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A MAXIMUM PRINCIPLE

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Abstract

Let K be a nonempty compact set in a Hausdorff locally convex space, and F a nonempty family of upper semicontinuous convex-like functions from K into $[-\infty, \infty)$. K is partially ordered by F in a natural manner. It is shown among other things that each isotone, upper semicontinuous and convex-like function $g: K \to [-\infty, \infty)$ attains its K-maximum at some extreme point of K which is also a maximal element of K.

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Let K be a nonempty compact set in a Hausdorff locally convex topological vector space E and F a nonempty family of convex, upper semi-continuous functions from K into $[-\infty, \infty)$. Recall that a function f on K is *convex* if

$$f(\lambda_1 k_1 + \lambda_2 k_2) \leqslant \lambda_1 f(k_1) + \lambda_2 f(k_2)$$

whenever $0 \leqslant \lambda_1,\ \lambda_2 \leqslant 1,\ \lambda_1 + \lambda_2 = 1$ and $k_1,\ k_2,\ \lambda_1 k_1 + \lambda_2 k_2$ are in K, and that f is said to be *affine* if both f and $-f$ are convex. Also K has a natural quasi-ordering induced by F:

$$x \leqslant y \quad \text{if and only if} \quad f(x) \leqslant f(y) \text{ for all } f \in F.$$

We write $x \sim y$ if $x \leqslant y$ and $y \leqslant x$. An element x of K is said to be *maximal* if $y \sim x$ whenever $x \leqslant y$. The *F-boundary* of K is, by definition, the set of all maximal extreme points of K and will be denoted by $\partial_F K$. In the special case when K is convex and each f in F is affine, a generalized Bauer's maximum principle proved by Lumer (1963) and Edwards (1970) asserts that each f in F attains its K-maximum on $\partial_F K$. In some situations one wishes to have a similar principle applicable to certain non-affine, even non-convex functions. For example, if K is taken to be the closed unit disc Δ in the complex plane and F the set $S(\Delta)$ of all continuous functions f on

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Δ such that f is subharmonic on the interior of Δ, then $\partial_F \Delta$ is precisely the topological boundary $\partial\Delta$, that is, the circumference of Δ; hence, by the classical maximum principle (see Conway (1973), p. 266), each f in F attains its Δ-maximum on the F-boundary $\partial_F \Delta$ of Δ, though f may not be convex on Δ. In this note, we extend the above theorem of Lumer and Edwards to the case when K and f in F are not necessarily convex.

Recall first that a non-empty subset A of K is *extreme* if $x, y \in A$ whenever $\lambda x + (1-\lambda) y \in A$ for some $\lambda \in (0,1)$ and $x, y \in K$. A function $f\colon K \to [-\infty, \infty)$ is said to be *convex-like* if, for each closed extreme subset A of K, the set

$$\{a \in A : f(a) = \sup f(A)\}$$

is either empty or else an extreme subset of A. Thus, each convex function is certainly convex-like; also each function f in $S(\Delta)$ is convex-like on Δ because proper extreme subsets of Δ are the sets contained in $\partial\Delta$. From now on we shall assume that F is a nonempty family of upper semi-continuous and convex-like functions from a compact (not necessarily convex) set K in E into $[-\infty, \infty)$ and that K is ordered by $\leqslant$ induced by F. An extended real-valued function g on K is said to be *isotone* if $g(x) \leqslant g(y)$ whenever $x \leqslant y$ in K. For example, each f in F is isotone; more generally, if g is the limit function of a pointwise convergent net in F, then g is isotone.

THEOREM 1. *Each isotone, upper semi-continuous and convex-like function* $g\colon K \to [-\infty, \infty)$ *attains its K-maximum on the F-boundary* $\partial_F K$ *of K.*

PROOF. Let $\mathscr{E}$ be the collection of all nonempty extreme, compact and increasing subsets of K (a subset A of K is *increasing* if $k \in A$ whenever $a \leqslant k$ for some $a \in A$). Since $K \in \mathscr{E}$, $\mathscr{E}$ is a non-empty set, partially ordered by set inclusion. By Zorn's Lemma, each member of $\mathscr{E}$ contains a minimal member of $\mathscr{E}$. By assumption on g, the set

$$G = \{x \in K\colon g(x) = \sup g(K)\}$$

is a member of $\mathscr{E}$ and hence contains a minimal member, say Q of $\mathscr{E}$. Since Q is a compact extreme subset of K, Q contains at least one extreme point x of K by virtue of the Krein–Milman theorem. It remains to show that x is maximal in the quasi-ordered set K. For a contradiction, let us assume that there exists y in K such that $x \leqslant y$ but $y \nleqslant x$. Then $y \in Q$ since Q is increasing, and there exists f in F such that $f(y) > f(x)$. Notice that the set

$$\{z \in Q : f(z) = \sup f(Q)\}$$

is a member of $\mathscr{E}$, properly contained in Q. This contradicts the minimality of Q.

