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THE CARTAN DETERMINANT AND GENERALIZATIONS OF QUASIHEREDITARY RINGS

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The Cartan determinant conjecture for left artinian rings was verified for quasihereditary rings showing $\det C(R) = \det C(R/I)$, where I is a projective ideal generated by a primitive idempotent. This article identifies classes of rings generalizing the quasihereditary ones, first by relaxing the "projective" condition on heredity ideals. These rings, called left k-hereditary are all of finite global dimension. Next a class of rings is defined which includes left serial rings of finite global dimension, quasihereditary and left 1-hereditary rings, but also rings of infinite global dimension. For such rings, the Cartan determinant conjecture is true, as is its *converse*. This is shown by matrix reduction. Examples compare and contrast these rings with other known families and a recipe is given for constructing them.

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Introduction

Let us recall that the Cartan determinant conjecture asserts that if R is a left artinian ring with gl dim $R < \infty$ then det $C(R) = 1$. (See the survey [6].) Among the various lines of attack on this conjecture there are two which involve matrix reduction. We suppose for the moment that R is left artinian and gl dim $R < \infty$. The first method was initiated by Zacharia in [12] and was used in [4], [9] and [1]. Here the idea is to find $e^2 = e \in R$ so that det $C(R) =$ det $C(eRe)$ and gl dim $eRe < \infty$. The second method is to find an ideal I so that det $C(R) = \det C(R/I)$ and gl dim $R/I < \infty$. The easiest case is where $I = ReR$, for some idempotent e, is an heredity ideal (see, e.g., [3]). Generalizations of this method are found in [9] and [11].

Our approach is to develop further the matrix reduction method where $I = ReR$, e a primitive idempotent, $\operatorname{p\,dim}_R I < \infty$, but R is not necessarily of finite global dimension. This leads to the identification of two classes of rings which generalize the quasihereditary rings. The first, called *left k-hereditary*, are always of finite global dimension, with the global dimension bounded above by a bound generalizing that for quasihereditary rings. The second, called rings with *left matrix reducing series*, gives a class of rings for which the Cartan determinant conjecture *and* its converse hold. Rings of finite global dimension in this class include all left serial rings of finite global dimension, the left 1-hereditary and the quasihereditary rings. Examples are then presented which illustrate these classes. A method of constructing rings with left matrix reducing series from two given such rings is exhibited.

Throughout R will be a semiprimary ring with a basic orthogonal set of primitive idempotents idem $(R) = \{e_1 \ldots, e_n\}$. The radical is denoted $J(R)$, or just J. The Loewy length of a module M is LL (M). If R is left artinian, the (left) Cartan matrix of R will be denoted $C(R) = (c_{ij})$.

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1. Definitions and preliminary results

We begin by considering a class of rings of finite global dimension which contains the quasihereditary rings. Their defining condition is strong enough to yield a bound on that dimension which generalizes [5, Statement 9].

Proposition 1.1. *Let R be a semiprimary ring. Suppose $0 = I_0 \subset I_1 \subset \cdots \subset I_t = R$ is a chain of idempotent ideals such that for $j = 1, \ldots, t$,*

(1) $I_j J I_j \subseteq I_{j-1}$*; and*

(2) $\mathrm{p\,dim}\,({}_{R/I_{j-1}} I_j/I_{j-1}) = l_j < \infty$.

Then $\mathrm{gl\,dim}\, R \leq \Sigma_{j=1}^{t-1} l_j + 2t - 2$.

Proof. We first get an estimate on projective dimension which will be applied to the simples of R.

Let I be any ideal of R and put $B = R/I$. If $\mathrm{p\,dim}\,{}_R I = l < \infty$ and M is a left B-module, then $\mathrm{p\,dim}\,{}_R M \leq l + 1 + \mathrm{p\,dim}\,{}_B M$. To see this, the exact sequence $0 \to {}_R I \to {}_R R \to {}_R B \to 0$ shows that $\mathrm{p\,dim}\,{}_R B = l + 1$. In particular, if ${}_B P$ is projective then $\mathrm{p\,dim}\,{}_R P \leq l + 1$. Thus if

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to M \to 0$$

is a projective resolution of M over B, then $\mathrm{p\,dim}\,{}_R M \leq d + l + 1$.

