DESCRIPTIVE PROPERTIES OF I2-EMBEDDINGS

VINCENZO DIMONTE, MARTINA IANNELLA, AND PHILIPP LÜCKE

Abstract. We contribute to the study of generalizations of the Perfect Set Property and the Baire Property to subsets of spaces of higher cardinalities, like the power set $\mathcal{P}(\lambda)$ of a singular cardinal λ of countable cofinality or products $\prod_{i<\omega} \lambda_i$ for a strictly increasing sequence $\langle \lambda_i \mid i < \omega \rangle$ of cardinals. We consider the question under which large cardinal hypothesis classes of definable subsets of these spaces possess such regularity properties, focusing on rank-into-rank axioms and classes of sets definable by Σ_1-formulas with parameters from various collections of sets. We prove that ω-many measurable cardinals, while sufficient to prove the perfect set property of all Σ_1-definable sets with parameters in $V_\lambda \cup \{V_\lambda\}$, are not enough to prove it if there is a cofinal sequence in λ in the parameters. For this conclusion, the existence of an I2-embedding is enough, but there are parameters in $V_{\lambda+1}$ for which I2 is still not enough. The situation is similar for the Baire property: under I2 all sets that are Σ_1-definable using elements of V_λ and a cofinal sequence as parameters have the Baire property, but I2 is not enough for some parameter in $V_{\lambda+1}$. Finally, the existence of an I0-embedding implies that all sets that are Σ^1_n-definable with parameters in $V_{\lambda+1}$ have the Baire property.

§1. Introduction. Fundamental results of descriptive set theory show that simply definable sets of real numbers, e.g., Borel sets, possess a rich and canonical structure theory and these structural results have various applications in other areas of mathematics. Moreover, seminal results show that canonical extensions of the axioms of ZFC allow us to extend these structural conclusions to much larger classes of definable sets of reals. Since the developed theory is limited to the study of mathematical objects that can be encoded as definable sets of real numbers, there has been a recent interest to develop a *generalized descriptive set theory* that allows the study of definable objects of much higher cardinalities. While it is already known that several key results of the classical theory cannot be directly generalized to all higher cardinalities (see, for example, [14]), the research conducted so far in this area isolated several settings in which rich structure theories for definable sets of higher cardinalities can be developed. The work presented in this paper contributes to the study of one of these settings that originates in Hugh Woodin‘s work on large cardinal assumptions close to the *Kunen Inconsistency* (see [18]).

Remember that a non-trivial elementary embedding $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ for some ordinal λ is an *I0-embedding* if $\mathrm{crit}(j) < \lambda$ holds. Kunen's analysis of elementary embeddings in [10] then directly shows that $\lambda = \sup_{n<\omega} \lambda_n$ holds for

Received October 31, 2023.
2020 *Mathematics Subject Classification.* Primary 03E55, Secondary 03E35, 03E45, 03E47.
Key words and phrases. I2-embeddings, definability, perfect subsets, Baire property, absoluteness.

0022-4812/00/0000-0000
DOI:10.1017/jsl.2024.75

every I0-embedding $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ with critical sequence[1] $\langle \lambda_n \mid n < \omega \rangle$. Embeddings of this type produce a setting in which descriptive concepts can be developed fruitfully. More specifically, several deep results show that the structural properties of the collection of subsets of $\mathcal{P}(\lambda)$ contained in $L(V_{\lambda+1})$ strongly resembles the behavior of the collection of sets of reals in $L(\mathbb{R})$ in the presence of the *Axiom of Determinacy* AD in $L(\mathbb{R})$. In the following, we will focus on generalizations of the *Perfect Set Property* to definable subsets of higher power sets. Given a non-empty set X and an infinite cardinal μ, we equip the set ${}^{\mu}X$ of all functions from μ to X with the topology whose basic open sets consists of all functions that extend a given function $s : \xi \longrightarrow X$ with $\xi < \mu$. In addition, we equip the set $\mathcal{P}(\nu)$ of all subsets of an infinite cardinal ν with the topology whose basic open sets consist of all subsets of ν whose intersection with a given ordinal $\eta < \nu$ is equal to a fixed subset of η. Finally, we say that a map $\iota : X \longrightarrow Y$ between topological spaces is a *perfect embedding* if it induces a homeomorphism between X and the subspace $\mathrm{ran}(\iota)$ of Y.

It is easy to see that for every infinite cardinal λ, there is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that does not contain the range of a perfect embedding of ${}^{\mathrm{cof}(\lambda)}\lambda$ into $\mathcal{P}(\lambda)$.[2] In contrast, classical results show that if AD holds in $L(\mathbb{R})$, then every uncountable subset of $\mathcal{P}(\omega)$ in $L(\mathbb{R})$ contains the range of a perfect embedding of ${}^{\omega}\omega$ into $\mathcal{P}(\omega)$. The work of Hugh Woodin, Xianghui Shi, and Scott Cramer now shows that I0-embeddings imply an analogous dichotomy at the supremum of the corresponding critical sequence (see [1, Section 5], [16, Section 4], and [18, Section 7]).

Theorem 1.1 [1]. *If $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ is an I0-embedding and X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is an element of $L(V_{\lambda+1})$, then there is a perfect embedding $\iota : {}^{\omega}\lambda \longrightarrow \mathcal{P}(\lambda)$ with* $\mathrm{ran}(\iota) \subseteq X$.

The work presented in this paper is motivated by the question whether the restriction of this implication to smaller classes of definable sets can be derived from weaker large cardinal assumptions. It is motivated by the results of Sandra Müller and the third author in [13] that analyze simply definable sets at limits of measurable cardinals. In the following, we say a class C is *definable by a formula* $\varphi(v_0, \ldots, v_n)$ *and parameters* $z_0, \ldots, z_{n-1}$ if $C = \{y \mid \varphi(y, z_0, \ldots, z_{n-1})\}$ holds. We now distinguish classes of definable sets using the *Levy hierarchy of formulas*[3] and the rank of parameters. The following result is the starting point of our work.

Theorem 1.2 [13]. *If λ is a limit of measurable cardinals and X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{\lambda\}$, then there is a perfect embedding $\iota : {}^{\mathrm{cof}(\lambda)}\lambda \longrightarrow \mathcal{P}(\lambda)$ with* $\mathrm{ran}(\iota) \subseteq X$.

---

[1] We say that a sequence $\langle \lambda_n \mid n < \omega \rangle$ of ordinals is the *critical sequence* of a non-trivial elementary embedding $j : M \longrightarrow N$ between transitive classes if $\lambda_0 = \mathrm{crit}(j)$ and $j(\lambda_n) = \lambda_{n+1}$ holds for all $n < \omega$.

[2] First, observe that for every $\gamma < \lambda$, the set $\mathcal{P}(\gamma)$ is discrete in $\mathcal{P}(\lambda)$ and therefore it does not contain the range of a perfect embedding of ${}^{\mathrm{cof}(\lambda)}\lambda$ into $\mathcal{P}(\lambda)$. In particular, if $2^{<\lambda} > \lambda$, then there is a subset of $\mathcal{P}(\lambda)$ with the desired property. In the other case, if $2^{<\lambda} = \lambda$, then the set of perfect embeddings of ${}^{\mathrm{cof}(\lambda)}\lambda$ into $\mathcal{P}(\lambda)$ has cardinality 2^λ and we can build the desired subset through a standard recursive construction.

[3] See [9, p. 5].

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

Given an infinite cardinal λ, the Σ_1-*Reflection Principle* shows that all Σ_1-formulas with parameters in H_{λ^+} are absolute between V and H_{λ^+}. Therefore, it follows that a subset of H_{λ^+} is definable by a Σ_1-formula with parameters in H_{λ^+} if and only if the given set is definable in this way in H_{λ^+}. This shows that, if λ is an infinite cardinal with $H_\lambda = V_\lambda$, then $L(V_{\lambda+1})$ contains all subsets of $\mathcal{P}(\lambda)$ that are definable by a Σ_1-formula with parameters in H_{λ^+}, because H_{λ^+} is contained in $L(V_{\lambda+1})$. In particular, it follows that the conclusion of the implication stated in Theorem 1.1 directly implies the conclusion of the implication stated in Theorem 1.2.

The theorems cited above directly raise the question if stronger perfect set theorems can be proven for limits of countably many measurable cardinals. In particular, it is natural to ask if the implication of Theorem 1.2 still holds true if we allow more elements of $\mathcal{P}(\lambda)$ in our Σ_1-definitions. A natural candidate for such an additional parameter in $\mathcal{P}(\lambda) \setminus (V_\lambda \cup \{\lambda\})$ is an ω-sequence of measurable cardinals that is cofinal in the given supremum λ. Our first result, proven in Section 2, shows that we no longer get a provable implication if we are allowed to use such a sequence as a parameter in our Σ_1-definitions:

THEOREM 1.3. *If $\vec{\lambda}$ is a strictly increasing sequence of measurable cardinals with limit λ, then the following statements hold in an inner model M containing $\vec{\lambda}$:*

(i) *The sequence $\vec{\lambda}$ consists of measurable cardinals.*
(ii) *If $\vec{\nu}$ is a strictly increasing ω-sequence of regular cardinals with limit λ, then there is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that does not contain the range of a perfect embedding of ${}^\omega\lambda$ into $\mathcal{P}(\lambda)$ and is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{\vec{\nu}\}$.*

We now proceed by showing that a large cardinal axiom strictly weaker than the existence of an I0-embedding implies the perfect set property discussed above. Remember that an elementary embedding $j : V \longrightarrow M$ with critical sequence $\langle \lambda_n \mid n < \omega \rangle$ is an *I2-embedding* if $V_\lambda \subseteq M$, where $\lambda = \sup_{n<\omega} \lambda_n$. The existence of such an embedding is equivalent to the existence of a non-trivial elementary embedding $i : V_\lambda \longrightarrow V_\lambda$ with critical sequence $\langle \nu_n \mid n < \omega \rangle$ such that $\lambda = \sup_{n<\omega} \nu_n$ and the canonical map

$$i_+ : V_{\lambda+1} \longrightarrow V_{\lambda+1};\ A \longmapsto \bigcup \{ i(A \cap V_{\lambda_n}) \mid n < \omega \}$$

extending i to $V_{\lambda+1}$ maps well-founded relations on V_λ to well-founded relations on V_λ (see [9, Proposition 24.2]). The results of [11] show that, if $i : L(V_{\nu+1}) \longrightarrow L(V_{\nu+1})$ is an I0-embedding, then there is an embedding $j : V_\lambda \longrightarrow V_\lambda$ for some $\lambda < \nu$ with the given property. Since ν is a limit of inaccessible cardinals in this setting, it follows that the existence of an I0-embedding has strictly higher consistency strength than the existence of an I2-embedding. The next result, proven in Section 3, shows that I2-embeddings imply the desired perfect set property.

THEOREM 1.4. *Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and set $\lambda = \sup_{n<\omega} \lambda_n$. If X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{V_\lambda, \vec{\lambda}\}$, then there is a perfect embedding $\iota : {}^\omega\lambda \longrightarrow \mathcal{P}(\lambda)$ with* $\operatorname{ran}(\iota) \subseteq X$.

The proof of this theorem will show that its conclusion holds for subsets of $\mathcal{P}(\lambda)$ that are definable from a significantly larger set of parameters in $V_{\lambda+1}$ (see

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

Theorem 3.1 below). However, in Section 2, we will observe that an assumption strictly stronger than the existence of an I2-embedding is necessary to obtain this perfect set property for all subsets of $\mathcal{P}(\lambda)$ that are definable by Σ_1-formulas with parameters in $\mathcal{P}(\lambda)$.

THEOREM 1.5. *If $j : V \longrightarrow M$ is an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and $\lambda = \sup_{n<\omega} \lambda_n$, then the following statements hold in an inner model:*
(i) *There is an I2-embedding whose critical sequence has supremum λ.*
(ii) *There is a subset z of λ and a subset X of $\mathcal{P}(\lambda)$ of cardinality greater than λ such that X does not contain the range of a perfect embedding of ${}^{\omega}\lambda$ into $\mathcal{P}(\lambda)$ and the set X is definable by a Σ_1-formula with parameter z.*

The five results discussed above suggest the intriguing possibility of studying large cardinal assumptions canonically inducing singular cardinals λ of countable cofinality through the provable regularity properties of simply definable subsets of $\mathcal{P}(\lambda)$. More specifically, they suggest that for each large cardinal axiom of this form, we want to uniformly assign as large subsets P_λ of $V_{\lambda+1}$ as possible to each singular cardinal λ, in a way that ensures that ZFC proves that whenever λ is a singular cardinal of countable cofinality induced by a cardinal of the given type, then all subsets of $\mathcal{P}(\lambda)$ that are definable by Σ_1-formulas using parameters from P_λ either have cardinality at most λ or contain the range of a perfect embedding of ${}^{\omega}\lambda$ into $\mathcal{P}(\lambda)$. Note that, since this approach is based on provable implications and not consistency strength, it is less affected by the current technical limitations of inner model theory and therefore provides a new angle to study strong large cardinal axioms.

In addition to Σ_1-definable subsets of power sets, we will also study spaces and complexity classes that more closely resemble the objects studied in classical descriptive set theory. More specifically, for a given strictly increasing sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ of infinite cardinals with supremum λ, we will study subsets of the closed subspace $C(\vec{\lambda})$ of ${}^{\omega}\lambda$ consisting of all functions in the set $\prod_{n<\omega} \lambda_n$, i.e., all functions $x : \omega \longrightarrow \lambda$ satisfying $x(n) < \lambda_n$ for all $n < \omega$. Note that the map

$$\iota_{\vec{\lambda}} : C(\vec{\lambda}) \longrightarrow \mathcal{P}(\lambda);\ x \longmapsto \{\prec\lambda_n, x(n)\succ \mid n < \omega\}$$

yields a homeomorphism between $C(\vec{\lambda})$ and a closed subset of $\mathcal{P}(\lambda)$.[4] Moreover, since the map $\iota_{\vec{\lambda}}$ is definable by a Δ_0-formula with parameter $\vec{\lambda}$, Theorem 1.4 immediately implies a perfect set theorem for subsets of $C(\vec{\lambda})$ definable by Σ_1-formulas with parameters in the set $V_\lambda \cup \{\vec{\lambda}\}$. Finally, the sets produced in the proofs of Theorems 1.3 and 1.5 will actually be subsets of $\mathrm{ran}(\iota_{\vec{\lambda}})$ and therefore yield analogous negative results for Σ_1-definable subsets of $C(\vec{\lambda})$ (see Theorems 2.3 and 2.4 below).

