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A NOTE ON INTERMEDIATE NORMALISING EXTENSIONS

S. DĂSCĂLESCU AND L. VAN WYK

We prove that the following ring-theoretic properties are shared by the two rings involved in a normalising extension $R \subset S$, and that these properties are inherited by any intermediate extension: semilocal, left perfect, semiprimary. This transfer fails for the nilpotency of the Jacobson radical. However, if the normalising set is a basis for the left R-module S, then the nilpotency of the Jacobson radical behaves in the same way as the three properties mentioned above.

Every ring herein is associative with identity, and a subring is assumed to inherit the identity. We denote the Jacobson radical of a ring R by $\mathcal{J}(R)$.

The Morita equivalence of a ring R and the full matrix ring $\mathbb{M}_n(R)$ ensures that many properties are shared by these two rings. Various kinds of subrings of $\mathbb{M}_n(R)$ have provided a wealth of interesting examples and counterexamples. Moreover, these rings play a significant role in the structure theory of rings. For example, in [**8**] left Artinan CI-prime rings were characterised as complete blocked triangular matrix rings over division rings. A study of triangular matrix rings, or in general structural matrix rings, in their own right has also recently received considerable attention (see [**2**, **6**, and **13**]). The motivation for the sequel is the natural question whether R and a structural matrix ring over R share classical ring-theoretic properties in spite of the fact that they are in general not Morita equivalent. Furthermore, there are, of course, numerous examples of subrings of $\mathbb{M}_n(R)$ which are not structural matrix rings. Our results will shed light on them too.

The same generic problem for the relationship between a ring R and the skew group ring RG, G a finite group, has been considered by several authors. Nevertheless, the study of subrings of skew group rings has not been very extensive. Another application of our results will be the transfer of certain ring-theoretic properties between a ring R and the semigroup ring RS, S a finite monoid, and the inheritance of these properties by an intermediate extension. We refer the reader to [**9**] for a survey of semigroup rings.

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A suitable framework for the investigation of the transfer of the following ring-theoretic properties in the three cases discussed above is that of a normalising extension:

(i) R is *semilocal*, that is, $R/\mathcal{J}(R)$ is a left (or right) Artinian ring;
(ii) R is *left perfect*, that is, R is semilocal and $\mathcal{J}(R)$ is left T-nilpotent;
(iii) R is *semiprimary*, that is, R is semilocal and $\mathcal{J}(R)$ is nilpotent;
(iv) $\mathcal{J}(R)$ is nilpotent.

Recall from [4] that a ring S is called a *normalising extension of* a subring R if S is finitely generated as a left R-module by elements $s_1, s_2, \dots, s_n$ (for some n) which normalise R, that is, $s_i R = R s_i$ for $i = 1, 2, \dots, n$. The set $\{s_1, s_2, \dots, s_n\}$ is called a *normalising set*, and any ring A such that $R \subset A \subset S$ is called an *intermediate normalising extension of* R. The purpose of this note is to prove the following

Theorem. *Let $R \subset A \subset S$, with $R \subset S$ a normalising extension.*

1. *If either of the rings R, S is (i) semilocal, or (ii) left perfect, or (iii) semiprimary, then so is the other, and moreover, the same property is inherited by A.*
2. *If the normalising set $\{s_1, s_2, \dots, s_n\}$ is a basis for the left R-module S, then $\mathcal{J}(R)$ is nilpotent if and only if $\mathcal{J}(S)$ is nilpotent, and in this case $\mathcal{J}(A)$ is nilpotent too.*

The transfer of the properties in (ii) and (iii) between R and the skew group ring RG was established by Park in [10]. However, his proofs for passing from R to RG can be simplified considerably; actually this simplification applies to any ring extension $R \subset S$ with S finitely generated as a right R-module. To be more precise, assuming that $R \subset S$ with S finitely generated as a right R-module, then, first, in order to see that S is left perfect if R is left perfect, it suffices to recall that R is left perfect if and only if every right R-module satisfies the DCC on finitely generated submodules [**7**, Theorem 23.20], and, second, [**1**, Corollary 0.1] shows that S is semiprimary if R is semiprimary. This was also noticed by Resco in [**12**], where he showed that if $R \subset S$ is a normalising extension, then R is left perfect (respectively semiprimary) if and only S is left perfect (respectively semiprimary). Although we are mainly interested in intermediate normalising extensions, we included these results in our Theorem for the sake of a holistic picture.

Let $R \subset S$ be a normalising extension, and let A be an intermediate normalising extension. We note that A is not necessarily finitely generated as a left R-module, as can be easily seen by taking

$$S = \begin{bmatrix} R & R \\ 0 & R \end{bmatrix} \quad \text{(with the diagonal embedding of } R\text{) and} \quad A = \begin{bmatrix} R & I \\ 0 & R \end{bmatrix},$$

with I a two-sided ideal of R which is not finitely generated as a left ideal, and so the above mentioned techniques are not applicable for passing from R to A.