The following notation will be used: for $j = 0, \ldots, t-1$, $R_j = R/I_j$ and $I_{j+1}/I_j = \bar{I}_{j+1}$. Since $J(R_{t-1}) = 0$, R_{t-1} is semisimple. If $t = 1$ we are done. Otherwise, assume for purpose of induction, that for $1 \leq j < t-1$ that the theorem is true for R_{j+1}. Let the bound on $\mathrm{gl\,dim}\, R_{j+1}$ given by the theorem be d. The simple R_j-modules annihilated by $\bar{I}_{j+1}$ are of projective dimension at most $l_j + 1 + d$, by the lemma. Pick $e^2 = e \in R_j$ with $\bar{I}_{j+1} = R_j e R_j$ ([5, Statement 6]). Consider a presentation

$$0 \to J(R_j)e \to R_j e \to R_j e/J(R_j)e \to 0.$$

Since $eJ(R_j)e = 0$, $J(R_j)e$ has composition factors only among the simples annihilated by $\bar{I}_{j+1}$. This means that $\mathrm{p\,dim}\,{}_{R_j} R_j e/J(R_j)e \leq l_j + 1 + d + 1$, and completes the induction. □

Note that if R has a chain of ideals as in (1.1) then LL $(R) \leq 2^t - 1$, just as in [**5**, Statement 9]. Notice also that if R has such a chain, then the same chain satisfies the right hand version of the hypotheses, given the conclusion of (1.1). It is readily seen that the rings of (1.1) have no loops in their quivers.

Definition 1.2. Let R be semiprimary and $k \geq 0$. We say R is *left k-hereditary* whenever idem (R) can be ordered $e_1, \ldots, e_n$, so that letting $I_0 = 0$ and $I_j = R(e_1 + \cdots + e_j)R$, $i = 1, \ldots, n$, we have

(1) $e_jJe_j \subseteq I_{j-1}$; and

(2) p dim $({}_{R/I_{j-1}}I_j/I_{j-1}) = l_j \leq k$, for $j = 1, \ldots, n$.

(The right hand version is defined similarly.)

Examples will show that "left k-hereditary" is not left-right symmetric. However, (1.1) shows that R left k-hereditary implies R right l-hereditary, for some l. Of course, "left 0-hereditary" means " quasihereditary", which is a symmetric concept.

Corollary 1.3. *If R is left k-hereditary, using the notation of* (1.2), *then* gl dim $R \leq \Sigma_{j=1}^{n-1} l_j + 2n - 2 \leq (n-1)(k+2)$.

The Cartan determinant conjecture was verified for a quasihereditary left artinian ring R in [3, Proposition 3.1] by observing that since idem (R) can be ordered $e_1, \ldots, e_n$ so that if $I_0 = 0$, $I_j = R(e_1 + \cdots + e_j)R$, $j = 1, \ldots, n$, then I_j/I_{j-1} is projective over R/I_{j-1} and $e_jJe_j \subseteq I_{j-1}$, it follows that $\det C(R) = \det C(R/I_1) = \cdots = \det C(R/I_{n-1}) = 1$. In [**1**], generalizing quasihereditary, an algebra R is called *neat* in case idem (R) can be ordered $e_1, \ldots, e_n$ so that if $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$ then the terms in the $\varepsilon_i R \varepsilon_i$-minimal projective resolution of $\varepsilon_i J e_i$ all belong to add $(\varepsilon_i R \varepsilon_{i+1})$; and the authors observed that, as in [**9**, Proposition 4], based on [**12**], $\det C(R) = \det C(\varepsilon_1 R \varepsilon_1) = \cdots = C(\varepsilon_{n-1} R \varepsilon_{n-1}) = 1$. These results suggest Definition 1.4, below.