The theorems above extends beyond $\mathcal{P}(\lambda)$ and $C(\vec{\lambda})$: In [4] a whole classes of spaces is introduced: the λ-Polish spaces, i.e., spaces that are completely metrizable and with weight λ, and it is easy to prove analogous results for them. For example, ${}^{\lambda}2$, with the bounded topology, is homeomorphic to $\mathcal{P}(\lambda)$, and therefore Theorems 1.2, 1.3, 1.4, and 1.5 hold in there. The space ${}^{\omega}\lambda$, with the product topology,

[4]Here, we let $\prec\cdot,\cdot\succ : \mathrm{Ord} \times \mathrm{Ord} \longrightarrow \mathrm{Ord}$ denote the *Gödel pairing function*.

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

is homeomorphic to a closed subset of $\mathcal{P}(\lambda)$ via the map $x \mapsto \prec n, x(n) \succ$, and it contains $C(\vec{\lambda})$ as a closed set, therefore Theorems 1.3, 1.4, and 1.5 hold in there. If (X, d) is any λ-Polish space, then there is a $\Sigma_1(d)$ continuous bijection between a closed set $F \subseteq {}^{\lambda}\omega$ and X ([4]). By pulling back with the bijection, we can therefore prove Theorems 1.2, 1.3, 1.4, and 1.5 also in there. Finally, if d is more complicated, the negative results of Theorems 1.3 and 1.5 hold anyway, but respectively with a witness in $\Sigma_1(V_\lambda \cup \{V_\lambda, \vec{\lambda}, d\})$ and in $\Sigma_1(z, d)$.

In another direction, we will not only study subsets of $\mathcal{P}(\lambda)$, ${}^{\omega}\lambda$ or $C(\vec{\lambda})$ that are definable in V by formulas of a given complexity, but also sets that are definable over V_λ by higher-order formulas in the classes Σ^m_n and Π^m_n (see, for example, [8, p. 295]) using certain parameters contained in $V_{\lambda+1}$. The following results (whose proof is a routine adaptation of the proof of [8, Lemma 25.25] to higher cardinals of countable cofinality) connects this form of definability to Σ_1-definitions:

Proposition 1.6. *For every Σ_1-formula $\varphi(v_0, \dots, v_{k-1})$ in the language of set theory, there exists a Σ^1_2-formula $\psi(w_0, \dots, w_{k-1})$ in the language of set theory with free second-order parameters $w_0, \dots, w_{k-1}$ such that* ZFC *proves that*

$$\varphi(A_0, \dots, A_{k-1}) \iff \langle V_\lambda, \in \rangle \models \psi(A_0, \dots, A_{k-1})$$

holds for every singular cardinal λ of countable cofinality with $H_\lambda = V_\lambda$ and all $A_0, \dots, A_{k-1} \in V_{\lambda+1}$.

We will later show (see Corollary 3.3) that, in certain contexts, it is also possible to translate Σ^1_2-formulas into Σ_1-formulas. Moreover, note that, in [4], Luca Motto Ros and the first author prove that, analogous to the classical setting, for every singular strong limit cardinal λ of countable cofinality, every $\mathbf{\Sigma}^1_1$-subset of ${}^{\omega}\lambda$ (i.e., every subset of ${}^{\omega}\lambda$ that is definable over V_λ by a Σ^1_1-formula with parameters in $V_{\lambda+1}$) of cardinality greater than λ contains the range of a perfect embedding of ${}^{\omega}\lambda$ into itself. In addition, still completely analogous to the classical setting, they show that, if $V = L$ holds and λ is a singular cardinal of countable cofinality, then there is a subset of ${}^{\omega}\lambda$ of cardinality λ^+ that is definable over V_λ by a Π^1_1-formula without parameters.

In addition, we later will consider an analog of the Baire Property to λ, that we call *$\vec{\mathcal{U}}$-Baire property* (see Definition 4.3 below). In analogy with Theorem 1.4, the existence of an I2-embedding with supremum of the critical sequence λ implies that every subset of $\mathrm{C}(\vec{\lambda})$ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{V_\lambda, \vec{\lambda}\}$ has the $\vec{\mathcal{U}}$-Baire property (see Theorem 4.12 below). Moreover, in analogy with Theorem 1.1, the existence of an I0-embedding with supremum of the critical sequence λ implies that every subset of $\mathrm{C}(\vec{\lambda})$ in $L_1(V_{\lambda+1})$ has the $\vec{\mathcal{U}}$-Baire property (see Theorem 4.14 below). Finally, as a negative result, we show that, in the inner model constructed in the proof of Theorem 1.5, there exists an I2-embedding with supremum of the critical sequence λ and a subset of $\mathrm{C}(\vec{\lambda})$ without the $\vec{\mathcal{U}}$-Baire property that is definable by a Σ_1-formula with parameters in $V_{\lambda+1}$ (see Theorem 4.10 below).

§2. Negative results. In this section, we will prove the restricting results stated in the introduction (Theorems 1.3 and 1.5). Theorem 1.3 motivates the formulation of

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

the main result of this paper (Theorem 1.4) by showing that its conclusion cannot be derived from the weaker large cardinal assumptions used in Theorem 1.2. On the other hand, Theorem 1.5 shows that the statement of Theorem 1.4 cannot be strengthened to affect all sets that are Σ_1-definable from arbitrary subsets of the given singular cardinal. In the following, we use arguments based on ideas and notions that were already used in [13, Section 4].

DEFINITION 2.1. Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of cardinals with supremum λ and let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ be a sequence of elements of V_λ.

(i) Given $x \subseteq \lambda$, we define $\lhd_x$ to be the unique binary relation on λ with the property that

$$\alpha \lhd_x \beta \iff \prec\alpha, \beta\succ \in x$$

holds for all $\alpha, \beta < \lambda$.

(ii) We define $\mathcal{WO}_\lambda$ to be the set of all $x \in \mathcal{P}(\lambda)$ with the property that $\lhd_x$ is a well-ordering of λ.

(iii) We let $\mathsf{WO}(\vec{\lambda}, \vec{a})$ denote the set of all $b \in {}^\omega\lambda$ with the property that there exists $x \in \mathcal{WO}_\lambda$ such that $x \cap \lambda_n = a_{b(n)}$ holds for all $n < \omega$.

(iv) Given an element b of $\mathsf{WO}(\vec{\lambda}, \vec{a})$, we let $\|b\|_{\vec{a}}$ denote the order-type of the resulting well-order $\langle \lambda, \lhd_{\bigcup\{a_{b(n)} \mid n<\omega\}} \rangle$.

The following *Boundedness Lemma* now follows from the theory developed in [13, Section 4] that generalizes classical arguments from descriptive set theory to singular strong limit cardinals of countable cofinality.

LEMMA 2.2 [13, Lemma 4.5]. *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of inaccessible cardinals with supremum λ and let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ be an enumeration of H_λ. If $f : {}^\omega\lambda \longrightarrow {}^\omega\lambda$ is a continuous function with* $\operatorname{ran}(f) \subseteq \mathsf{WO}(\vec{\lambda}, \vec{a})$, *then there exists an ordinal $\gamma < \lambda^+$ with $\|f(c)\|_{\vec{a}} < \gamma$ for all $c \in {}^\omega\lambda$.*

We start by limiting the provable structural consequences of I2-embeddings by proving the following strengthening of Theorem 1.5 that shows that the statement of Theorem 1.4 cannot be strengthened to show that the existence of an I2-embedding at a cardinal λ implies that every subset of $\mathcal{P}(\lambda)$ that is definable by a Σ_1-formula with parameters in $V_{\lambda+1}$ either has cardinality λ or contains the range of a perfect embedding:

THEOREM 2.3. *If $j : V \longrightarrow M$ is an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and $\lambda = \sup_{n<\omega} \lambda_n$, then the following statements hold in an inner model:*

(i) *There is an I2-embedding whose critical sequence has supremum λ.*

(ii) *There is a subset z of λ and a subset X of $C(\vec{\lambda})$ of cardinality greater than λ such that X does not contain the range of a perfect embedding of ${}^\omega\lambda$ into $C(\vec{\lambda})$ and the set X is definable by a Σ_1-formula with parameter z.*

PROOF. Since λ is a limit of inaccessible cardinals, we can find a subset y of λ with the property that $V_\lambda \cup \{\vec{\lambda},\ j \restriction V_\lambda\} \subseteq L[y]$. Since this setup ensures that

$$(j \restriction V_\lambda)_+^{L[y]} = (j \restriction V_\lambda)_+ \restriction V_{\lambda+1}^{L[y]},$$

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

we know that $(j \upharpoonright V_\lambda)_+^{L[y]}$ maps well-founded relations on V_λ in $L[y]$ to well-founded relations on V_λ in $L[y]$ and it follows that $j \upharpoonright V_\lambda$ witnesses that, in $L[y]$, there is an I2-embedding whose critical sequence has supremum λ.

Now, work in $L[y]$. First, observe that the set $\mathcal{WO}_\lambda$ consists of all subsets x of λ with the property that there exists an ordinal γ and an order isomorphism between $\langle \lambda, \lhd_x \rangle$ and $\langle \gamma, < \rangle$. In addition, the set $\mathcal{WO}_\lambda$ also consists of all subsets x of λ such that $\langle \lambda, \lhd_x \rangle$ is a linear ordering with the property that no injective sequence $\langle \alpha_n \mid n < \omega \rangle$ is decreasing with respect to $\lhd_x$. This shows that $\mathcal{WO}_\lambda$ is Δ_1-definable[5] from the parameter λ. Pick an enumeration $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ of V_λ with $V_{\lambda_n} = \{a_\alpha \mid \alpha < \lambda_n\}$ for all $n < \omega$. Then there exists an unbounded subset z of λ with the property that the sets $\{\vec{a}\}$, $\{y\}$ and $\{\vec{\lambda}\}$ are all definable by Σ_1-formulas with parameter z. Note that this implies that these sets are actually Δ_1-definable from the parameter z. Note that an element b of ${}^\omega\lambda$ is not contained in $\mathsf{WO}(\vec{\lambda}, \vec{a})$ if and only if either there are $m < n < \omega$ with $a_{b(n)} \cap \lambda_m \neq a_{b(m)}$ or there exists $x \in \mathcal{P}(\lambda) \setminus \mathcal{WO}_\lambda$ with $x \cap \lambda_n = a_{b(n)}$ holds for all $n < \omega$. Together with our earlier observations, this shows that the set $\mathsf{WO}(\vec{\lambda}, \vec{a})$ is Δ_1-definable from the parameter z. Given $\lambda \leq \gamma < \lambda^+$, we now let b_γ denote the $<_{L[y]}$-least element of $\mathsf{WO}(\vec{\lambda}, \vec{a})$ with $\|b_\gamma\|_{\vec{a}} = \gamma$ and $b_\gamma(n) < \lambda_{n+1}$ for all $n < \omega$. Note that our setup ensures that such a set exists for all $\lambda \leq \gamma < \lambda^+$. Moreover, since the basic structure theory of $L[y]$ ensures that the class of proper initial segments of $<_{L[y]}$ is definable by a Σ_1-formula with parameter z, the fact that $\mathsf{WO}(\vec{\lambda}, \vec{a})$ is Δ_1-definable from the parameter z yields a Σ_1-formula $\varphi(v_0, v_1, v_2)$ with the property that $\varphi(\gamma, b, z)$ holds if and only if γ is an ordinal in the interval $[\lambda, \lambda^+)$ and $b = b_\gamma$. Let X denote the set of all $b \in {}^\omega\lambda$ with the property that $b(0) = 0$ and there exists $\lambda \leq \gamma < \lambda^+$ with $b(n+1) = b_\gamma(n)$ for all $n < \omega$. We then know that X is a subset of $C(\vec{\lambda})$ of cardinality greater than λ that is definable by a Σ_1-formula with parameter z.

Assume, towards a contradiction, that there is a perfect embedding $\iota : {}^\omega\lambda \longrightarrow C(\vec{\lambda})$ with $\mathrm{ran}(\iota) \subseteq X$. Set $Y = \{b_\gamma \mid \lambda \leq \gamma < \lambda^+\}$ and let $\zeta : X \longrightarrow Y$ denote the unique map with $\zeta(b)(n) = b(n+1)$ for all $b \in X$ and $n < \omega$. Then ζ is a homeomorphism of the subspace X of $C(\vec{\lambda})$ and the subspace Y of $C(\vec{\lambda})$. In particular, it follows that $\zeta \circ \iota$ is a perfect embedding of ${}^\omega\lambda$ into $C(\vec{\lambda})$ with $\mathrm{ran}(\zeta \circ \iota) \subseteq Y \subseteq \mathsf{WO}(\vec{\lambda}, \vec{a})$. In this situation, Lemma 2.2 yields $c, d \in {}^\omega\lambda$ with $c \neq d$ and $\|(\zeta \circ \iota)(c)\|_{\vec{a}} = \|(\zeta \circ \iota)(d)\|_{\vec{a}}$. By the definition of Y, this is a contradiction. $\dashv$

Note that, in order to construct an inner model N with $V_\lambda \subseteq N$ and the property that (ii) of the above theorem holds, it suffices to assume that λ is the supremum of ω-many inaccessible cardinals in order to carry out the construction made in the proof of the theorem.

In the remainder of this section, we further develop the arguments used in the above proof to obtain the following strengthening of Theorem 1.3.

THEOREM 2.4. *If $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ is a strictly increasing sequence of measurable cardinals with limit λ, then the following statements hold in an inner model M containing $\vec{\lambda}$:*

[5] Given a natural number $n > 0$, a class C is Δ_n-*definable* from a parameter p if the classes C and $V \setminus C$ are both definable by Σ_n-formulas with parameter p.

