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ON THE INFLUENCE OF THE INITIAL DATA IN A COMBUSTION PROBLEM

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Abstract

The combustion of a material can be modelled by two coupled parabolic partial differential equations for the temperature and concentration of the material. This paper deals with properties of the solution of these equations inside a cylinder or a sphere and under given initial conditions. Bounds for the variation of the temperature with the initial conditions are first established by considering a decoupled form of the equations. Then the coupled system is used to obtain approximate expressions for the temporal evolution of temperature and concentration.

1. Introduction

A simple model governing the combustion of a material can be formulated in the non-dimensional form as follows

$$\frac{\partial \theta}{\partial t} = \nabla^2 \theta + H\chi \exp\left(\frac{\alpha\theta}{\alpha+\theta}\right), \tag{1}$$

$$\frac{\partial \chi}{\partial t} = \nabla^2 \chi - \varepsilon\chi \exp\left(\frac{\alpha\theta}{\alpha+\theta}\right), \tag{2}$$

$$\theta(\mathbf{x}, 0) = h(\mathbf{x}), \quad \theta = 0 \quad \text{on } \partial D, \tag{3}$$

$$\chi(\mathbf{x}, 0) = g(\mathbf{x}), \quad \frac{\partial \chi}{\partial \nu} = 0 \quad \text{on } \partial D. \tag{4}$$

Here, θ is the temperature, χ the concentration of the combustible material, $\mathbf{x}, t$ are respectively the spatial and time variables, H, α are positive parameters and $\varepsilon = \exp(-\alpha)$. Typically, the value of α is between 20 and 100 so that $\varepsilon \ll 1$. The equations (1) and (2) are considered in a bounded domain D with initial and boundary values given in (3) and (4). The derivation of the above system can be found in Frank–Kamenetskii [1], and discussions on the system can be found in Gelfand [3], Parks [5], Sattinger [6], among others. It is known that, if the initial concentration and the initial temperature are small, χ decays very slowly and θ remains of order one. Such a situation is referred to as the subcritical. However, if the initial temperature and/or the initial concentration is sufficiently large, χ decays rapidly and θ becomes extremely large before both finally decay to zero. Such a situation is referred to as the supercritical state. For the subcritical case, Sattinger obtained an asymptotic development for θ and χ based on $\varepsilon \ll 1$. For the super-critical case, no accurate approximation to the solution of (1) to (4) has been obtained.

Recently, Tam [7] considered the case of $\nabla^2 = \partial^2/\partial x^2$, $0 < x < 1$, and used a comparison theorem to construct upper and lower solutions for the multiple steady state solutions. For the time dependent case, when the initial data are $\theta(x,0) = 0$, $\chi(x,0) = N$, upper and lower solutions for both the sub- and super-critical cases were obtained, although the time interval for the latter case is limited. When $\chi(x,0) = N$ and HN is such that the steady state of

$$\frac{\partial\theta}{\partial t} = \nabla^2\theta + HN\exp\left(\frac{\alpha\theta}{\alpha+\theta}\right)$$

has multiple solutions, the role of the initial temperature was further examined by Tam [8], again for $\nabla^2 = \partial^2/\partial x^2$, $0 < x < 1$.

In this paper, we examine the problem for a sphere and a cylinder each of unit radius. In Section 2, we consider the θ-equation by itself and obtain upper and lower steady state solutions, from which we determine some bounds for the critical parameter. When the steady state has multiple solutions, we obtain in Section 3 a criterion to indicate how large an arbitrary initial temperature has to be for the system to become super-critical. Finally, in Section 4, we consider the coupled system and obtain approximate expressions for the temporal evolution of θ and χ. The large parameter α, which contributes to the existence of multiple steady state solutions for the θ-equation, also induces a multiple time scale phenomenon. This feature has been exploited by some authors (see Kassoy [4]) who have treated similar problems by singular perturbation methods.

2. The θ equation

When χ is treated as a constant, only the θ-equation remains to be considered. If we write $H\chi = \delta$, we have

$$P\theta = \frac{\partial\theta}{\partial t} - \nabla^2\theta - \delta\exp\left(\frac{\alpha\theta}{\alpha+\theta}\right) = 0, \tag{5}$$

$$\theta(\mathbf{x},0) = h(\mathbf{x}), \quad \theta = 0 \quad \text{on } \partial D. \tag{6}$$

A function $U(\mathbf{x},t)$ is an upper solution of (5) and (6), that is, $\theta(\mathbf{x},t) \leqslant U(\mathbf{x},t)$, if (see [6], page 50)

$$PU = U_t - \nabla^2 U - \delta \exp\left(\frac{\alpha U}{\alpha + U}\right) \geqslant 0.$$

$$U(\mathbf{x},0) \geqslant \theta(\mathbf{x},0), \quad U \geqslant 0 \quad \text{on } \partial D.$$

A lower solution is similarly defined with the inequality signs reversed.