The notions "left k-hereditary" and "neat" coincide for left serial rings (see (2.4)), but not for more general rings (Example 3.2). We shall see that if R is left 1-hereditary then $\det C(R) = 1$. The example in Section 3 of [**3**] is left 1-hereditary, using, for example, e_1, e_2, e_4, e_3, e_5 and right 1-hereditary using e_4, e_5, e_3, e_2, e_1. It is of global dimension 3 and is not quasihereditary. (It is also neat via the ordering e_5, e_1, e_2, e_3, e_4.)

Definition 1.4. Let R be left artinian. A primitive idempotent e, which may be taken in idem (R), is called *left a matrix reducing idempotent* whenever idem (R) may be ordered, $e = e_1, \ldots, e_n$ in such a way that

(1) p dim ${}_R Re_1 R < \infty$, and

(2) for each $k > 1$, Re_1Re_k has a projective resolution with terms from add $R(e_1 + \cdots + e_{k-1})$.

An ordering $e_1, \ldots, e_n$ of idem (R) is called a *left matrix reducing series* if each $e_j + I_{j-1}$, for $j = 1, \ldots, n-1$, is matrix reducing in R/I_{j-1}, where $I_0 = 0$ and $I_j = R(e_1 + \cdots + e_j)R, j = 1, \ldots, n$.

If $e_1, \ldots, e_n$ is a left matrix reducing series, then the 1,1-entry of $C(R/I_{j-1})$ is denoted $\bar{c}_{jj}$.

We shall soon see that every left 1-hereditary ring and every left serial ring of finite global dimension has a matrix reducing series.

Lemma 1.5. *If R is left artinian and has a left matrix reducing idempotent e_1 then elementary column operations of adding a multiple of a column to a column strictly to its right will reduce the Cartan matrix $C(R)$ to the form*

$$\begin{pmatrix} c_{11} & 0 \cdots 0 \\ \vdots & C \\ c_{n1} & \end{pmatrix},$$

where $C = C(R/Re_1R)$. Moreover, $\det C(R) = c_{11} \det C(R/Re_1R)$.

Proof. Order idem (R) as in (1.4). Consider a minimal projective resolution of Re_1Re_i. Each term of it is isomorphic to a direct sum of indecomposable projectives Re_j, where, by hypothesis, $j < i$. This says that the Cartan matrix (column) $C(Re_1Re_i)$ is a linear combination of the columns $C(Re_j), j < i$. Then the i column of $C(R)$ can be converted into $C(Re_i) - C(Re_1Re_i) = C(Re_i/Re_1Re_i)$ by the column operations indicated. □

Theorem 1.6. *Let R be a left artinian ring which has a left matrix reducing series, $e_1, \ldots, e_n$. Then*

(a) $\det C(R) = \Pi_{i=1}^{n} \bar{c}_{ii}$. *In particular* $\det C(R) > 0$ *and so if* gl dim $R < \infty$ *then* $\det C(R) = 1$. (*i.e., the Cartan determinant conjecture is verified for these rings.*)

(b) *If* $\det C(R) = 1$ *then* gl dim $R < \infty$. (*i.e., the converse of the Cartan determinant conjecture is true for these rings.*)

Proof. (a) This part follows immediately by successively applying (1.5). (b) In this case R is left k-hereditary since the condition $\bar{c}_{jj} = 1$ is precisely the condition (1) of (1.1). Hence, R has finite global dimension by (1.1). □

We now look at the 1-hereditary case.