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

(i) *The sequence $\vec{\lambda}$ consists of measurable cardinals.*
(ii) *If $\vec{v}$ is a strictly increasing ω-sequence of cardinals of uncountable cofinality with limit λ, then for some $x \in H_{\aleph_1}$, there is a subset of $C(\vec{\lambda})$ of cardinality greater than λ that does not contain the range of a perfect embedding of ${}^{\omega}\lambda$ into $C(\vec{\lambda})$ and is definable by a Σ_1-formula with parameters $\vec{v}$ and x.*

PROOF. Pick a sequence $\langle U_n \mid n < \omega\rangle$ with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$ and define

$$\mathcal{U} = \{\langle n, A\rangle \mid n < \omega,\ A \in U_n\}.$$

Then $\vec{\lambda} \in L[\mathcal{U}]$ and for every $n < \omega$, the cardinal λ_n is measurable in $L[\mathcal{U}]$.

Now, work in $L[\mathcal{U}]$ and fix a strictly increasing sequence $\vec{v} = \langle v_n \mid n < \omega\rangle$ of cardinals of uncountable cofinality with limit λ. Using standard arguments about iterated measurable ultrapowers (see [9, Lemma 19.5] and [17, Section 3]), we can find

- a transitive class M,
- an elementary embedding $j : V \longrightarrow M$ with $j(\lambda) = \lambda$,
- a function $x : \omega \longrightarrow \omega$, and
- a sequence $\langle C_n \mid n < \omega\rangle$

such that the following statements hold for all $n < \omega$:

(i) $j(\lambda_n) = v_{x(n)}$.
(ii) $v_{x(n+1)} > |H_{v^+_{x(n)}}|$.
(iii) C_n is a closed unbounded subset of $v_{x(n)}$.
(iv) $j(U_n) = \{A \in M \cap \mathcal{P}(v_{x(n)}) \mid \exists \xi < v_{x(n)}\ C_n \setminus \xi \subseteq A\}$.

Now, set $\mathcal{V} = j(\mathcal{U})$ and define $\mathcal{N}$ to be the class of all pairs $\langle N, \vec{F}\rangle$ with the property that N is a transitive set of cardinality λ, $\vec{F} = \langle F_n \mid n < \omega\rangle$ is a sequence of length ω and there exists a sequence $\langle D_n \mid n < \omega\rangle$ such that the following statements hold:

(a) D_n is a closed unbounded subset of $v_{x(n)}$ for all $n < \omega$.
(b) If $n < \omega$, then F_n is an element of N, $v_{x(n)}$ is a regular cardinal in N and F_n is a normal ultrafilter on $v_{x(n)}$ in N.
(c) If $n < \omega$, then $F_n = \{A \in N \cap \mathcal{P}(v_{x(n)}) \mid \exists \xi < v_{x(n)}\ D_n \setminus \xi \subseteq A\}$.
(d) If $\mathcal{F} = \{\langle n, A\rangle \mid n < \omega,\ A \in F_n\}$, then $\mathcal{F} \in N$ and $N = L_{N\cap \mathrm{Ord}}[\mathcal{F}]$.

It is easy to see that the class $\mathcal{N}$ is definable by a Σ_1-formula with parameters $\vec{v}$ and x. Moreover, our assumptions ensure that for every $x \in M \cap \mathcal{P}(\lambda)$, there exists $\gamma < \lambda^+$ with $x \in L_\gamma[\mathcal{V}]$ and $\langle L_\gamma[\mathcal{V}], \langle j(U_n) \mid n < \omega\rangle\rangle \in \mathcal{N}$.

CLAIM. *If $\langle N, \langle F_n \mid n < \omega\rangle\rangle \in \mathcal{N}$ and $\mathcal{F} = \{\langle n, A\rangle \mid n < \omega,\ A \in F_n\}$, then we have $\mathcal{F} \cap N = \mathcal{V} \cap L_{N\cap\mathrm{Ord}}[\mathcal{V}]$ and $N = L_{N\cap\mathrm{Ord}}[\mathcal{V}]$.*

PROOF OF THE CLAIM. Let $\langle D_n \mid n < \omega\rangle$ be a sequence that witnesses that $\langle N, \langle F_n \mid n < \omega\rangle\rangle$ is contained in $\mathcal{N}$. Set $\gamma = N \cap \mathrm{Ord}$. By induction, we now show that

$$\mathcal{F} \cap L_\beta[\mathcal{F}] = \mathcal{V} \cap L_\beta[\mathcal{V}]$$

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

holds for all $\beta \leq \gamma$. Hence, assume that $\beta \leq \gamma$ with $\mathcal{F} \cap L_\alpha[\mathcal{F}] = \mathcal{V} \cap L_\alpha[\mathcal{V}]$ for all $\alpha < \beta$. Then $L_\beta[\mathcal{F}] = L_\beta[\mathcal{V}]$. Pick $n < \omega$ and $A \in F_n$ with $\langle n, A\rangle \in L_\beta[\mathcal{F}]$. Then there exists $\xi < v_{x(n)}$ with $D_n \setminus \xi \subseteq A$. Since $C_n \cap D_n$ is unbounded in $v_{x(n)}$, we know that $A \cap C_n$ is unbounded in $v_{x(n)}$ and hence there is no $\zeta < v_{x(n)}$ with the property that $C_n \setminus \zeta \subseteq v_{x(n)} \setminus A$. In this situation, the fact that $j(U_n)$ is an ultrafilter on $v_{x(n)}$ in $L[\mathcal{V}]$ implies that $A \in j(U_n)$ and hence $\langle n, A\rangle \in j(\mathcal{U}) \cap L_\beta[\mathcal{V}] = \mathcal{V} \cap L_\beta[\mathcal{V}]$. The dual argument then shows that we also have $\mathcal{V} \cap L_\beta[\mathcal{V}] \subseteq \mathcal{F} \cap L_\beta[\mathcal{F}]$. This completes the induction and we know that $\mathcal{F} \cap N = \mathcal{V} \cap L_\gamma[\mathcal{V}]$. This allows us to conclude that

$$N \;=\; L_\gamma[\mathcal{F}] \;=\; L_\gamma[\mathcal{F} \cap N] \;=\; L_\gamma[\mathcal{V} \cap L_\gamma[\mathcal{V}]] \;=\; L_\gamma[\mathcal{V}],$$

completing the proof of the claim. ⊣

Now, note that (ii) above ensures that there is a sequence $\langle a_\alpha \mid \alpha < \lambda\rangle$ in M with the property that

$$M \cap \mathcal{P}(v_{x(n)}) \;=\; \{a_\alpha \mid \alpha < v_{x(n+1)}\} \tag{1}$$

holds for all $n < \omega$. Define $\vec{a}$ to be the $<_{L[\mathcal{V}]}$-least sequence in M with this property.

Claim. *The set $\{\vec{a}\}$ is definable by a Σ_1-formula with parameters $\vec{v}$ and x.*

Proof of the Claim. First, note that our previous claim implies that, if $\langle N, \langle F_n \mid n < \omega\rangle\rangle$ is an element of $\mathcal{N}$ with $\lambda \in N$, then $N = L_{N \cap \mathrm{Ord}}[\mathcal{V}]$ and N contains all bounded subsets of λ in M. It follows that $\vec{a}$ is the unique sequence of length λ with the property that there exists $\langle N, \langle F_n \mid n < \omega\rangle\rangle$ in $\mathcal{N}$ and $\mathcal{F} = \{\langle n, A\rangle \mid n < \omega, A \in F_n\}$ such that $\vec{a}$ is the $<_{L[\mathcal{F}]}$-least element of N with (1) for all $n < \omega$. This characterization directly yields the desired Σ_1-definition. ⊣

Next, notice that, if y is an element of $\mathcal{WO}^M_\lambda$, then M contains an order-isomorphism between $\langle \lambda, \lhd_y\rangle$ and $\langle \gamma, <\rangle$ for some ordinal $\gamma \in [\lambda, \lambda^+)$ and this isomorphism witnesses that x is an element of $\mathcal{WO}_\lambda$ in V. This shows that $\mathcal{WO}^M_\lambda \subseteq \mathcal{WO}_\lambda$, $\mathsf{WO}(\vec{v}, \vec{a})^M \subseteq \mathsf{WO}(\vec{v}, \vec{a})$ and $\|b\|_{\vec{a}} = \|b\|^M_{\vec{a}}$ for all $b \in \mathsf{WO}(\vec{v}, \vec{a})^M$. Moreover, using (1) and the fact that $\lambda^+ = (\lambda^+)^M$, we can pick a sequence $\langle b_\gamma \mid \lambda \leq \gamma < \lambda^+\rangle$ with the property that for all $\gamma < \lambda^+$, the set b_γ is the $<_{L[\mathcal{V}]}$-least element of $\mathsf{WO}(\vec{v}, \vec{a})^M$ with the property that $\|b_\gamma\|_{\vec{a}} = \gamma$ and $b_\gamma(x(n)) < v_{x(n+1)}$ for all $n < \omega$. The following statement now follows from a combination of the above claims:

Claim. *The set $B = \{b_\gamma \mid \lambda \leq \gamma < \lambda^+\}$ is definable by a Σ_1-formula with parameters $\vec{v}$ and x.*

Given $\lambda \leq \gamma < \lambda^+$, we let c_γ denote the unique element of ${}^\omega\lambda$ such that the following statements hold for all $n < \omega$:

- If n is of the form $x(m+1)$ for some $m < \omega$, then $c_\gamma(n) = b_\gamma(x(m))$.
- If $n \neq x(m+1)$ for all $m < \omega$, then $c_\gamma(n) = 0$.

We then know that $c_\gamma \in C(\vec{v})$ for all $\lambda \leq \gamma < \lambda^+$.

Claim. *The set $C = \{c_\gamma \mid \lambda \leq \gamma < \lambda^+\}$ has cardinality λ^+ and is definable by a Σ_1-formula with parameters $\vec{v}$ and x.*

Let $\zeta : B \longrightarrow C$ denote the unique function with $\zeta(b_\gamma) = c_\gamma$ for all $\lambda \leq \gamma < \lambda^+$.

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

Claim. *The map ζ is a homeomorphism of the subspace B of $C(\vec{v})$ and the subspace C of $C(\vec{v})$.*

Now, assume, towards a contradiction, that there is a perfect embedding $\iota : {}^{\omega}\lambda \longrightarrow C(\vec{v})$ with the property that $\mathrm{ran}(\iota) \subseteq C$. Then $\zeta^{-1} \circ \iota$ is a perfect embedding of ${}^{\omega}\lambda$ into $C(\vec{v})$ and

$$\mathrm{ran}(\zeta^{-1} \circ \iota) \ \subseteq \ B \ \subseteq \ \mathsf{WO}(\vec{v}, \vec{a})^M \ \subseteq \ \mathsf{WO}(\vec{v}, \vec{a}).$$

An application of Lemma 2.2 now yields $c, d \in {}^{\omega}\lambda$ with $c \neq d$ and

$$\|(\zeta^{-1} \circ \iota)(c)\|_{\vec{a}} \ = \ \|(\zeta^{-1} \circ \iota)(d)\|_{\vec{a}},$$

contradicting the definition of B. ⊣

§3. Σ_1-definability at I2-cardinals. Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and set $\lambda = \sup_{n<\omega} \lambda_n$. Classical results (see [15]) then show that j is *ω-iterable*, i.e., there exists a commuting system

$$\langle \langle M^j_\alpha \mid \alpha \leq \omega \rangle, \langle j : M^j_\alpha \longrightarrow M^j_\beta \mid \alpha \leq \beta \leq \omega \rangle \rangle$$

of inner models and elementary embeddings such that the following statements hold:

(i) $M^j_0 = V$ and $j_{0,1} = j$.
(ii) If $n < \omega$, then $j_{n+1,n+2} = \bigcup \{ j_{n,n+1}(j_{n,n+1} \restriction V_\alpha) \mid \alpha \in \mathrm{Ord} \}$.
(iii) $\langle M^j_\omega, \langle j_{n,\omega} \mid n < \omega \rangle \rangle$ is a direct limit of

$$\langle \langle M^j_n \mid n < \omega \rangle, \langle j_{m,n} : M^j_m \longrightarrow M^j_n \mid m \leq n < \omega \rangle \rangle.$$

Given $m \leq n < \omega$, we then have $V_\lambda \subseteq M^j_\omega \subseteq M^j_n \subseteq M^j_m$, $\mathrm{crit}(j_{n,n+1}) = \lambda_n = j_{m,n}(\lambda_m)$, $j_{m,n}(\lambda) = \lambda$ and $j_{n,\omega}(\lambda_n) = \lambda$. Moreover, it is easy to see that $j_{0,\omega}(\lambda^+) = \lambda^+$ holds and therefore $(2^\lambda)^{M^j_\omega} < \lambda^+$. Note that the *Mathias criterion* shows that the sequence $\vec{\lambda}$ is Prikry-generic over M^j_ω and, by the theory of Prikry-type forcings, this implies that $(2^\lambda)^{M^j_\omega[\vec{\lambda}]} < \lambda^+$.

The following theorem is the main result of this section. We will later show that it is a direct strengthening of Theorem 1.4.

Theorem 3.1. *Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and let N be an inner model of* ZFC *with $M^j_\omega \cup \{\vec{\lambda}\} \subseteq N$. Set $\lambda = \sup_{n<\omega} \lambda_n$. If X is a subset of $C(\vec{\lambda})$ with $|X| > (2^\lambda)^N$ that is definable over V_λ by a Σ^1_2-formula with parameters in $V^N_{\lambda+1}$, then there is a perfect embedding $\iota : {}^{\omega}\lambda \longrightarrow C(\vec{\lambda})$ with* $\mathrm{ran}(\iota) \subseteq X$.

The proof of the above theorem closely follows the proof of Solovay's classical result showing that every $\mathbf{\Sigma}^1_2$-set of reals has the perfect set property if ω_1 is inaccessible to the reals (see [9, Theorem 14.10]). The key ingredient that makes this adaptation possible is an absoluteness theorem proven by Laver in [11]. We start this argument by obtaining tree representations for the sets in the given definability class.