We first deal with the spherical case. Assuming only radial dependence, we have $\nabla^2 = (\partial^2/\partial r^2) + (2/r)\,\partial/\partial r$. It is readily verified that a steady state upper solution is given by

$$\bar{\theta} = A(1-r^2), \tag{7}$$

when A is chosen as a solution of the equation

$$\frac{6}{\delta} A = \exp\left(\frac{\alpha A}{\alpha + A}\right). \tag{8}$$

A steady state lower solution is given by

$$\tilde{\theta} = C(1-r^2)^2, \tag{9}$$

when C is chosen as a solution of the equation

$$\frac{12}{\delta} C = \exp\left(\frac{\alpha C}{6.25\alpha + C}\right). \tag{10}$$

Now the exponential function $\exp[\alpha A/(\alpha + A)]$ in (8) is shaped like a logistic curve, which has the value 1 at $A = 0$, and tends to e^α as $A \to \infty$. This curve is intersected by the straight line $(6/\delta)\,A$ at one or more points, depending on the value of δ. It is readily seen that equation (8) has only one solution if $\delta < \delta_1$ or $\delta > \delta_3$. The numbers δ_1 and δ_3 are obtained by solving simultaneously equation (8) and the equation obtained by differentiating (8) with respect to A. We have

$$\delta_1 = 6A_1 \exp\left(-\frac{\alpha A_1}{\alpha + A_1}\right) \quad \text{and} \quad \delta_3 = 6A_3 \exp\left(-\frac{\alpha A_3}{\alpha + A_3}\right),$$

where

$$A_1 = \frac{\alpha}{2}\{(\alpha-2) + \sqrt{[\alpha(\alpha-4)]}\} \quad \text{and} \quad A_3 = \frac{\alpha}{2}\{(\alpha-2) - \sqrt{[\alpha(\alpha-4)]}\}.$$

In what follows, we write $\phi(r) = O(\psi(r))$ if there exists a constant A such that $|\phi| < A|\psi|$ for all r in the set R under consideration. If we compare two numerical

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constants, $A = O(B)$ simply means A and B are of comparable magnitude. When the order symbol is used in an asymptotic sense, it would be qualified explicitly. We note that the solution of equation (8) is $O(1)$ if $\delta < \delta_1$, and $O(e^\alpha)$ if $\delta > \delta_3$. For $\delta_1 < \delta < \delta_3$, equation (8) admits three solutions. Each solution of (8), when substituted into equation (7), yields an upper solution for the steady state solution. A similar consideration as given to (8) shows that, if $\delta < \delta_2$ or $\delta > \delta_4$, equation (10) has one solution, while it has three solutions for $\delta_2 < \delta < \delta_4$. The numbers δ_2 and δ_4 are given by

$$\delta_2 = 12C_2 \exp\left[-\frac{\alpha C_2}{6.25\alpha + C_2}\right] \quad \text{and} \quad \delta_4 = 12C_4 \exp\left[-\frac{\alpha C_4}{6.25\alpha + C_4}\right]$$

where

$$C_2 = 3.125\alpha\{\Pi\alpha - 2) + \sqrt{[(\alpha - 2.8)(\alpha - 1.2)]}\} \quad \text{and}$$
$$C_4 = 3.125\alpha\{(\alpha - 2) - \sqrt{[(\alpha - 2.8)(\alpha - 1.2)]}\}.$$

Again we note that the solution of equation (10) is $O(1)$ if $\delta < \delta_2$, and is $O(e^\alpha)$ if $\delta > \delta_4$. We can conclude from the above that, regardless of the initial data, the steady-state solution is sub-critical if $\delta < \delta_1$, and is super-critical if $\delta > \delta_4$. When $\delta_2 < \delta < \delta_3$, both the upper and lower steady state solutions admit three solutions. If we denote their maximum values by $U_1(\delta), U_2(\delta), U_3(\delta)$ and $w_1(\delta), w_2(\delta), w_3(\delta)$, respectively, $(U_1 < U_2 < U_3,\ w_1 < w_2 < w_3)$, where U_i, w_i, $i = 1, 2, 3$, are determined from (8) and (10), respectively, we find that $w_1 < U_1$, $w_3 < U_3$, whereas $w_2 > U_2$. Thus, for this interval of δ, there is a sub-critical steady-state solution of equation (5) such that