Proposition 1.7. *Suppose R is left artinian and has a primitive idempotent e_1 so that $e_1Je_1 = 0$ and* p dim ${}_R Re_1R \leq 1$. *Then e_1 is a left matrix reducing idempotent.*

Proof. We can number the idempotents so that if some $Re_1Re_i = 0$, then $Re_1Re_2 = \cdots = Re_1Re_k = 0$ and $Re_1Re_i \neq 0$ for $i > k$. (The case where all Re_1Re_i are projective is where Re_1R is a heredity ideal.) If there is $k < i \leq n$ with $\operatorname{p\,dim} Re_1Re_i = 1$, suppose its minimal projective resolution has the form

$$0 \to P \oplus Re_j \to Re_1^{(m)} \to Re_1Re_i \to 0,$$

for some j. Since $e_1Je_1 = 0$, it follows that $1 < j \leq k$. Hence, the ordering $e_1, \ldots, e_k, \ldots, e_n$ is left matrix reducing. □

Corollary 1.8. *If the left artinian ring R is left 1-hereditary then it has a left matrix reducing series, and, hence,* $\det C(R) = 1$.

2. Applications to left serial rings.

In this section some of the ideas of the first section are explored for left serial rings. The following will be useful for building examples and for the reduction of the left serial case to where the (left) quiver is a cycle.

Proposition 2.1. *Suppose R is a left artinian ring which has orthogonal idempotents E, F such that* $E + F = 1$ *and* $FRE = 0$. *Suppose further that ERE is of finite global dimension and that ERE and FRF have left matrix reducing series,* $e_1, \ldots, e_k$ *and* $f_1, \ldots, f_l$, *respectively. Then* $e_1, \ldots, e_k, f_1, \ldots, f_l$ *is a left matrix reducing series for R.*

Proof. The methods of [6, Proposition 2.5] are used.

Since e_1 is a left reducing idempotent for ERE, changing labels, if necessary, we can assume that the ordering $e_1, \ldots, e_k$ satisfies the first part of Definition 1.4. Consider a minimal projective resolution of Re_1Re_i, for some i, $2 \leq i \leq k$. Since $Re_1Re_i = ERe_1Re_k$, a minimal R-projective resolution coincides with a minimal ERE-projective resolution. Hence the terms of it are in $\operatorname{add}(Re_1 \oplus \cdots \oplus Re_{i-1})$. Now consider Re_1Rf_j for some $j, 1 \leq j \leq l$. Once again we have that $Re_1Rf_j = ERe_1Rf_j$, so a minimal projective resolution will have terms from $\operatorname{add}(Re_1 \oplus \cdots \oplus Re_k)$. Hence, e_1 is a matrix reducing idempotent for R.

Next, we need only note that R/Re_1R is either FRF, if $k = 1$, or another ring of the same type as the original. Hence for the first $k - 1$ steps, we proceed as above. At this stage, the next factor ring is FRF, which has a matrix reducing series, by hypothesis. □

Lemma 2.2. *Let R be an indecomposable basic left artinian, left serial ring. Then there are orthogonal idempotents E and F, with* $E + F = 1$, *so that* $FRE = 0$, *the quiver of FRF is acyclic and the quiver of ERE is a cycle. Moreover, FRF is quasihereditary and if* $\operatorname{gl\,dim} R < \infty$ *then* $\operatorname{gl\,dim} EFE < \infty$.

Proof. As in [**4**] or [**6**, page 58], the quiver of R is like a rooted tree, with arrows going into the root, where the root may be replaced by an oriented cycle. If the quiver is already acyclic, put $E=0$. Otherwise, let the vertices in the "branches" correspond to the idempotents $e_1,\dots,e_k$, and those in the cycle, $e_{k+1},\dots,e_n$. The simples coming from $e_1,\dots,e_k$ do not appear in the projectives $Re_{k+1},\dots,Re_n$. Put $E=e_{k+1}+\cdots+e_n$ and $F=e_1+\cdots+e_k$. Then $FRE=0$.

Each component of the quiver of FRF has a sink; hence, FRF has a simple projective, FRe_i and a heredity ideal FRe_iRF. Dividing out by FRe_iRF gives another ring of the same type. In this way we get a heredity chain for FRF.