Given non-empty sets $a_0, \ldots, a_k$, a subset T of ${}^{<\omega}a_0 \times \ldots \times {}^{<\omega}a_k$ is a *(descriptive) tree* if the following statements hold for all elements $\langle t_0, \ldots, t_k \rangle$ of T:

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

- $\mathrm{dom}(t_0) = \ldots = \mathrm{dom}(t_k)$.
- If $\ell \in \mathrm{dom}(t_k)$, then $\langle t_0 \upharpoonright \ell, \ldots, t_k \upharpoonright \ell \rangle \in T$.

In addition, if $T \subseteq {}^{<\omega}a_0 \times \ldots \times {}^{<\omega}a_k$ is a tree, then we let $[T]$ denote the set of all tuples $\langle x_0, \ldots, x_k \rangle$ in ${}^{\omega}a_0 \times \ldots \times {}^{\omega}a_k$ with the property that $\langle x_0 \upharpoonright \ell, \ldots, x_k \upharpoonright \ell \rangle \in T$ holds for all $\ell < \omega$. Finally, for every tree $T \subseteq {}^{<\omega}a_0 \times \ldots \times {}^{<\omega}a_{k+1}$, we define

$$p[T] \;=\; \{\langle x_0, \ldots, x_k \rangle \in {}^{\omega}a_0 \times \ldots \times {}^{\omega}a_k \mid \exists x_{k+1} \in {}^{\omega}a_{k+1} \; \langle x_0, \ldots, x_{k+1} \rangle \in [T]\}.$$

As outlined in [11, Section 1], for singular strong limit cardinals λ and $0 < k < \omega$, there is a direct correspondence between subsets of $V^k_{\lambda+1}$ that are $\mathbf{\Sigma}^1_1$-definable over V_λ and sets of the form $p[T]$ for trees $T \subseteq ({}^{<\omega}V_\lambda)^{k+1}$. Several key arguments in this section rely on the absoluteness properties of this correspondence that can be isolated from the arguments in [11, Section 1]:

Lemma 3.2. *For every Σ^1_1-formula $\varphi(w_0, \ldots, w_{k+2})$ in the language of set theory with free second-order variables $w_0, \ldots, w_{k+2}$, there is a first-order formula $\psi(v_0, \ldots, v_{k+1})$ in the language of set theory expanded by two unary relation symbols with free variables $v_0, \ldots, v_{k+1}$ such that* ZFC *proves that for every strictly increasing sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ of strong limit cardinals with supremum λ and every $B \subseteq V_\lambda$, the set*

$$T_{\varphi,B,\vec{\lambda}} \;=\; \{\langle t_0, \ldots, t_{k+1} \rangle \in ({}^{<\omega}V_\lambda)^{k+2} \mid \langle V_\lambda, \in, B, \vec{\lambda} \rangle \models \psi(t_0, \ldots, t_{k+1})\} \tag{2}$$

is a tree and the following statements hold:

(i) $T_{\varphi,B,\vec{\lambda}} \cap ({}^{n}V_\lambda)^{k+2} \in V_\lambda$ *for all* $n < \omega$.

(ii) *If* $\langle x_0, \ldots, x_k \rangle \in p[T_{\varphi,B,\vec{\lambda}}]$, $i \leq k$ *and* $m < n < \omega$, *then* $x_i(m) = x_i(n) \cap V_{\lambda_m}$.

(iii) *The following statements are equivalent for all* $A_0, \ldots, A_k \subseteq V_\lambda$:

  (a) $\langle V_\lambda, \in \rangle \models \varphi(A_0, \ldots, A_k, B, \vec{\lambda})$.

  (b) *There is* $\langle x_0, \ldots, x_k \rangle \in p[T_{\varphi,B,\vec{\lambda}}]$ *with* $x_i(n) = A_i \cap V_{\lambda_n}$ *for all* $i \leq k$ *and* $n < \omega$.

The above lemma provides a setting in which a converse of Proposition 1.6 holds.

Corollary 3.3. *For every Σ^1_2-formula $\psi(w_0, \ldots, w_{k-1})$ in the language of set theory with free second-order parameters $w_0, \ldots, w_{k-1}$, there exists a Σ_1-formula $\varphi(v_0, \ldots, v_k)$ in the language of set theory such that* ZFC *proves that*

$$\varphi(A_0, \ldots, A_{k-1}, V_\lambda, \vec{\lambda}) \iff \langle V_\lambda, \in \rangle \models \psi(A_0, \ldots, A_{k-1})$$

holds for every strictly increasing sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ of strong limit cardinals with supremum λ and all $A_0, \ldots, A_{k-1} \in V_{\lambda+1}$.

Following [11, Section 1], we now generalize the concept of *Shoenfield trees* (i.e., tree representations for $\mathbf{\Sigma}^1_2$-sets of real numbers) to higher cardinals of countable cofinality. Given a tree $T \subseteq {}^{<\omega}a_0 \times \ldots \times {}^{<\omega}a_k$, $i < k$ and $\langle s_0, \ldots, s_i \rangle \in {}^{<\omega}a_0 \times \ldots \times {}^{<\omega}a_i$ with $\mathrm{dom}(s_0) = \ldots = \mathrm{dom}(s_i)$, we define $T^{\langle s_0, \ldots, s_i \rangle}$ to be the set of all tuples $\langle t_{i+1}, \ldots, t_k \rangle$ in ${}^{<\omega}a_{i+1} \times \ldots \times {}^{<\omega}a_k$ with the property that

$$\mathrm{dom}(t_{i+1}) \;=\; \ldots \;=\; \mathrm{dom}(t_k) \;\leq\; \mathrm{dom}(s_0)$$

and

$$\langle s_0 \upharpoonright \mathrm{dom}(t_k), \ldots, s_i \upharpoonright \mathrm{dom}(t_k), t_{i+1}, \ldots, t_k \rangle \in T.$$

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

Note that

$$T^{\langle s_0\restriction\ell,\dots,s_i\restriction\ell\rangle} \;=\; T^{\langle s_0,\dots,s_i\rangle}\cap({}^{\leq\ell}a_{i+1}\times\dots\times{}^{\leq\ell}a_k)$$

holds for all $\ell\in\mathrm{dom}(s_0)$. In addition, for every ordinal θ, we let $R_{T,\theta}(s_0,\dots,s_i)$ denote the set of functions

$$r : T^{\langle s_0,\dots,s_i\rangle}\longrightarrow\theta$$

satisfying

$$r(\langle t_{i+1},\dots,t_k\rangle) \;<\; r(\langle t_{i+1}\restriction\ell,\dots,t_k\restriction\ell\rangle)$$

for all $\langle t_{i+1},\dots,t_k\rangle\in T^{\langle s_0,\dots,s_i\rangle}$ and $\ell<\mathrm{dom}(t_k)$. It is then easy to see that $r\restriction T^{\langle s_0\restriction\ell,\dots,s_i\restriction\ell\rangle}$ is an element of $R_{T,\theta}(s_0\restriction\ell,\dots,s_i\restriction\ell)$ for all $r\in R_{T,\theta}(s_0,\dots,s_i)$ and $\ell<\mathrm{dom}(s_0)$.

Now, let λ be an infinite ordinal, let $T\subseteq({}^{<\omega}V_\lambda)^{k+3}$ be a tree and let $\theta>\lambda$ be an ordinal. We then define $S_{T,\theta}$ to be the subset of $({}^{<\omega}V_{\theta+\omega})^{k+2}$ consisting of all tuples $\langle s_0,\dots,s_k,t\rangle$ such that the following statements hold:

- $s_0,\dots,s_k\in{}^{<\omega}V_\lambda$.
- $\mathrm{dom}(s_0)=\dots=\mathrm{dom}(s_k)=\mathrm{dom}(t)$.
- There exists $s_{k+1}\in{}^{\mathrm{dom}(s_0)}V_\lambda$ and $r\in R_{T,\theta}(s_0,\dots,s_{k+1})$ such that

$$t(\ell) \;=\; \langle s_{k+1}\restriction(\ell+1),\; r\restriction T^{\langle s_0\restriction(\ell+1),\dots,s_{k+1}\restriction(\ell+1)\rangle}\rangle \tag{3}$$

  holds for all $\ell\in\mathrm{dom}(t)$.

It is then easy to check that $S_{T,\theta}$ is a tree. The following lemma from [11, Section 1] shows how these constructions yield tree representations of $\mathbf{\Sigma}^1_2$-subsets of $V_{\lambda+1}$:

LEMMA 3.4. *Let $\varphi(w_0,\dots,w_{k+3})$ be a Σ^1_1-formula in the language of set theory with free second-order variables $w_0,\dots,w_{k+3}$. Then the following statements are equivalent for every strictly increasing sequence of inaccessible cardinals $\vec\lambda=\langle\lambda_n\mid n<\omega\rangle$ with supremum λ, every limit ordinal $\theta\geq\lambda^+$ and all $A_0,\dots,A_k,B\subseteq V_\lambda$:*

(i) $\langle V_\lambda,\in\rangle\models\exists C\,\neg\varphi(A_0,\dots,A_k,B,C,\vec\lambda)$.
(ii) *There is $\langle x_0,\dots,x_k\rangle\in p[S_{T_{\varphi,B,\vec\lambda}},\theta]$ with $x_i(n)=A_i\cap V_{\lambda_n}$ for all $i\leq k$ and $n<\omega$.*

Still following Laver's arguments, we now show that the structural properties of higher Shoenfield trees can be fruitfully combined with the combinatorics of I2-embeddings. The proof of the next lemma is a reformulation of the proof of [11, Theorem 1.4].

LEMMA 3.5. *Let $\varphi(w_0,\dots,w_{k+3})$ be a Σ^1_1-formula in the language of set theory with free second-order variables $w_0,\dots,w_{k+3}$, let $j:V\longrightarrow M$ be an I2-embedding with critical sequence $\vec\lambda=\langle\lambda_n\mid n<\omega\rangle$ and let N be an inner model of* ZFC *with $M^j_\omega\cup\{\vec\lambda\}\subseteq N$. Set $\lambda=\sup_{n<\omega}\lambda_n$ and $\theta=(\lambda^+)^V$. Fix $B\in V^N_{\lambda+1}$ and define $T=T^V_{\varphi,B,\vec\lambda}$. Then the following statements hold:*

(i) *$T=T^N_{\varphi,B,\vec\lambda}$ and $S^N_{T,\theta}\subseteq S^V_{T,\theta}$.*

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

(ii) *There is an inclusion-preserving embedding* $\Lambda : S^V_{T,\theta} \longrightarrow S^N_{T,\theta}$ *with the property that for all* $\langle s_0, \dots, s_k, t\rangle \in S^V_{T,\theta}$, *there exists u with*

$$\Lambda(\langle s_0, \dots, s_k, t\rangle) = \langle s_0, \dots, s_k, u\rangle.$$

(iii) $p[S^V_{T,\theta}]^V = p[S^N_{T,\theta}]^V$.
(iv) $p[S^V_{T,\theta}]^V \cap N = p[S^N_{T,\theta}]^N$.

Proof. (i) Since $V_\lambda \cup \{B, \vec{\lambda}\} \subseteq N$, the fact that (2) holds in both V and N directly implies that $T = T^N_{\varphi,B,\vec{\lambda}}$. In addition, if $s_0, \dots, s_{k+1} \in {}^{<\omega}V_\lambda$ with $\mathrm{dom}(s_0) = \dots = \mathrm{dom}(s_{k+1})$, then

$$R_{T,\theta}(s_0, \dots, s_{k+1})^N \subseteq R_{T,\theta}(s_0, \dots, s_{k+1})^V.$$

In particular, we know that $S^N_{T,\theta} \subseteq S^V_{T,\theta}$.

(ii) The proof of [11, Theorem 1.4] shows that for every $d \in V_\lambda$ and every function $f : d \longrightarrow \mathrm{Ord}$, the function $j_{0,\omega} \circ f : d \longrightarrow \mathrm{Ord}$ is an element of M^j_ω. In particular, if $s_0, \dots, s_{k+1} \in {}^{<\omega}V_\lambda$ with $\mathrm{dom}(s_0) = \dots = \mathrm{dom}(s_{k+1})$ and $r \in R_{T,\theta}(s_0, \dots, s_{k+1})^V$, then the fact that $j_{0,\omega}(\theta) = \theta$ implies that $j_{0,\omega} \circ r \in R_{T,\theta}(s_0, \dots, s_{k+1})^N$. This inclusion allows us to define $\Lambda : S^V_{T,\theta} \longrightarrow S^N_{T,\theta}$ to be the unique function with the property that for all $\langle s_0, \dots, s_k, t\rangle \in S^V_{T,\theta}$ and all $s_{k+1} \in {}^{\mathrm{dom}(s_0)}V_\lambda$ and $r \in R_{T,\theta}(s_0, \dots, s_{k+1})$ such that (3) holds for all $\ell \in \mathrm{dom}(t)$, we have $\Lambda(\langle s_0, \dots, s_k, t\rangle) = \langle s_0, \dots, s_k, u\rangle$, where

$$u(\ell) = \langle s_{k+1} \restriction (\ell + 1), (j_{0,\omega} \circ r) \restriction T^{\langle s_0 \restriction (\ell+1), \dots, s_{k+1} \restriction (\ell+1)\rangle}\rangle$$

for all $\ell \in \mathrm{dom}(u)$. This definition directly ensures that Λ is an inclusion-preserving embedding.

(iii) Since $S^N_{T,\theta} \subseteq S^V_{T,\theta}$, we know that $p[S^N_{T,\theta}]^V \subseteq p[S^V_{T,\theta}]^V$. Pick a tuple $\langle x_0, \dots, x_k, y\rangle$ in $[S^V_{T,\theta}]^V$ and let z be the unique element of ${}^\omega V_{\theta+\omega}$ with

$$\Lambda(\langle x_0 \restriction n, \dots, x_k \restriction n, y \restriction n\rangle) = \langle x_0 \restriction n, \dots, x_k \restriction n, z \restriction n\rangle \tag{4}$$

for all $n < \omega$. We then know that $\langle x_0, \dots, x_k, z\rangle$ is an element of $[S^N_{T,\theta}]^V$. This shows that we also have $p[S^V_{T,\theta}]^V \subseteq p[S^N_{T,\theta}]^V$.

