define a mapping $\phi_{i,j} : S_i \to G_j$ by:

$$a\phi_{i,j} = u_j^{-1} \circ (u_j a) \quad (a \in S_i).$$

Since B is a left regular band, then $u_j a \in S_j$, so $u_j^{-1} \circ (u_j a) \in G_j$. Clearly (1) holds. Assume

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$i, j \in B$, $a \in S_i, b \in S_j$. Further, since B is a left regular band, then $u_{ij}, u_{ij}a \in S_{ij}$, so $vu_{ij}a = wu_{ij}$, for some $v, w \in S_{ij}$. Now

$$\begin{aligned}(a\phi_{i,ij}) \circ (b\phi_{j,ij}) &= u_{ij}^{-1} \circ (u_{ij}a) \circ u_{ij}^{-1} \circ (u_{ij}b) \\ &= u_{ij}^{-1} \circ v^{-1} \circ v \circ (u_{ij}a) \circ u_{ij}^{-1} \circ (u_{ij}b) = u_{ij}^{-1} \circ v^{-1} \circ w \circ u_{ij} \circ u_{ij}^{-1} \circ (u_{ij}b) \\ &= u_{ij}^{-1} \circ v^{-1} \circ w \circ (u_{ij}b) = u_{ij}^{-1} \circ v^{-1} \circ (wu_{ij}b) = u_{ij}^{-1} \circ v^{-1} \circ (vu_{ij}ab) \\ &= u_{ij}^{-1} \circ v^{-1} \circ v \circ (u_{ij}ab) = u_{ij}^{-1} \circ (u_{ij}ab) = u_{ij}^{-1} \circ u_{ij} \circ (ab) = ab.\end{aligned}$$

Therefore, (2) and (4) hold.

Assume $i, j, k \in B$, $ij \geqslant k$, $a \in S_i$, $b \in S_j$. Since B is a left regular band, then $u_k a, u_k \in S_k$, whence $vu_k a = wu_k$, for some $v, w \in S_k$. Now

$$\begin{aligned}[(a\phi_{i,ij}) \circ (b\phi_{j,ij})]\phi_{ij,k} &= (ab)\phi_{ij,k} = u_k^{-1} \circ (u_k ab) = u_k^{-1} \circ v^{-1} \circ (vu_k ab) \\ &= u_k^{-1} \circ v^{-1} \circ (wu_k b) = u_k^{-1} \circ v^{-1} \circ w \circ (u_k b) \\ &= u_k^{-1} \circ v^{-1} \circ w \circ u_k \circ u_k^{-1} \circ (u_k b) = u_k^{-1} \circ v^{-1} \circ v \circ (u_k a) \circ u_k^{-1} \circ (u_k b) \\ &= u_k^{-1} \circ (u_k a) \circ u_k^{-1} \circ (u_k b) = (a\phi_{i,k}) \circ (b\phi_{j,k}).\end{aligned}$$

Therefore, $S = (B; S_i, \phi_{i,j}, G_i)$.

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REFERENCES

1. M. Ćirić and S. Bogdanović, Spined products of some semigroups, *Proc. Japan Acad. Ser. A Math. Sci.* **69** (1993), 357–362.

2. M. Ćirić and S. Bogdanović, Subdirect products of a band and a semigroup, to appear.

3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol I (American Mathematical Society, 1961).

4. A. El-Qallali, Left regular bands of groups of left quotients, *Glasgow Math. J.* **33** (1991), 29–40.

5. J. B. Fountain and V. Gould, Completely 0-simple semigroups of quotients. II. *Contributions to general algebra*, 3 (Hölder-Pichler-Tempsky, 1985), 115–124.

6. J. B. Fountain and M. Petrich, Brandt semigroups of quotients, *Math. Proc. Cambridge Philos. Soc.* **98** (1985), 413–426.

7. J. B. Fountain and M. Petrich, Completely 0-simple semigroups of quotients, *J. Algebra* **101** (1986), 365–402.

8. J. B. Fountain and M. Petrich, Completely 0-simple semigroups of quotients. III, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), 263–275.

9. V. Gould, Bisimple inverse ω-semigroups of left quotients, *Proc. London. Math. Soc.* (3) **52** (1986), 95–118.

10. V. Gould, Clifford semigroups of left quotients, *Glasgow Math. J.* **28** (1986), 181–191.

11. V. Gould, Orders in semigroups, *Contributions to general algebra*, 5 (Hölder-Pichler-Tempsky, 1987), 163–169.

12. V. Gould, Semigroups of left quotients—the uniqueness problem, *Proc. Edinburgh Math. Soc.* **35** (1992), 213–226.

13. J. M. Howie, *An introduction to semigroup theory* (Academic Press, 1976).

14. M. Petrich, Regular semigroups which are subdirect products of a band and a semilattice of groups, *Glasgow Math. J.* **14** (1973), 27–49.

15. M. Petrich, *Introduction to semigroups* (Merill, 1973).
16. B. M. Schein, Bands of monoids, *Acta Sci. Math.* (*Szeged*) **36** (1974), 145–154.
17. M. Yamada, Strictly inversive semigroups, *Bull. Shimane Univ.* **13** (1964), 128–138.

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