The last statement follows from [**6**, Proposition 2.5]. □

If R is a left serial ring whose quiver is a cycle, then R has a Kupisch series ([**2**, §32]), and then, for our needs, behaves like a serial ring.

Proposition 2.3. *If R is a left serial artinian ring which is of finite global dimension, then R has a left matrix reducing series.*

Proof. Assume, as we may, that R is basic and indecomposable. Using the language of (2.2), Proposition 2.1 says that R has a left matrix reducing series if ERE does. Thus we may assume that the quiver of R is a cycle and, hence, that R has a Kupisch series, $Re_1,\dots,Re_n$.

Methods and notation from Gustafson [**8**] will be used, but modified to the left hand version to conform to the conventions of this note. Hence the function $f:\{1,\dots,n\}\to\{1,\dots,n\}$ will here be defined $f(i)=[i-c_i]$, where "$[j]$" means the least positive residue of j modulo n and c_i is the composition length of Re_i. Put $X=\{i\mid 1\le i\le n \text{ and } f^s(i)=i \text{ for some } s\ge 0\}$. It is clear that $f(X)\subseteq X$. By [**8**, (5)], the minimal projective resolution of Re_iRe_j, insofar as the terms are non-zero, looks like

$$\cdots\to Re_{f^2(i)}\to Re_{f^2(j)}\to Re_{f(i)}\to Re_{f(j)}\to Re_i\to Re_iRe_j\to 0.$$

Also, by [**8**, (6)], if $i,j\in X$, any homomorphism $g:Re_j\to Re_i$ is either 0 or is an isomorphism. Hence, in particular, if $i\in X$, LL $(Re_i)\le n$.

The claim is that if $m\in X$ then there is a matrix reducing series starting with m. Let $A_0=\{i\mid Re_mRe_i=0 \text{ or } Re_mRe_i\cong Re_m\}$, ordered in any way which starts with m. Note that $X\subseteq A_0$. Next, for any $i\notin A_0$, look at a minimal projective resolution of Re_mRe_i:

$$\cdots\to Re_{f(i)}\to Re_m\to Re_mRe_i\to 0.$$

The last non-zero terms are either

$$0\to Re_{f^k(m)}\to Re_{f^k(i)} \text{ or } 0\to Re_{f^k(i)}\to Re_{f^{k-1}(m)}.$$

This means that $f^{k+1}(i)=f^{k+1}(m)$ or $f^{k+1}(i)=f^k(m)$ ([**8**, (3)]). In either case, $f^{k+1}(i)\in$

$X \subseteq A_0$. Let the elements of $\{i, f(i), f^2(i), \ldots\} \backslash A_0$ be ordered $f^{a_1}(i)$, $f^{a_2}(i), \ldots, i$, with $a_1 > a_2 > \cdots > 0$. Add these elements to A_0, following the elements of A_0 in the order, to form A_1. Notice that for any $f^l(i)$ with $Re_m Re_{f(i)} \neq 0$, the minimal projective resolution of $Re_m Re_{f^l(i)}$ has terms with indices in A_0 or $f^{l+1}(i)$, $f^{l+2}(i), \ldots$; these latter are either among the new indices added to A_0 or are already in A_0.

If $A_1 \neq \{1, \ldots, n\}$, then the process may be repeated by picking $i \notin A_1$ and adjoining elements from $\{i, f(i), f^2(i) \ldots\} \backslash A_1$ in the same way as in the previous step, to form A_2. This process continues until the indices are included in some A_k. Then A_k is ordered to form a matrix reducing series for R. □

Proposition 2.4. *Let R be a left serial, left artinian ring. Then the following are equivalent: (a) R is left k-hereditary for some k, (b) R is neat, and (c)* $\operatorname{gl\,dim} R < \infty$.