(iv) First, the fact that $S^N_{T,\theta} \subseteq S^V_{T,\theta}$ directly implies that $p[S^N_{T,\theta}]^N \subseteq p[S^V_{T,\theta}]^V \cap N$. Now, fix $\langle x_0, \dots, x_k\rangle \in p[S^V_{T,\theta}]^V \cap N$ and pick $y \in V$ with $\langle x_0, \dots, x_k, y\rangle \in [S^V_{T,\theta}]^V$. Let z denote the unique element of ${}^{<\omega}V_{\theta+\omega}$ such that (4) holds for all $n < \omega$. This shows that $\langle x_0, \dots, x_k, z\rangle \in [S^N_{T,\theta}]^V$. Since the tuple $\langle x_0, \dots, x_k\rangle$ is an element of N, we know that

$$U = \{t \in {}^{<\omega}V_{\theta+\omega} \mid \langle x_0 \restriction \mathrm{dom}(t), \dots, x_k \restriction \mathrm{dom}(t), t\rangle \in S^N_{T,\theta}\}$$

is a tree of height ω in N and $z \in [U]^V$. In this situation, the fact that the ill-foundedness of U is absolute between N and V yields an element z' of $[U]^N$. We then have $\langle x_0, \dots, x_k, z'\rangle \in [S^N_{T,\theta}]^N$ and $\langle x_0, \dots, x_k\rangle \in p[S^N_{T,\theta}]^N$. ⊣

Corollary 3.6 [11, Theorem 1.4]. *Let* $\varphi(w_0, \dots, w_{n-1})$ *be a* Σ^1_2*-formula in the language of set theory with free second-order variables* $w_0, \dots, w_{n-1}$. *If* $j : V \longrightarrow M$

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

is an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and N is an inner model of ZFC *with $M^j_\omega \cup \{\vec{\lambda}\} \subseteq N$, then the statement*

$$\langle V_\lambda, \in \rangle \models \varphi(A_0, \dots, A_{n-1})$$

is absolute between V and N for all $A_0, \dots, A_{n-1} \in V^N_{\lambda+1}$, where $\lambda = \sup_{n<\omega} \lambda_n$.

In order to connect the above concepts with the existence of perfect subsets, we now adapt a classical result of Mansfield (see [9, Theorem 14.7]) to our setting:

Lemma 3.7. *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of infinite cardinals with limit λ and let $T \subseteq {}^{<\omega}a \times {}^{<\omega}b$ be a tree with the property that $p[T]$ does not contain the range of a perfect embedding of ${}^{\omega}\lambda$ into ${}^{\omega}a$. If N is an inner model of* ZFC *with $V_\lambda \cup \{T, \vec{\lambda}\} \subseteq N$, then $p[T]^V \subseteq N$.*

Proof. Given a tree $S \subseteq {}^{<\omega}a \times {}^{<\omega}b$, we define S' to be the set of all $\langle t, u \rangle \in S$ with the property that for all $n < \omega$, there exists $\mathrm{dom}(t) < \ell < \omega$ such that the set

$$\{v \in {}^{\ell}a \mid \exists w \in {}^{\ell}b\ [t \subseteq v \wedge u \subseteq w \wedge \langle v, w \rangle \in S]\}$$

has cardinality at least λ_n. Then it is easy to see that for every such tree S, the set S' is again a tree with $S' \subseteq S$ and, if S is an element of N, then S' is also contained N. Now, let $\langle T_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ denote the unique sequence of trees with $T_0 = T$, $T_{\alpha+1} = T'_\alpha$ for all $\alpha \in \mathrm{Ord}$ and $T_\beta = \bigcap_{\alpha<\beta} T_\alpha$ for all $\beta \in \mathrm{Lim}$. Then it is easy to see that $T_\alpha \in N$ holds for all $\alpha \in \mathrm{Ord}$. Moreover, there exists $\alpha_* \in \mathrm{Ord}$ with $T_{\alpha_*} = T_\beta$ for all $\alpha_* \leq \beta \in \mathrm{Ord}$. Set $T_* = T_{\alpha_*}$.

Claim. $T_* = \emptyset$.

Proof of the Claim. Assume, towards a contradiction, that $T_* \neq \emptyset$. Let $S_{\vec{\lambda}}$ denote the subtree of ${}^{<\omega}\lambda$ consisting of all $s \in {}^{<\omega}\lambda$ with $s(\ell) < \lambda_\ell$ for all $\ell \in \mathrm{dom}(s)$. We inductively construct a system $\langle \langle s_u, t_u \rangle \in T_* \mid u \in S_{\vec{\lambda}} \rangle$ such that the following statements hold for all $u, v \in S_{\vec{\lambda}}$:

- If $u \subsetneq v$, then $s_u \subsetneq s_v$ and $t_u \subsetneq t_v$.
- If $\alpha < \beta < \lambda_{\mathrm{dom}(u)}$, then $\mathrm{dom}(s_{u^\frown\langle\alpha\rangle}) = \mathrm{dom}(s_{u^\frown\langle\beta\rangle})$ and $s_{u^\frown\langle\alpha\rangle} \neq s_{u^\frown\langle\beta\rangle}$.

First, define $s_\emptyset = t_\emptyset = \emptyset$. Now, assume that $u \in S_{\vec{\lambda}}$ and $\langle s_u, t_u \rangle \in T_*$ is already constructed. Since $\langle s_u, t_u \rangle \in T'_* = T_*$, we can find $\mathrm{dom}(s_u) < \ell < \omega$ and a sequence $\langle \langle s_\xi, t_\xi \rangle \in T_* \mid \xi < \lambda_{\mathrm{dom}(u)} \rangle$ with the property that for all $\xi < \rho < \lambda_{\mathrm{dom}(u)}$, we have $\mathrm{dom}(s_\xi) = \mathrm{dom}(s_\rho) = \ell$ and $s_\xi \neq s_\rho$. Given $\xi < \lambda_{\mathrm{dom}(u)}$, we then define $s_{u^\frown\langle\xi\rangle} = s_\xi$ and $t_{u^\frown\langle\xi\rangle} = t_\xi$. It then directly follows that the constructed sets satisfy all required properties. This completes the inductive construction of our system. If we now define

$$\iota : C(\vec{\lambda}) \longrightarrow {}^{\omega}a;\ x \longmapsto \bigcup \{s_{x\restriction\ell} \mid \ell < \omega\},$$

then our setup ensures that ι is a perfect embedding. Moreover, we have

$$\langle \iota(x), \bigcup \{t_{x\restriction i} \mid i < \omega\} \rangle \in [T]$$

for all $x \in C(\vec{\lambda})$ and this shows that $\mathrm{ran}(\iota)$ is a subset of $p[T]$. Since there exists a perfect embedding of ${}^{\omega}\lambda$ into $C(\vec{\lambda})$, this yields a contradiction to our assumptions on T. ⊣

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

Now, fix $\langle x, y\rangle \in [T]$. Then there is an $\alpha < \alpha_*$ with $\langle x, y\rangle \in [T_\alpha] \setminus [T_{\alpha+1}]$ and we can find $k < \omega$ with the property that $\langle x \restriction k, y \restriction k\rangle \notin T_{\alpha+1} = T'_\alpha$. Hence, there is $n < \omega$ with the property that for all $k < \ell < \omega$, the set

$$E_\ell = \{s \in {}^\ell a \mid \exists t \in {}^\ell b \, [x \restriction k \subseteq s \wedge y \restriction k \subseteq t \wedge \langle s, t\rangle \in T_\alpha]\}$$

has cardinality less than λ_n. Note that $x \restriction \ell \in E_\ell$ holds for all $k < \ell < \omega$. Moreover, since N contains the sequence $\langle E_\ell \mid k < \ell < \omega\rangle$ and each E_ℓ has cardinality less than λ_n in N, we can find a sequence $\langle \tau_\ell : \lambda_n \longrightarrow E_\ell \mid k < \ell < \omega\rangle$ of surjections that is an element of N. If we pick $z \in {}^\omega\lambda_n$ with $\tau_\ell(z(\ell)) = x \restriction \ell$ for all $k < \ell < \omega$, then the fact that $V_\lambda \subseteq N$ ensures that z is an element of N and hence we can conclude that x is also contained in the inner model N. ⊣

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega\rangle$ and let N be an inner model of ZFC with $M^j_\omega \cup \{\vec{\lambda}\} \subseteq N$. Set $\lambda = \sup_{n<\omega} \lambda_n$. Fix a Σ^1_1-formula $\varphi(w_0, \dots, w_3)$ with second-order variables $w_0, \dots, w_3$ and $B \in V^N_{\lambda+1}$ such that the set

$$X = \{A \in V_{\lambda+1} \mid \langle V_\lambda, \in\rangle \models \exists C \, \neg\varphi(A, B, C, \vec{\lambda})\}$$

is a subset of $C(\vec{\lambda})$ of cardinality greater than $(2^\lambda)^N$. Set $T = T^V_{\varphi,B,\vec{\lambda}}$, $\theta = (\lambda^+)^V$, $S_1 = S^V_{T,\theta}$ and $S_0 = S^N_{T,\theta} \subseteq S_1$. An application of Lemma 3.5.iii then shows that $p[S_0]^V = p[S_1]^V$. In particular, since Lemma 3.4 ensures that every element of X is of the form $\bigcup\{y(n) \mid n < \omega\}$ for some $y \in p[S_1]^V$, we know that $p[S_0]^V$ has cardinality greater than $(2^\lambda)^N$ in V and we can conclude that $p[S_0]^V \nsubseteq N$. In this situation, an application of Lemma 3.7 shows that, in V, there exists a perfect embedding $\iota : {}^\omega\lambda \longrightarrow {}^\omega V_\lambda$ satisfying $\mathrm{ran}(\iota) \subseteq p[S_0] = p[S_1]$.

Now, work in V and define Y to be the set of all $y \in {}^\omega V_\lambda$ with the property that $y(m) = y(n) \cap V_{\lambda_m}$ holds for all $m \leq n < \omega$ and $\bigcup\{y(n) \mid n < \omega\}$ is an element of $C(\vec{\lambda})$. Then Y is a closed subset of ${}^\omega V_\lambda$ and $p[S_0] \subseteq Y$. Moreover, the map

$$\tau : Y \longrightarrow C(\vec{\lambda}); \; y \longmapsto \bigcup\{y(n) \mid n < \omega\}$$

is a homeomorphism of the subspace Y of ${}^\omega V_\omega$ and the space $C(\vec{\lambda})$ with $\tau[p[S_0]] = X$. In particular, there is a perfect embedding of ${}^\omega\lambda$ into $C(\vec{\lambda})$ whose range is contained in X. ⊣

Proof of Theorem 1.4. Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega\rangle$ and set $\lambda = \sup_{n<\omega} \lambda_n$. Let X be a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{V_\lambda, \vec{\lambda}\}$. In $M^j_\omega[\vec{\lambda}]$, there is an injective enumeration $\vec{e} = \langle d_\alpha \mid \alpha < \lambda\rangle$ of V_λ with the property that $V_{\lambda_n} = \{d_\alpha \mid \alpha < \lambda_n\}$ holds for all $n < \omega$. Define Y to be the set of all $y \in C(\vec{\lambda})$ with the property that $y(0) = 0$ and there exists $A \in X$ with $d_{y(n+1)} = A \cap V_{\lambda_n}$ for all $n < \omega$. Then Y is a subset of $C(\vec{\lambda})$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $M^j_\omega[\vec{\lambda}] \cap V_{\lambda+1}$. Since $(2^\lambda)^{M^j_\omega[\vec{\lambda}]} < \lambda^+$, we can now combine Theorem 3.1 with Proposition 1.6 to find a perfect embedding of ${}^\omega\lambda$ into $C(\vec{\lambda})$ whose range is contained in Y. Using the fact

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

that the subspace X of $\mathcal{P}(\lambda)$ is homeomorphic to the subspace Y of $C(\vec{\lambda})$, we can now conclude that there is a perfect embedding of ${}^{\omega}\lambda$ into $\mathcal{P}(\lambda)$ whose range is contained in X. ⊣

§4. The $\vec{\mathcal{U}}$-Baire Property. In [5], a new type of regularity property for higher function spaces is introduced: the λ-*Baire property*. We can formalize this regularity property in a natural way as the λ-generalization of the classical Baire category notions:

Definition 4.1 [5]. Let X be a topological space and let A be a subset of X.

(i) The set A is λ-*meager in* X if it is a λ-union of nowhere dense sets.
(ii) The set A is λ-*comeager in* X if it is the complement of a λ-meager set, i.e., if it contains the intersection of λ-many open dense subsets of X.
(iii) The space X is a λ-*Baire space* if every non-empty open subset of X is not λ-meager.
(iv) The set A has the λ-*Baire property in* X if there exists an open set U in X such that the symmetric difference $A \Delta U$ is λ-meager.

Note that a space X is a λ-Baire space if and only if the intersection of λ-many open dense sets is dense. The definition of the $\vec{\mathcal{U}}$-Baire property is more complex, as a direct generalization is unfruitful[6]. It is strictly correlated to *diagonal Prikry forcing* (see, for example, [7, Section 1.3]). In the following, fix a strictly increasing sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega\rangle$ of measurable cardinals with limit λ and a sequence $\vec{\mathcal{U}} = \langle U_n \mid n < \omega\rangle$ with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$.

Definition 4.2. The *diagonal Prikry forcing with* $\vec{\mathcal{U}}$ is the partial order $\mathbb{P}_{\vec{\mathcal{U}}}$ defined by the following clauses:

(i) Conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$ are sequences $p = \langle p_n \mid n < \omega\rangle$ with the property that there exists $n < \omega$ such that $p_i < \lambda_i$ for all $i < n$ and $p_i \in U_i$ for all $n \leq i < \omega$. In this case, we set $s^p = \langle p_0, \dots, p_{n-1}\rangle$, $\mathrm{lh}(p) = \mathrm{lh}(s^p)$, and $A^p_i = p_i$ for all $n \leq i < \omega$. The sequence s^p is also called the *stem of p*.
(ii) Given conditions p and q in $\mathbb{P}_{\vec{\mathcal{U}}}$, we have $p \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} q$ if and only if the following statements hold:
   - $\mathrm{lh}(p) \geq \mathrm{lh}(q)$.
   - s^p is an end-extension of s^q.
   - If $\mathrm{lh}(q) \leq i < \mathrm{lh}(p)$, then $s^p(i) \in A^q_i$.
   - If $\mathrm{lh}(p) \leq i < \omega$, then $A^p_i \subseteq A^q_i$.