The following theorem was proved by Bauer in the special case when K is convex.

THEOREM 2. *Let K be a compact subset of E. Then each upper semi-continuous convex function $f: K \to [-\infty, \infty)$ attains its K-maximum on the extreme boundary $\partial_e K$ of K.*

Indeed, if we take F to consist of all upper semi-continuous convex functions on K, then the partial ordering induced by F is simply equality and $\partial_e K = \partial_F K$.

An immediate consequence of Theorem 2 is the following strong version of the Krein–Milman theorem: each compact subset K in E is contained in the closed convex hull of its extreme points. Moreover, Theorem 1 may also be used to prove the following generalization of Dini's theorem.

THEOREM 3. *Let $\{g_i: i \in I\}$ be a downward directed family of isotone, upper semi-continuous convex functions on K into $[0, \infty)$ such that $\lim_i g_i(x) = 0$ for each x in the F-boundary $\partial_F K$. Then $\{g_i\}$ converges to 0 uniformly on K.*

PROOF. Let g_0 denote the limit function of $\{g_i\}$. Then g_0 satisfies the conditions in Theorem 1 and $g_0 \geqslant 0$ on K with equality on $\partial_F K$. By Theorem 1, we must have $g_0 = 0$ on K. Consequently, it follows from the classical theorem of Dini that $\{g_i\}$ converges to $g_0 = 0$ uniformly on K.

Finally, we note an interesting application of Theorem 3 to the theory of ordered vector spaces. Let V be a partially ordered normed space with a normal cone C and (V', C') the partially ordered Banach dual space (with the natural dual ordering). Let $K = \{v' \in C': \|v'\| \leqslant 1\}$. Then K is a compact convex subset of V' under the $\sigma(V', V)$-topology. Let F be the set of all continuous affine functions on K of the form

$$\tilde{x}: v' \to \langle x, v' \rangle,$$

where $x \in C$. Then the ordering in K induced by F is simply the dual ordering. Moreover, since C is normal, the Krein–Grosberg theorem (see Wong and Ng (1973)) asserts that if $\{v_i: i \in I\}$ is a directed upward family of elements in V and if $v \in V$ is such that $\lim_i \langle v_i, v' \rangle = \langle v, v' \rangle$ for each v' in K then $v = \sup_i v_i$ and $\lim_i \|v_i - v\| = 0$.

The following theorem extends this result.

THEOREM 4. *Let $\{v_i\}$ be a directed upward family of elements in V and $v \in V$ be such that $v_i \leqslant v$ for each i. Suppose that $\lim_i \langle v_i, v' \rangle = \langle v, v' \rangle$ for each v' in $\partial_F K$. Then $v = \sup_i v_i$ and $\lim_i \|v_i - v\| = 0$.*

PROOF. Let $g_i = \tilde{v} - \tilde{v}_i$. Applying Theorem 3, we conclude that

$$\lim_i \langle v_i, v' \rangle = \langle v, v' \rangle$$

for each $v' \in K$.

REMARK. In Schaefer (1974), p. 89, it is inferred that the condition $v_i \leqslant v$ for all i can be dropped. Unfortunately this in fact is not correct. For example, let $V = l_1$. Then it is easily verified that $\partial_F K$ is a singleton consisting of $e = (1, 1, \ldots) \in l_\infty = V'$. For each n, let

$$v_n = \left(1, \frac{1}{2}, \ldots \frac{1}{2^n}, 0, 0, \ldots\right) \in l_1.$$

Then $\lim_n \langle v_n, e \rangle = \langle v, e \rangle$ where $v = (2, 0, 0, \ldots)$, say; but $\lim_n \| v_n - v \| \neq 0$.

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