Proof. Both "neat" and "left k-hereditary" imply finite global dimension ([**1**, Corollary 4] and (1.1)). Now assume R is left serial and of finite global dimension. Then, by (2.3), R has a left matrix reducing series and is *a fortiori* left k-hereditary for some k. Moreover, R has a simple Re/Je of projective dimension 1 ([**4**, Lemma 3]). It is immediate that e is a neat idempotent and $\operatorname{gl\,dim}(1-e)R(1-e) < \infty$ by [**12**, Lemma 2] again. Hence, a neat sequence may be built and R is neat. □

The next result shows that for a serial ring R, "left 1-hereditary" (in fact an apparently weaker condition) and "quasihereditary" coincide. This is not true for more general rings, as we have seen.

Proposition 2.5. *If R is an indecomposable artinian serial ring such that there is i, $1 \leq i \leq n$, with* $\operatorname{p\,dim}_R Re_i R \leq 1$ *and $e_i J e_i = 0$, then R is quasihereditary.*

Proof. We may assume that R has no simple projectives. We again use the notation of the proof of (2.3). Let $Re_1, \ldots, Re_n$ be a Kupisch series for R with Re_i of composition length c_i. If $\operatorname{p\,dim}_R Re_i R = 0$ then R is quasihereditary by [**3**, Proposition 2.3].

Suppose that $\operatorname{p\,dim}_R Re_i R = 1$ and that $e_i J e_i = 0$ (the latter is equivalent to $c_i \leq n$). Then there is some $m > 0$ so that $Re_i Re_{[i+1]} \cong \cdots \cong Re_i Re_{[i+m-1]} \cong Re_i$, while $\operatorname{p\,dim} Re_i Re_{[i+m]} = 1$. It follows that $c_i < c_{[i+1]} < \cdots < c_{[i+m-1]}$. Thus, renumbering the indices cyclically, we may assume that $Re_1, \ldots, Re_k$ is a chain of maximal length satisfying $c_1 \leq n$, $c_1 < c_2 \cdots < c_k$ and $\operatorname{p\,dim} Re_1 Re_{[k+1]} \leq 1$.

According to [**3**, Proposition 2.4], in order to complete the proof we need only show that $c_{[k+1]} \leq k$. But, if this is not the case, there is an exact sequence

$$0 \to J^l e_1 \to Re_1 \to Re_1 Re_{[k+1]} \to 0$$

with $0 \neq J^l e_1$ projective. Then $Je_1 \cong Re_n$, so $c_n < c_1 < \cdots < c_k$ and we have a commutative diagram

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$$\begin{array}{ccccccccc} 0 & \to & J^{l}e_1 & \to & Je_1 & \to & Je_1Re_{[k+1]} & \to & 0 \\ & & \downarrow(\cong) & & \downarrow(\cong) & & \downarrow(=) & & \\ 0 & \to & J^{l-1}e_n & \to & Re_n & \to & Re_nRe_{[k+1]} & \to & 0 \end{array} .$$

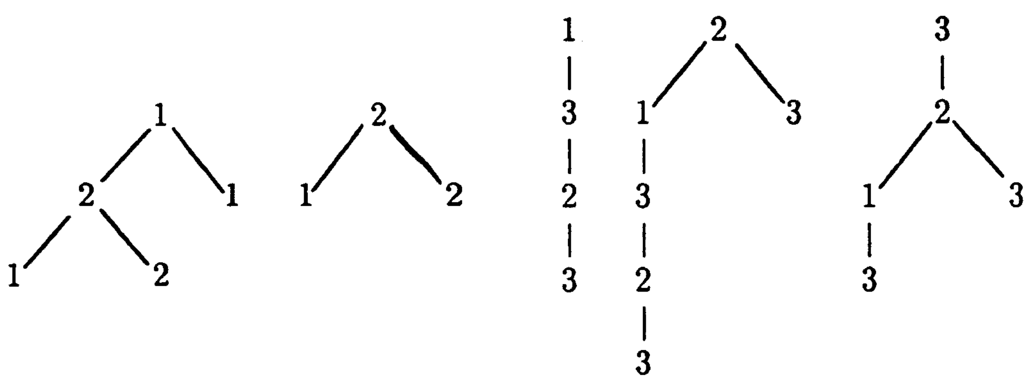
This contradicts the maximality of $Re_1, \ldots, Re_k$ since $c_n < c_1 < \cdots < c_k$ and $\text{p dim } Re_nRe_{[k+1]} = \text{p dim } Re_1Re_{[k+1]}$. □