Our main tools will be the results of Heinicke and Robson in [**5**] concerning the relationship between the Jacobson radicals of R, A and S. We point out that part 2 of our Theorem does not hold for arbirary normalising extensions, as the next example shows.

EXAMPLE. Let $R = k[X_1, X_2, \dots]$ be the ring of polynomials in a countable set of indeterminates over a field k, and let I be the ideal of R generated by $X_1, X_2^2, \dots, X_n^n, \dots$. Then the ideal of R/I generated by the images of $X_2, \dots, X_n, \dots$ in R/I is nil, and so it is contained in $\mathcal{J}(R/I)$; however, it is not nilpotent. Set $S = R \times R/I$ and embed R in S by $r \mapsto (r, \bar{r})$, where $\bar{r}$ denotes the images of R in R/I Then $\mathcal{J}(S)$ is not nilpotent, but $\mathcal{J}(R) = \{0\}$.

We also note that part 2 of our theorem was proved in [**11**, Theorem 7.2.5] under stronger hypotheses on the normalising extension $R \subset S$.

PROOF OF THE THEOREM: Recall first from [**5**, Corollaries 4.8 and 5.5] that

$$\text{(I)} \qquad \mathcal{J}(R) = \mathcal{J}(S) \cap R,$$

$$\text{(II)} \qquad (\mathcal{J}(A))^n \subset \mathcal{J}(S) \cap A \subset \mathcal{J}(A).$$

1. (i) By (I) and the second part of (II) we obtain the ring extension

$$R/\mathcal{J}(R) \subset A/(\mathcal{J}(S) \cap A) \subset S/\mathcal{J}(S),$$

with $R/\mathcal{J}(R) \subset S/\mathcal{J}(S)$ a normalising extension. Now by [**3**, Theorem 4] $R/\mathcal{J}(R)$ is left Artinian if and only if $S/\mathcal{J}(S)$ is left Artinian. Standard arguments show that in this case $A/(\mathcal{J}(S) \cap A)$ is left Artinian, and so $A/\mathcal{J}(A)$ is too.

(ii) and (iii). Since R is left perfect if and only if every right R-module satisfies the DCC on finitely generated R-submodules, and since S is a finitely generated right R-module, it follows that if R is left perfect, then S is too. Next, if R is semiprimary, then by [**1**, Corollary 0.1] S is semiprimary.

Conversely, the first part of (II) shows that $\mathcal{J}(A)$ is left T-nilpotent (respectively nilpotent) whenever $\mathcal{J}(S)$ is left T-nilpotent (respectively nilpotent); in particular if $A = R$. The result now follows from (I).

2. If $\mathcal{J}(S)$ is nilpotent, then again by the first part of (II) $\mathcal{J}(A)$ is nilpotent; in particuar $\mathcal{J}(R)$ is nilpotent. Conversely, suppose now that $\mathcal{J}(R)$ is nilpotent. Then $\mathcal{J}(\mathbb{M}_n(R))$ is nilpotent too. Let $1 \leqslant i \leqslant n$. For every $r \in R$ there is a unique $f_i(r) \in R$ such that $f_i(r) = a_i r$. It is easy to check that the f_i's are surjective endomorphisms of the ring R. Let $\varphi : S \to \mathrm{End}({}_R S)$ be the injective ring morphism defined by $\varphi(s)(x) =$

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xs, $x, s \in S$, where the multiplication in the ring $\mathrm{End}(_R S)$ is inverse composition. Let $\psi : \mathrm{End}(_R S) \to \mathbb{M}_n(R)$ be the ring isomorphism induced by the basis $\{a_1, \dots, a_n\}$. Since for $r \in R$ the matrix $\psi\varphi(r)$ is diagonal with $f_1(r), \dots, f_n(r)$ on the diagonal, the ring extension $\psi\varphi(R) \subset \mathbb{M}_n(R)$ is normalising with the set $\{e_{ij} : 1 \leqslant i, j \leqslant n\}$ as a normalising set. Moreover, $\psi\varphi(S)$ is an intermedaite normalising extension, and so by the implication we already proved its Jacobson radical is nilpotent. ∎

We conclude that if R has any of the properties mentioned in (i)-(iv), then every subring of $\mathbb{M}_n(R)$ containing the scalar matrices has that property too. The same result applies to any ring A such that $R \subset A \subset RG$ or $R \subset A \subset RS$, where RG denotes the skew group ring, G a finite group, and RS denotes the semigroup ring, S a finite monoid.

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Facultatea de Matematica
University of Bucharest
Str. Academiai 14
R70109 Bucharest 1
Romania

Department of Mathematics
University of Stellenbosch
Stellenbosch 7600
South Africa
E-mail: lvw@maties.sun.ac.za