   Moreover, we say that $p \leq^*_{\mathbb{P}_{\vec{\mathcal{U}}}} q$ if $p \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} q$ and $\mathrm{lh}(p) = \mathrm{lh}(q)$.

The intuition behind the definitions below is the following: it is easy to see that the product topology on the classical Baire space is isomorphic to the topology of the maximal filters on Cohen forcing. Thus, we are going to define a topology on $C(\vec{\lambda})$ that is isomorphic to the topology of the maximal filters on the diagonal Prikry forcing. Note that we can define this only if λ is limit of measurable cardinals,

[6]In [5], it is proven that the space $C(\vec{\lambda})$ is the ω_1-union of nowhere dense sets

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

therefore this will be the only setting for which to consider our new regularity property.

DEFINITION 4.3. (i) Given a condition p in $\mathbb{P}_{\vec{\mathcal{U}}}$, we define

$$N_p = \{x \in \mathrm{C}(\vec{\lambda}) \mid \forall i < \omega\, [i < \mathrm{lh}(p) \to x(i) = s^p(i) \wedge i \geq \mathrm{lh}(p) \to x(i) \in A^p_j]\}.$$

(ii) The *Ellentuck-Prikry $\vec{\mathcal{U}}$-topology* on $\mathrm{C}(\vec{\lambda})$ (briefly, $\vec{\mathcal{U}}$-EP topology) is the topology whose basic open sets are of the form N_p for some condition p in $\mathbb{P}_{\vec{\mathcal{U}}}$.

(iii) A subset A of $\mathrm{C}(\vec{\lambda})$ has the *$\vec{\mathcal{U}}$-Baire property* if it has the λ-Baire property in the $\vec{\mathcal{U}}$-EP topology.

The results of [5] now show that the constructed topological spaces possess properties that generalize key properties of the classical Baire space to λ:

PROPOSITION 4.4 (λ-Baire Category, [5]). *The space $\mathrm{C}(\vec{\lambda})$ endowed with the $\vec{\mathcal{U}}$-EP topology is a λ-Baire space. Moreover, every subset of $\mathrm{C}(\vec{\lambda})$ that is λ-comeager in the $\vec{\mathcal{U}}$-EP topology contains a basic open set of this topology.*

To motivate the main results of this section, we first show that the above property is non-trivial.

THEOREM 4.5. *There exists a subset of $C(\vec{\lambda})$ without the $\vec{\mathcal{U}}$-Baire property.*

The fact that the $\vec{\mathcal{U}}$-EP topology is build using 2^λ-many basic open subsets stops the proof of the above result from being a routine diagonalization argument. Instead, we have to use strong combinatorial properties of $\mathbb{P}_{\vec{\mathcal{U}}}$ to reduce the class of relevant open subsets.

LEMMA 4.6 (Strong Prikry condition). *If D is a dense open subset of $\mathbb{P}_{\vec{\mathcal{U}}}$ and p is a condition in $\mathbb{P}_{\vec{\mathcal{U}}}$, then there exists a condition $q \leq^*_{\mathbb{P}_{\vec{\mathcal{U}}}} p$ and $n < \omega$ such that $r \in D$ holds for every condition $r \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} q$ with $\mathrm{lh}(r) \geq n$.*

COROLLARY 4.7. *If O is an open subset of $\mathbb{P}_{\vec{\mathcal{U}}}$ and p is a condition in $\mathbb{P}_{\vec{\mathcal{U}}}$, then there exists a condition $\bar{p} \leq^*_{\mathbb{P}_{\vec{\mathcal{U}}}} p$ such that if there exists a condition $q \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}$ with $q \in O$, then $r \in O$ holds for every $r \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}$ with $\mathrm{lh}(r) \geq \mathrm{lh}(q)$.*

Given a set P of conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$, we let

$$U_P = \bigcup\{N_p \mid p \in P\} \subseteq C(\vec{\lambda})$$

denote the corresponding open set in the $\vec{\mathcal{U}}$-EP topology.

PROPOSITION 4.8. *A set P of condition in $\mathbb{P}_{\vec{\mathcal{U}}}$ is predense in the partial order $\mathbb{P}_{\vec{\mathcal{U}}}$ if and only if U_P is dense in the $\vec{\mathcal{U}}$-EP topology.*

PROOF. First, assume that P is predense in $\mathbb{P}_{\vec{\mathcal{U}}}$ and fix a condition p in $\mathbb{P}_{\vec{\mathcal{U}}}$. Then there exists a condition q in P and a condition r in $\mathbb{P}_{\vec{\mathcal{U}}}$ with $r \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} p, q$. We now know that $\emptyset \neq N_r \subseteq N_p \cap N_q \subseteq N_p \cap U_P$.

Now, assume that U_P is dense in the $\vec{\mathcal{U}}$-EP topology and fix a condition p in $\mathbb{P}_{\vec{\mathcal{U}}}$. Since $N_p \cap U_P \neq \emptyset$, we can find a condition q in P and an element x of $C(\vec{\lambda})$

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

with $x \in N_p \cap N_q$. Then there exists a condition r in $\mathbb{P}_{\vec{\mathcal{U}}}$ with $r_i = x(i)$ for all $i < \max(\mathrm{lh}(p), \mathrm{lh}(q))$ and $A^r_i = A^p_i \cap A^q_i$ for all $\max(\mathrm{lh}(p), \mathrm{lh}(q)) \leq i < \omega$. We then know that $r \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} p, q$ holds. These computations show that P is predense in $\mathbb{P}_{\vec{\mathcal{U}}}$. ⊣

Lemma 4.9. *If U is an open set in the $\vec{\mathcal{U}}$-EP topology, then there exists a set P of at most λ-many conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$ such that $U_P \subseteq U$ and $U \setminus U_P$ is nowhere dense in the $\vec{\mathcal{U}}$-EP topology.*

Proof. Define O to be the set of all conditions p in $\mathbb{P}_{\vec{\mathcal{U}}}$ with the property that $N_p \subseteq U$. Then O is an open subset of $\mathbb{P}_{\vec{\mathcal{U}}}$. In addition, define S to be the set of all conditions p in $\mathbb{P}_{\vec{\mathcal{U}}}$ such that $p_i = \lambda_i$ holds for all $\mathrm{lh}(p) \leq i < \omega$. In this situation, Corollary 4.7 shows that for every $p \in P$, we can then find a condition $\bar{p} \leq^*_{\mathbb{P}_{\vec{\mathcal{U}}}} p$ with the property that if there is a condition $q \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}$ with $q \in O$, then $r \in O$ holds for every $r \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}$ with $\mathrm{lh}(r) \geq \mathrm{lh}(q)$. Define

$$P \ = \ \{\bar{p} \mid p \in S, \ \exists q \ [q \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p} \ \wedge \ q \in O]\}.$$

The fact that the set S has cardinality λ then ensures that P consists of at most λ-many conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$.

Claim. *$U_P \subseteq U$.*

Proof of the Claim. Pick $p \in S$ with the property that there is $q \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}$ with $q \in O$ and fix $x \in N_{\bar{p}}$. Then there exists a condition $r \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}$ with $r_i = x(i)$ for all $i < \mathrm{lh}(q)$ and $r_i = A^{\bar{p}}_i$ for all $\mathrm{lh}(r) \leq i < \omega$. Since $\mathrm{lh}(q) = \mathrm{lh}(r)$, we then know that $r \in O$ and $x \in N_r \subseteq U$. ⊣

Claim. *If $p \in S$ with $q \notin O$ for all $q \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}$, then $N_{\bar{p}} \cap U = \emptyset$.*

Proof of the Claim. Assume, towards a contradiction, that there is an $x \in N_{\bar{p}} \cap U$. Pick a condition q in $\mathbb{P}_{\vec{\mathcal{U}}}$ with $x \in N_q \subseteq U$. Then there exists a condition r in $\mathbb{P}_{\vec{\mathcal{U}}}$ with $r_i = x(i)$ for all $i < \max(\mathrm{lh}(\bar{p}), \mathrm{lh}(q))$ and $r_i = A^{\bar{p}}_i \cap A^q_i$ for all $\max(\mathrm{lh}(\bar{p}), \mathrm{lh}(q)) \leq i < \omega$. We then know that $r \leq_{\mathbb{P}_{\vec{\mathcal{U}}}} \bar{p}, q$ and $x \in N_r \subseteq N_q \subseteq U$. But this implies that r is an element of O below $\bar{p}$, a contradiction. ⊣

Define u to be the unique condition in $\mathbb{P}_{\vec{\mathcal{U}}}$ with $\mathrm{lh}(u) = 0$ and

$$A^u_i \ = \ \bigcap\{A^{\bar{p}}_i \mid p \in S, \ \mathrm{lh}(p) \leq i\}$$

for all $i < \omega$. In addition, set

$$N \ = \ \{x \in C(\vec{\lambda}) \mid \forall j < \omega \ \exists j \leq i < \omega \ x(i) \notin A^u_i\}.$$

Claim. *The set N is nowhere dense in the $\vec{\mathcal{U}}$-EP topology.*

Proof of the Claim. Assume, towards a contradiction, that N is dense in N_p for some condition p in $\mathbb{P}_{\vec{\mathcal{U}}}$. Let $q \leq^*_{\mathbb{P}_{\vec{\mathcal{U}}}} p$ be the unique condition with $A^q_i = A^p_i \cap A^u_i$ for all $\mathrm{lh}(p) \leq i < \omega$. Then there is $x \in N \cap N_q$ and we can find $\mathrm{lh}(q) \leq i < \omega$ with $x(i) \notin A^u_i$. But, this implies that $x(i) \notin A^q_i$, a contradiction. ⊣

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

CLAIM. $U \setminus U_P \subseteq N$.

PROOF OF THE CLAIM. Pick $x \in U \setminus U_P$ and fix $j < \omega$. Let p denote the unique element of S with $s^p = x \restriction j$. Then $x \notin N_{\bar{p}}$, because otherwise we would have $x \in N_{\bar{p}} \cap U \neq \emptyset$ and our second claim would imply that $N_{\bar{p}} \subseteq U_P$. Since $\bar{p} \leq^*_{\mathbb{P}_{\vec{\mathcal{U}}}} p$, we can now find $j \leq i < \omega$ with $x(i) \notin A^{\bar{p}}_i$. Our definitions then ensure that $A^u_i \subseteq A^{\bar{p}}_i$ and we can conclude that $x(i) \notin A^u_i$. These computations show that x is an element of N. ⊣

This last claim completes the proof of the lemma. ⊣

PROOF OF THEOREM 4.5. Define $\mathcal{O}$ to be the collection of all subsets of $C(\vec{\lambda})$ of the form U_P for some set P of at most λ-many conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$. Then the set $\mathcal{O}$ has cardinality at most 2^λ and we can fix an enumeration $\langle\langle U_\gamma, M_\gamma\rangle \mid \gamma < 2^\lambda\rangle$ of all pairs $\langle U, M\rangle$ such that $U \in \mathcal{O}$ and there exists a sequence $\langle O_\alpha \mid \alpha < \lambda\rangle$ of dense elements of $\mathcal{O}$ with $M = \bigcup_{\alpha<\lambda}(C(\vec{\lambda}) \setminus O_\alpha)$. We inductively define increasing sequences $\langle A_\gamma \mid \gamma < 2^\lambda\rangle$ and $\langle B_\gamma \mid \gamma < 2^\lambda\rangle$ of subsets of $C(\lambda)$ with $A_\gamma \cap B_\gamma = \emptyset$ and $|A_\gamma \cup B_\gamma| \leq |\gamma|$ for all $\gamma < 2^\lambda$. Fix $\gamma < 2^\lambda$ and assume that we already defined A_β and B_β for all $\beta < \gamma$. Set $A = \bigcup_{\beta<\gamma} A_\beta$ and $B = \bigcup_{\beta<\gamma} B_\beta$. Then $A \cap B = \emptyset$ and both sets have cardinality less than 2^λ. First, assume that U_γ is empty. Since Proposition 4.4 ensures that $C(\vec{\lambda}) \setminus M_\gamma$ has cardinality 2^λ, we can find $x \in C(\vec{\lambda}) \setminus (B \cup M_\gamma)$. We then define $A_\gamma = A \cup \{x\}$ and $B_\gamma = B$. Next, assume that U_γ is non-empty. Then Proposition 4.4 shows that $U_\gamma \setminus M_\gamma$ has cardinality 2^λ and we can find $x \in U_\gamma \setminus (A \cup M_\gamma)$. We now define $A_\gamma = A$ and $B_\gamma = B \cup \{x\}$. This completes our construction.

Define $A = \bigcup_{\gamma<2^\lambda} A_\gamma$ and $B = \bigcup_{\gamma<2^\lambda} B_\gamma$. Then $A \cap B = \emptyset$. Assume, towards a contradiction, that the set A has the $\vec{\mathcal{U}}$-Baire property. Pick an open subset U in the $\vec{\mathcal{U}}$-EP topology such that $A \,\Delta\, U$ is λ-meager in this topology. Then Lemma 4.9 shows that there exists $W \in \mathcal{O}$ with $W \subseteq U$ and $U \setminus W$ nowhere dense. It follows that $A \,\Delta\, W$ is also λ-meager. Another application of Lemma 4.9 then yields a sequence $\langle O_\alpha \mid \alpha < \lambda\rangle$ of dense elements of $\mathcal{O}$ with $A \,\Delta\, W \subseteq \bigcup_{\alpha<\lambda}(C(\vec{\lambda}) \setminus O_\alpha)$. In this situation, there exists a $\gamma < 2^\lambda$ with $U_\gamma = W$ and $M_\gamma = \bigcup_{\alpha<\lambda}(C(\vec{\lambda}) \setminus O_\alpha)$. Then $U_\gamma \neq \emptyset$, because otherwise our construction would ensure that there is $x \in A \setminus W$ with $x \notin M_\gamma$. But this means that there is $x \in B \cap U_\gamma$ with $x \notin M_\gamma$ and therefore $x \in A \cap B$, a contradiction. ⊣

We now proceed by showing that, in the model constructed in the proof of Theorem 2.3, the above constructions can also be used to find a simply definable set without the $\vec{\mathcal{U}}$-Baire property.