The conclusion of the proposition does not extend to serial rings which are left 2-hereditary. The $n = 3$ case of the set of examples in [**8**] is serial of global dimension four. It is left (and right) 2-hereditary but not quasihereditary. (This family is discussed in more detail in (3.3).) For serial rings, global dimension 3 implies quasihereditary by [**3**, Proposition 2.7] or [**10**, Theorem 3.2].

3. Examples

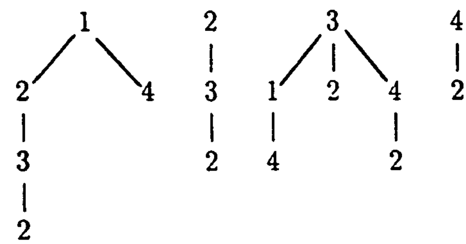
In the following examples, the diagram notation of [**7**] is used freely. All algebras are assumed to be over some field F.

Example 3.1. There are algebras R which have a left but no right matrix reducing series. Two examples are presented, the first of infinite global dimension, the second of global dimension 4.



The first algebra has a left matrix reducing series, e_2, e_1, where the ideals to be used are, in fact, projective. However, both Re_1R_R and Re_2R_R are of infinite projective dimension. The second algebra has a left matrix reducing series, e_1, e_2, e_3 but no right matrix reducing series (it is left and right 2-hereditary).

Example 3.2. There is an algebra R which is left 1-hereditary (and, hence, has a matrix reducing series) but not right 1-hereditary. This algebra is right 2-hereditary. Moreover, it is not *neat*, in the sense of [1].



The global dimension is 5. A left matrix reducing series is e_1, e_4, e_3, e_2. A right matrix reducing series is e_4, e_3, e_1, e_2.

While every left artinian ring of global dimension 2 is quasihereditary, one of global dimension 3 need not be even left 1-hereditary. The example [**10**, (2) page 172] is of global dimension 3; it is left and right 2-hereditary, but neither left nor right 1-hereditary. It has a matrix reducing series on each side and is also neat.

Example 3.3. For any $n > 2$ there is a basic serial algebra R with n primitive idempotents, global dimension $2n-2$, $(2n-4)$-hereditary but not k-hereditary for any $k < 2n-4$.

This can be done by letting the composition length of Re_i be $n+1$ for $i = 1, \ldots, n-1$ and Re_n of composition length n. Then the only candidate for a first idempotent in Definition 1.2 is e_n and $\operatorname{p\,dim}{}_R Re_n R = 2n-4$. The series $e_n, e_1, \ldots, e_{n-1}$ is matrix reducing. This is the family of examples of [**8**], which exhibit maximal possible global dimension for a serial ring with n idempotents.

We conclude with a method for building a new ring with left matrix reducing series from existing ones.

Example 3.4. Suppose C and D are left artinian rings which have left matrix reducing series, and that $\operatorname{gl\,dim} C < \infty$. Suppose also that ${}_C M_D$ is a bimodule such that ${}_C M$ is finitely generated. Then $R = \begin{pmatrix} D & 0 \\ M & C \end{pmatrix}$ has a left matrix reducing series.

This is a corollary of Proposition 2.1. The easiest example is to start with a left artinian ring A of finite global dimension which has a left matrix reducing series. Then $R = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ also has one.

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