$$w_1(1 - r^2)^2 < \theta(r, \infty) < U_1(1 - r^2),$$

and a super-critical steady-state solution of equation (5) such that

$$w_3(1 - r^2)^2 < \theta(r, \infty) < U_3(1 - r^2).$$

Using the construction procedure considered, we can also obtain the following results regarding the influence of the initial data when $\delta_2 \leqslant \delta \leqslant \delta_3$. The proofs of these are similar to that given in Tam [7], and will not be presented.

LEMMA 1. *Let $U_1(\delta)$ and $U_2(\delta)$ be the smallest and the middle solution, respectively, of equation* (8) *when $\delta_1 < \delta < \delta_3$ and the smaller and larger solution when $\delta = \delta_1$ or $\delta = \delta_3$. If*

$$\theta(r, 0) \leqslant U_2(\delta)(1 - r^2), \tag{11}$$

then the solution of the problem (5) *and* (6) *is such that*

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$$\theta(r,\infty) \leqslant U_1(\delta)(1-r^2).$$

LEMMA 2. *Let $w_2(\delta)$ and $w_3(\delta)$ be the middle and the largest solution, respectively, of equation* (9) *and the smaller and larger solution when $\delta = \delta_2$ or $\delta = \delta_4$. If*

$$\theta(r,0) > w_2(\delta)(1-r^2)^2 \tag{12}$$

then the solution of the problem (5) *and* (6) *is such that*

$$\theta(r,\infty) > w_3(\delta)(1-r^2)^2.$$

Since $U_1(\delta) = O(1)$ and $w_3(\delta) = O(e^{\alpha})$, we conclude that condition (11) leads to a sub-critical solution for (5) and (6), while condition (12) leads to a super-critical solution. In particular, if the initial condition is $\theta(x,0) = 0$, then, for $\delta < \delta_3$, the solution is sub-critical and indeed we have

$$\theta(r,t) \leqslant U_1(\delta)(1-r^2) < A_3(1-r^2),$$

where $A_3 = (\alpha/2)\{(\alpha-2)-\sqrt{[\alpha(\alpha-4)]}\}$ was obtained above. Clearly, the value δ_3 can be considered as a lower bound for the critical value of δ.

The cylindrical case is treated in an entirely analogous manner. Assuming only radial dependence, we have

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

We shall quote the results, using the same notations as in the spherical case. We have

$$\bar{\theta} = A(1-r^2),$$

where A is a solution of

$$\frac{4}{\delta}A = \exp\left(\frac{A}{\alpha+A}\right),$$

and

$$\tilde{\theta} = C(1-r^2)^2,$$

where C is a solution of

$$\frac{8}{\delta}C = \exp\left(\frac{C}{4\alpha+C}\right).$$

Further, we have

$$\delta_1 = 4A_1\exp\left(\frac{-\alpha A_1}{\alpha+A_1}\right) \quad \text{and} \quad \delta_3 = 4A_3\exp\left(\frac{-\alpha A_3}{\alpha+A_3}\right)$$

where

$$A_1 = \frac{\alpha}{2}\{(\alpha-2)+\sqrt{[\alpha(\alpha-4)]}\} \quad \text{and} \quad A_3 = \frac{\alpha}{2}\{(\alpha-2)-\sqrt{[\alpha(\alpha-4)]}\},$$

and

$$\delta_2 = 8C_2 \exp\left(\frac{-\alpha C_2}{4\alpha+C_2}\right) \quad \text{and} \quad \delta_4 = 8C_4 \exp\left(-\frac{\alpha C_4}{4\alpha+C_4}\right)$$

where

$$C_2 = 2\alpha\{(\alpha-2)+\sqrt{[\alpha(\alpha-4)]}\} \quad \text{and} \quad C_4 = 2\alpha\{(\alpha-2)-\sqrt{[\alpha(\alpha-4)]}\}.$$

Results corresponding to Lemmas 1 and 2 hold.

3. Initial data and criticality for the θ-equation

A consequence of the considerations in Section 2 is that there is a certain range of values of δ for which the steady state solution may be sub- or super-critical, depending on the initial temperature $\theta(r, 0)$. We have also obtained some bounds on $\theta(r, 0)$ which would bring about a particular steady state.