THEOREM 4.10. *If $j : V \longrightarrow M$ is an I2-embedding whose critical sequence has supremum λ, then the following statements hold in an inner model*:

(i) *There is an I2-embedding whose critical sequence has supremum λ.*
(ii) *If $\vec{\lambda} = \langle \lambda_n \mid n < \omega\rangle$ is a strictly increasing sequence of measurable cardinal with limit λ and $\vec{\mathcal{U}} = \langle U_n \mid n < \omega\rangle$ is a sequence with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$, then there is a subset z of λ and a subset X of $C(\vec{\lambda})$ such that X does not have the $\vec{\mathcal{U}}$-Baire property and the set X is definable by a Σ_1-formula with parameter z.*

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

PROOF. As in the proof of Theorem 2.3, pick a subset y of λ with $V_\lambda \cup \{j \restriction V_\lambda\} \subseteq L[y]$ and work in $L[y]$. Then there is an I2-embedding whose critical sequence has supremum λ. Fix a strictly increasing sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ of measurable cardinals with limit λ and a sequence $\vec{\mathcal{U}} = \langle U_n \mid n < \omega \rangle$ with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$. We can now find an unbounded subset z of λ with the property that the $\{\vec{\mathcal{U}}\}$ is definable by a Σ_1-formula with parameter z and there is a well-ordering $\lhd$ of H_{λ^+} of order-type λ^+ with the property that the set of all proper initial segments of $\lhd$ is definable by a Σ_1-formula with parameter z. It then directly follows that the set $\{\vec{\mathcal{U}}\}$, the set of all conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$, the ordering of $\mathbb{P}_{\vec{\mathcal{U}}}$, the compatibility relation of $\mathbb{P}_{\vec{\mathcal{U}}}$ and the incompatibility relation of $\mathbb{P}_{\vec{\mathcal{U}}}$ are all Δ_1-definable from the parameter z.

Now, define $\mathcal{O}$ to be the set of all pairs $\langle P, \vec{Q} \rangle$ with the property that P is a set of at most λ-many conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$ and $\vec{Q} = \langle Q_\alpha \mid \alpha < \lambda \rangle$ is a sequence with the property that each Q_α is a set of at most λ-many conditions in $\mathbb{P}_{\vec{\mathcal{U}}}$. It is then easy to see that $\mathcal{O}$ is a subset of H_{λ^+} of cardinality λ^+ that is definable by a Σ_1-formula with parameter z. Let $\langle \langle P_\gamma, \langle Q^\gamma_\alpha \mid \alpha < \lambda \rangle \rangle \mid \gamma < \lambda^+ \rangle$ denote the enumeration of $\mathcal{O}$ induced by $\lhd$. We then again know that this sequence is definable by a Σ_1-formula with parameter z. Arguing as in the proof of Theorem 4.5, we can now use Proposition 4.8 to show that for every $\gamma < \lambda^+$ with the property that Q^γ_α is predense in $\mathbb{P}_{\vec{\mathcal{U}}}$ for all $\alpha < \lambda$, the set $\bigcap_{\alpha<\lambda} U_{Q^\gamma_\alpha}$ has cardinality λ^+. Moreover, we know that for every $\gamma < \lambda^+$ with the property that $P_\gamma \neq \emptyset$ and Q^γ_α is predense in $\mathbb{P}_{\vec{\mathcal{U}}}$ for all $\alpha < \lambda$, the set $U_P \cap \bigcap_{\alpha<\lambda} U_{Q^\gamma_\alpha}$ has cardinality λ^+. This shows that there is a unique sequence $\langle d_\gamma \mid \gamma < \lambda^+ \rangle$ with the property that for all $\gamma < \lambda^+$, the set d_γ is the $\lhd$-least element of H_{λ^+} such that one of the following statements hold:

- The set d_γ is of the form $\langle 0, p \rangle$, where p is a condition in $\mathbb{P}_{\vec{\mathcal{U}}}$ with the property that there exists an $\alpha < \lambda$ such that all conditions in Q^γ_α are incompatible with p in $\mathbb{P}_{\vec{\mathcal{U}}}$.
- P_γ is the empty set and the set d_γ is of the form $\langle 1, x \rangle$, where x is an element of $\bigcap_{\alpha<\lambda} U_{Q^\gamma_\alpha}$ with the property that $x \neq v$ holds whenever $\beta < \gamma$ and d_β is of the form $\langle 2, v \rangle$ for some v in $C(\vec{\lambda})$.
- The set d_γ is of the form $\langle 2, x \rangle$, where x is an element of $U_{P_\gamma} \cap \bigcap_{\alpha<\lambda} U_{Q^\gamma_\alpha}$ with the property that $x \neq u$ holds whenever $\beta < \gamma$ and d_β is of the form $\langle 1, u \rangle$ for some u in $C(\vec{\lambda})$.

This definition then ensures that the sequence $\langle d_\gamma \mid \gamma < \lambda^+ \rangle$ is definable by a Σ_1-formula with parameter z. We define

$$A = \{x \in C(\vec{\lambda}) \mid \exists \gamma < \lambda^+ \ d_\gamma = \langle 1, x \rangle\}.$$

Then A is definable by a Σ_1-formula with parameter z and, by repeating the computations made in the proof of Theorem 4.5, we can show that A does not have the $\vec{\mathcal{U}}$-Baire property. ⊣

Contrary to the perfect set property case, there are no previous results about the possibility of Σ_1- or Σ^1_2-definable sets to have this kind of regularity property. In the following, we will again focus on the structural consequences of large cardinal assumptions close to the Kunen inconsistency. The following lemma will allow us to prove an analogue to Theorem 3.1 for the $\vec{\mathcal{U}}$-Baire property.

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

LEMMA 4.11. *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of measurable cardinals with supremum λ and let N be an inner model of* ZFC *with $V_\lambda \cup \{\vec{\lambda}\} \subseteq N$ and $(2^\lambda)^N < \lambda^+$. If $\vec{\mathcal{U}} = \langle U_n \mid n < \omega \rangle$ is a sequence in N with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$ and*

$$C \ = \ \{x \in C(\lambda) \mid \text{“}x \textit{ is } \mathbb{P}^N_{\vec{\mathcal{U}}}\text{– generic over} N\text{”}\},$$

then C is λ-comeager in the $\vec{\mathcal{U}}$-EP topology.

PROOF. By the *Mathias condition* for the diagonal Prikry forcing (see [6]), the set C consists of all $x \in C(\vec{\lambda})$ with the property that for every sequence $\vec{A} = \langle A_n \in U_n \mid n < \omega \rangle$ in N, the function x belongs to the dense open set

$$\{x \in \mathrm{C}(\vec{\lambda}) \mid \exists m < \omega \ \forall m \leq n < \omega \ x(n) \in A_n\}.$$

Since $(2^\lambda)^N < \lambda^+$, there are only λ-many dense open sets of this form and Proposition 4.4 yields the desired conclusion. ⊣

We are now ready to prove our analogue to Theorem 3.1.

THEOREM 4.12. *Let $j : V \longrightarrow M$ be an I2-elementary embedding with λ being the supremum of its critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and let N be an inner model of* ZFC *with $M^j_\omega \cup \{\vec{\lambda}\} \subseteq N$ and $(2^\lambda)^N < \lambda^+$. Then there exists a sequence $\vec{\mathcal{F}} = \langle F_n \mid n < \omega \rangle$ in N such that each F_n is a normal ultrafilter on λ_n and every subset of $\mathrm{C}(\vec{\lambda})$ that is definable over V_λ by a Σ^1_2-formula with parameters in $V^N_{\lambda+1}$ has the $\vec{\mathcal{F}}$-Baire property.*

PROOF. Since $V_\lambda \subseteq M^j_\omega \subseteq N$, $\vec{\lambda} \in N$ and each λ_n is a measurable cardinal in N, we can pick a sequence $\vec{\mathcal{F}} = \langle F_n \mid n < \omega \rangle$ in N such that each F_n is a normal ultrafilter on λ_n. Note that every condition in $\mathbb{P}^N_{\vec{\mathcal{F}}}$ is a condition in $\mathbb{P}^V_{\vec{\mathcal{F}}}$. In V, we define

$$C \ = \ \{x \in C(\lambda) \mid \text{“}x \textit{ is } \mathbb{P}^N_{\vec{\mathcal{F}}}\text{– generic over} N\text{”}\}.$$

Then Lemma 4.11 shows that C is λ-comeager in the $\vec{\mathcal{F}}$-EP topology.

Fix a Σ^1_2-formula $\varphi(w_0, w_1)$ with second-order variables w_0 and w_1 and $B \in V^N_{\lambda+1}$ such that the set

$$X \ = \ \{A \in V_{\lambda+1} \mid \langle V_\lambda, \in \rangle \models \varphi(A, B)\}$$

is a subset of $C(\vec{\lambda})$. Define O to be the set of all conditions p in $\mathbb{P}^N_{\vec{\mathcal{F}}}$ with

$$p \Vdash^N_{\mathbb{P}^N_{\vec{\mathcal{F}}}} \text{“} \langle V_{\check{\lambda}}, \in \rangle \models \varphi(\dot{x}, \check{B}) \text{”}, \tag{5}$$

where $\dot{x}$ denotes the canonical $\mathbb{P}^N_{\vec{\mathcal{F}}}$-name for the generic sequence in N.

Work in V and define U to be the union of all sets of the form N_p with $p \in O$. Fix $x \in C$. First, assume that $x \in U$ and fix $p \in O$ with $x \in N_p$. Since

$$G_x \ = \ \{p \in \mathbb{P}^N_{\vec{\mathcal{F}}} \mid x \in N_p\}$$

is the filter on $\mathbb{P}^N_{\vec{\mathcal{F}}}$ induced by x, we then know that

$$\langle V_\lambda, \in \rangle \models \varphi(x, B) \tag{6}$$

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

holds in $N[x]$ and therefore Corollary 3.6 shows that x is an element of X. In the other direction, assume that $x \in X$. Then (6) holds in V and Corollary 3.6 ensures that this statement also holds in $N[x]$. Then there is a condition p in G_x with the property that (5) holds. But then $p \in O$, $x \in N_p$ and hence $x \in U$. These computations now show that the sets U and $C(\vec{\lambda}) \setminus C$ witness that X has the $\vec{\mathcal{F}}$-Baire property. ⊣

A quick analysis of the proof shows that the consequences of the above theorem hold for every $\vec{\mathcal{F}} \in N$.

Note that, since $(2^\lambda)^{M^j_\omega[\vec{\lambda}]} < \lambda^+$ holds in the situation of the above theorem, there exists a sequence $\vec{\mathcal{F}}$ of normal measures such that every subset of $\mathrm{C}(\vec{\lambda})$ that is definable over V_λ by a Σ^1_2-formula with parameters in $V_\lambda \cup \{\vec{\lambda}\}$ has the $\vec{\mathcal{F}}$-Baire property.

In the remainder of this paper, we study the interaction of I0-embeddings with the λ-Baire property of families of sets. One of the key ingredients of the proof of Theorem 4.12 is Corollary 3.6, that states that there is a certain amount of absoluteness between V and models that contain $M^j_\omega[\vec{\lambda}]$. Woodin and Cramer proved that I0-embeddings also entail absoluteness-like results.

Remember that, given a limit ordinal λ, we define

$$\Theta^{L(V_{\lambda+1})} \;=\; \sup\{\alpha \in \mathrm{Ord} \mid \textit{There is a surjection } \pi : V_{\lambda+1} \longrightarrow \alpha \textit{ in } L(V_{\lambda+1})\}.$$

This concept generalizes the definition of Θ for $L(\mathbb{R})$. Since $L(\mathbb{R})$ is not going to appear in this paper and there is no risk of confusion, we will below write Θ instead of $\Theta^{L(V_{\lambda+1})}$. An ordinal $\alpha < \Theta$ is called *good* if every element of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from an element of $V_{\lambda+1}$. The next theorem is called *Generic Absoluteness* in [18].

Theorem 4.13 (Woodin, [2, Theorem 82]). *Let $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ be an $I0$-embedding that is ω-iterable and let $j_{0,\omega} : L(V_{\lambda+1}) \longrightarrow M_\omega$ be the embedding into the ω-th iterate of $L(V_{\lambda+1})$ by j. Assume that $\mathbb{P} \in M_\omega$ is a partial order and $g \in V$ is $\mathbb{P}$-generic over M_ω with $\mathrm{cof}(\lambda)^{M_\omega[g]} = \omega$. If $\alpha < \Theta$ is good, then for some $\bar{\alpha} < \lambda$, there is an elementary embedding*

$$\pi : L_{\bar{\alpha}}(M_\omega[g] \cap V_{\lambda+1}) \longrightarrow L_\alpha(V_{\lambda+1})$$

that is the identity below λ.

Note that, in the situation of the above theorem the good ordinals are cofinal in Θ (see [12]). Moreover, if there exists an I0-embedding, then there exists an iterable I0-embedding (see [18, Lemmas 10 and 21]). Therefore, the hypothesis of the above result is not restrictive.

Theorem 4.14. *Let $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ be an $I0$-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$. Then there exists a sequence $\vec{\mathcal{F}} = \langle F_n \mid n < \omega \rangle$ such that each F_n is a normal ultrafilter on λ_n and every subset of $\mathrm{C}(\vec{\lambda})$ that is definable over V_λ by a Σ^1_n-formula with parameters in $V_{\lambda+1}$ has the $\vec{\mathcal{F}}$-Baire property.*

Proof. By earlier remarks, we may assume that j is ω-iterable. In the following, we let $j_{0,\omega} : L(V_{\lambda+1}) \longrightarrow M_\omega$ denote the embedding into the ω-th iterate of $L(V_{\lambda+1})$ by j. Then $\vec{\lambda}$ is Prikry-generic over M_ω and there is a sequence $\vec{\mathcal{F}} = \langle F_n \mid n < \omega \rangle$ in

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

$M_\omega[\vec{\lambda}]$ such that each F_n is a normal ultrafilter on λ_n. Finally, we define $\mathbb{P}$ to be the corresponding diagonal Prikry forcing $\mathbb{P}_{\vec{\mathcal{F}}}^{M_\omega[\vec{\lambda}]}$ in $M_\omega[\vec{\lambda}]$.

Given $n < \omega$, we fix a Σ^1_n-formula $\varphi(w_0, w_1)$ in the language of set theory with free second-order variables w_0 and w_1. Given $y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}$, we define

$$X_{\varphi,y} \;=\; \{x \in \mathrm{C}(\vec{\lambda}) \mid \langle V_\lambda, \in\rangle \models \varphi(x, y)\}.$$

As in the proof of Theorem 4.12, we now define $O_{\varphi,y}$ to be the open subset of $\mathbb{P}$ in $M_\omega[\vec{\lambda}]$ that consists of all conditions p with

$$p \Vdash_{\mathbb{P}}^{M_\omega[\vec{\lambda}]} \text{“}\langle V_{\check{\lambda}}, \in\rangle \models \varphi(\dot{x}, \check{y})\text{”},$$

where $\dot{x}$ denotes the canonical $\mathbb{P}$-name for the generic sequence in $M_\omega[\vec{j}]$. In addition, we let $U_{\varphi,y}$ denote the union of all sets N_p with $p \in O_{\varphi,y}$ in V. Finally, we define C to be the set of all x in $\mathrm{C}(\vec{\lambda})$ that are $\mathbb{P}$-generic over $M_\omega[\vec{\lambda}]$. Since $(2^\lambda)^{M_\omega[\vec{\lambda}]} < \lambda^+$, an application of Lemma 4.11 shows that C is λ-comeager in the $\vec{\mathcal{F}}$-EP topology.

Now, fix x in C. Still following the proof of Theorem 4.12, we then know that x is an element of $U_{\varphi,y}$ if and only if

$$\langle V_\lambda, \in\rangle \models \varphi(x, y)$$

holds in $M_\omega[\vec{\lambda}, x]$. The model $M_\omega[\vec{\lambda}, x]$ is a generic extension (via the forcing that is a two-step iteration of Prikry and diagonal Prikry forcing) of M_ω and $\mathrm{cof}(\lambda)^{M_\omega[\vec{\lambda},x]} = \omega$. Therefore, we can apply Generic Absoluteness to $M_\omega[\vec{\lambda}, x]$ to show that $x \in U_{\varphi,y}$ if and only if $x \in X_{\varphi,y}$. These computations show that $U_{\varphi,y} \,\Delta\, X_{\varphi,y} \subseteq C(\vec{\lambda}) \setminus C$ and we can conclude that the set $X_{\varphi,y}$ has the $\vec{\mathcal{F}}$-Baire property.

We now know that the statement

$$\text{“}X_{\varphi,y} \textit{ has the } \vec{\mathcal{F}}\textit{-Baire property}\text{”} \tag{7}$$

holds in $L(V_{\lambda+1})$ for every Σ^1_n-formula $\varphi(w_0, w_1)$ and for all $y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}$. We claim that this statement can be expressed by a formula that only uses a single existential quantifier bounded by the set $V_{\lambda+2}$ of all subsets of $V_{\lambda+1}$. Notice that, as a consequence of this, it follows that the $\vec{\mathcal{F}}$-Baire property is upward absolute. By definition of $\vec{\mathcal{F}}$-Baire property, the set $X_{\varphi,y}$ has the $\vec{\mathcal{F}}$-Baire property if and only if there exist an open subset U of $\mathrm{C}(\vec{\lambda})$ and a sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$ of closed nowhere dense subsets of $\mathrm{C}(\vec{\lambda})$ with the property that $A \,\Delta\, U \subseteq \bigcup_{\alpha<\lambda} C_\alpha$. Notice now that each open set W is determined by the subset $\{p \in \mathbb{P}_{\vec{\mathcal{F}}} \mid N_p \subseteq W\}$ of $V_{\lambda+1}$. Hence, the set U and the sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$ can be determined by a λ-sequence of subsets of $V_{\lambda+1}$, which in turn can be canonically identified with a subset of $V_{\lambda+1}$. It is now easy to see that the claim holds.

Now, given $y \in V_{\lambda+1}$ with the property that (7) holds in $L(V_{\lambda+1})$, we define α_y to be the least ordinal α below Θ such that (7) holds in $L_\alpha(V_{\lambda+1})$. Such an ordinal exists below Θ because all the subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ are elements of $L_\Theta(V_{\lambda+1})$ (see, for example, [3, Lemma 5.6]). In addition, we define $\alpha_y = 0$ for all $y \in V_{\lambda+1}$ with the property that (7) fails in $L(V_{\lambda+1})$. The resulting function $y \mapsto \alpha_y$ is then definable in $L(V_{\lambda+1})$. We now want to prove that

$$\alpha \;=\; \sup\{\alpha_y \mid y \in V_{\lambda+1}\} \;<\; \Theta.$$

For any $x \in V_{\lambda+1}$, we define $<_x$ to be the canonical well-ordering of $HOD_x^{L(V_{\lambda+1})}$, the inner model of all sets hereditarily definable in $L(V_{\lambda+1})$ with ordinals and x as parameters.[7] In addition, for all $x, y \in V_{\lambda+1}$, we let $g_x(y)$ denote the $<_x$-smallest surjection from $V_{\lambda+1}$ to α_y, if it exists and otherwise $g_x(y) = 0$. The map $x \mapsto g_x$ is also definable in $L(V_{\lambda+1})$. It is now easy to see that the function f defined by

$$f(x, y, z) = \begin{cases} g_x(y)(z), & \text{if } g_x(y) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

is a surjection from $V^3_{\lambda+1}$ to α and hence $\alpha < \Theta$.

In particular, we know that (7) holds in $L_\alpha(V_{\lambda+1})$ for all $y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}$. By the fact that the sequence of good ordinals is cofinal in Θ and that the $\vec{\mathcal{F}}$-Baire property is upward absolute, we can assume that α is good. Then, by Theorem 4.13, there exist $\bar{\alpha} < \lambda$ and an elementary embedding

$$\pi : L_{\bar{\alpha}}(M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) \longrightarrow L_\alpha(V_{\lambda+1})$$

such that $\pi \restriction (M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) = \mathrm{id}_{M_\omega[\vec{\lambda}] \cap V_{\lambda+1}}$. Thus, we can conclude that

$$\begin{aligned}
&\forall y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}\ L_\alpha(V_{\lambda+1}) \models \text{“}X_{\varphi,y} \textit{ has the } \vec{\mathcal{F}}\textit{-Baire property”}\\
\Longleftrightarrow\ &\forall y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}\ L_{\bar{\alpha}}(M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) \models \text{“}X_{\varphi,y} \textit{ has the } \vec{\mathcal{F}}\textit{-Baire property”}\\
\Longleftrightarrow\ &L_{\bar{\alpha}}(M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) \models \forall y \in V_{\lambda+1} \text{ “}X_{\varphi,y} \textit{ has the } \vec{\mathcal{F}}\textit{-Baire property”}\\
\Longleftrightarrow\ &L_\alpha(V_{\lambda+1}) \models \forall y \in V_{\lambda+1} \text{ “}X_{\varphi,y} \textit{ has the } \vec{\mathcal{F}}\textit{-Baire property”}.
\end{aligned}$$

These computation show that every set of the form $X_{\varphi,y}$ with $y \in V_{\lambda+1}$ has the $\vec{\mathcal{F}}$-Baire property. ⊣

§5. Open questions. We close this paper by stating two questions raised by the above results. As mentioned in the introduction, our results suggest that large cardinals assumptions can be studied through the provable validity of Perfect Set Theorems for simply definable sets at singular cardinals of countable cofinality. In particular, our results suggest that the existence of an I2-embedding with critical sequence $\vec{\lambda}$ naturally corresponds to the validity of a Perfect Set Theorem for subsets of $C(\vec{\lambda})$ that are definable by Σ_1-formulas with parameters in $V_\lambda \cup \{\vec{\lambda}\}$, where λ is the supremum of the sequence $\vec{\lambda}$. We therefore ask if the conclusion of Theorem 1.4 can also be derived from substantially weaker large cardinal assumptions:

QUESTION 5.1. *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of cardinals with supremum λ such that λ_n is a $<\lambda$-supercompact cardinal for all $n < \omega$. If X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{\vec{\lambda}\}$, is there a perfect embedding $\iota : {}^\omega\lambda \longrightarrow \mathcal{P}(\lambda)$ with* $\mathrm{ran}(\iota) \subseteq X$?

[7]It is a standard argument that $L(V_{\lambda+1}) = \bigcup\{HOD_x^{L(V_{\lambda+1})} \mid x \in V_{\lambda+1}\}$

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

If yes, what about subsets of $\mathcal{P}(\lambda)$ that are definable by Σ_1-formulas with parameters in $V_\lambda \cup \{V_\lambda, \vec{\lambda}\}$?

In another direction, we also ask which large cardinal assumptions are necessary to overcome the limitations to the influence of I2-embeddings given by Theorem 1.5. Note that, by Theorem 1.1, an I0-embedding suffices for this task. Remember that an I1-embedding is a non-trivial elementary embedding $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$.

QUESTION 5.2. *Let $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$ be an $I1$-embedding. If X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_{\lambda+1}$, is there a perfect embedding $\iota : {}^{\omega}\lambda \longrightarrow \mathcal{P}(\lambda)$ with $\mathrm{ran}(\iota) \subseteq X$? If yes, what about subsets of $\mathcal{P}(\lambda)$ that are definable over V_λ by Σ^1_n-formulas with parameters in $V_{\lambda+1}$?*

Acknowledgments. The authors are thankful to the anonymous referees for the careful reading of the manuscript and several helpful comments.

Funding. The first and second author partially supported by the Italian PRIN 2017 Grant "Mathematical Logic: models, sets, computability", the first author is further partially supported by the Italian PRIN 2022 Grant "Models, Sets and Classifications". The third author gratefully acknowledges support from the Deutsche Forschungsgemeinschaft (Project number 522490605) and the Spanish Government under grant EUR2022-134032.

REFERENCES

[1] S. CRAMER, *Inverse limit reflection and the structure of $L(V_{\lambda+1})$*. ***Journal of Mathematical Logic***, vol. 15 (2015), no. 1, 1550001, pp. 38.

[2] ———, *Implications of very large cardinals*, ***Foundations of Mathematics***, Contemporary Mathematics, vol. 690, American Mathematical Society, Providence, RI, 2017, pp. 225–257.

[3] V. DIMONTE, *I0 and rank-into-rank axioms*. ***Bollettino della Unione Matematica Italiana***, vol. 11 (2018), no. 3, pp. 315–361.

[4] V. DIMONTE and L. M. ROS, *Generalized descriptive set theory at singular cardinals*, in preparation.

[5] V. DIMONTE, L. M. ROS, and X. SHI, *Generalized descriptive set theory at singular cardinals II*, in preparation.

[6] G. FUCHS, *A characterization of generalized Příkrý sequences*. ***Archive for Mathematical Logic***, vol. 44 (2005), no. 8, pp. 935–971.

[7] M. GITIK, *Prikry-type forcings*, ***Handbook of Set Theory***, Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1351–1447.

[8] T. JECH, ***Set Theory***, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, the third millennium edition, revised and expanded.

[9] A. KANAMORI, ***The Higher Infinite: Large Cardinals in Set Theory from their Beginnings***, second ed, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[10] K. KUNEN, *Elementary embeddings and infinitary combinatorics*. ***The Journal of Symbolic Logic***, vol. 36 (1971), no. 3, pp. 407–413.

[11] R. LAVER, *Implications between strong large cardinal axioms*. ***Annals of Pure and Applied Logic***, vol. 90 (1997), no. 1–3, pp. 79–90.

[12] ———, Reflection of elementary embedding axioms on the $L[V_{\lambda+1}]$ hierarchy. ***Annals of Pure and Applied Logic***, vol. 107 (2001) no. 1-3, pp. 227–238.

[13] P. LÜCKE and S. MÜLLER, Σ_1 *-definability at higher cardinals: thin sets, almost disjoint families and long well-orders*. ***Forum of Mathematics, Sigma***, vol. 11 (2023): Paper No. e103, pp. 36.

[14] P. LÜCKE and P. SCHLICHT, *Continuous images of closed sets in generalized Baire spaces*. ***Israel Journal of Mathematics***, vol. 209 (2015), no. 1, pp. 421–461.

https://doi.org/10.1017/jsl.2024.75 Published online by Cambridge University Press

[15] D. A. Martin, *Infinite games*, ***Proceedings of the International Congress of Mathematicians (Helsinki, 1978)***, Annales Academiæ Scientiarum Fennicæ, Helsinki, 1980, pp. 269–273.

[16] X. Shi, *Axiom I_0 and higher degree theory*. ***Journal of Symbolic Logic***, vol. 80(2015) no. 3, pp. 970–1021.

[17] J. R. Steel, *An introduction to iterated ultrapowers*, ***Forcing, Iterated Ultrapowers, and Turing Degrees***, Lecture Notes Series Institute for Mathematical Sciences National University of Singapore, vol. 29, World Scientific Publication, Hackensack, NJ, 2016, pp. 123–174.

[18] W. Hugh Woodin, *Suitable extender models II: beyond ω -huge*. ***Journal of Mathematical Logic***, vol. 11 (2011), no. 2, pp. 115–436.

DIPARTIMENTO DI SCIENZE MATEMATICHE, INFORMATICHE E FISICHE
UNIVERSITÀ DEGLI STUDI DI UDINE
VIA DELLE SCIENZE, 206
UDINE 33100, ITALY

E-mail: vincenzo.dimonte@uniud.it

DEPARTMENT OF DISCRETE MATHEMATICS AND GEOMETRY TU WIEN
WIEDNER HAUPTSTRASSE 8–10
VIENNA 1040
AUSTRIA

E-mail: martina.iannella@tuwien.ac.at

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG
BUNDESSTRASSE 55
HAMBURG, 20146
GERMANY

E-mail: philipp.luecke@uni-hamburg.de