In this Section, we want to approach the problem via the integral equation obtainable from (5). We would like to answer the question that, for given δ and α, how large must $\theta(r, 0)$ be for the solution to be super-critical? Understandably, a number of simplifications and approximations will have to be made in getting the desired information from the non-linear integral equation. However, we can gauge part of them with the results of Section 2, which are exact.

Let $F(\theta) = \exp[\alpha\theta/(\alpha+\theta)]$, and $G(r, \xi, t)$ be the Green's function for the linear boundary value problem obtained from equation (5) and (6) by omitting $F(\theta)$. We have (see Duff and Naylor [1, page 289])

$$G(r, \xi, t) = \sum_{k=1}^{\infty} \exp[-\lambda_k^2 t]\, u_k(r)\, u_k(\xi),$$

where $u_k(r)$ and λ_k are, respectively, the normalized eigenfunctions and eigenvalues of

$$\nabla^2 u_k + \lambda_k^2 u_k = 0$$

subject to homogeneous boundary conditions. The solution of (5) and (6) can then be obtained from the integral equation

$$\theta(r, t) = G(r, \xi, t) \cdot h(\xi) + \delta \int_0^t G(r, \xi, t-s) \cdot F(\theta(\xi, s))\, ds, \tag{13}$$

where

$$G(r,\xi,t)\cdot\phi(\xi) \equiv \int_v G(r,\xi,t)\,\phi(\xi)\,dv.$$

and v is the region under consideration.

We define the iteration scheme

$$\theta_{n+1}(r,t) = G(r,\xi,t)\cdot h(\xi) + \delta\int_0^t G(r,\xi,t-s)\cdot F(\theta_n(\xi,s))\,ds,$$

with

$$\theta_0(\xi,s) \equiv h(\xi).$$

Clearly, it is not expected that we would be able to carry out the iteration analytically. We observe, however, that to answer questions regarding the steady state requires only a knowledge of the situation when $t \gg 1$. We are thus led to the following asymptotic considerations.

Let T be sufficiently large so that, for $(t-s) > T$, we have

$$G(r,\xi,t-s) \sim \exp[-\lambda_1^2(t-s)]\,u_1(r)\,u_1(\xi).$$

Then, for $t > T$, *we have*

$$\begin{aligned}
\theta_{n+1} &\sim \delta u_1(r)\int_0^{t-T} \exp[-\lambda_1^2(t-s)]\,u_1(\xi)\cdot F(\theta_n(\xi,s))\,ds \\
&\quad + \delta\int_{t-T}^{t} G(r,\xi,t-s)\cdot F(\theta_n(\xi,s))\,ds \\
&= \delta u_1(r)\int_0^{t} \exp[-\lambda_1^2(t-s)]\,u_1(\xi)\cdot F(\theta_n(\xi,s))\,ds \\
&\quad + \delta\int_{t-T}^{t} [G(r,\xi,t-s) - \exp[-\lambda_1^2(t-s)]\,u_1(r)\,u_1(\xi)]\cdot F(\theta_n(\xi,s))\,ds \\
&= \delta u_1(r)\int_T^{t} \exp[-\lambda_1^2(t-s)]\,u_1(\xi)\cdot F(\theta_n(\xi,s))\,ds \\
&\quad + \delta u_1(r)\int_0^{T} \exp[-\lambda_1^2(t-s)]\,u_1(\xi)\cdot F(\theta_n(\xi,s))\,ds \\
&\quad + \delta\int_{t-T}^{t} [G(r,\xi,t-s) - \exp[-\lambda_1^2(t-s)]\,u_1(r)\,u_1(\xi)]\cdot F(\theta_n(\xi,s))\,ds.
\end{aligned} \tag{14}$$

For $t > T$, the second term on the right is $O(\exp[-\lambda_1^2(t-T)])$. The third term on the right is equal to

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$$\delta \sum_{k=2}^{\infty} \frac{u_k(r)}{\lambda_k^2}[1-\exp(-\lambda_k^2 T)] \int_0^1 4\pi u_k(\xi) F(\theta_n(\xi, \bar{s}))\, \xi^2\, d\xi, \tag{15}$$

where $t-T < \bar{s} < t$. To estimate the above, we make the following observations. Since the iteration scheme is convergent, $\theta_n(\xi, s)$ will be close to $\theta(\xi, s)$ when n is sufficiently large. In addition, for $(t-T)$ sufficiently large, and $(t-T) < s < t$, $\theta_n(\xi, s)$ will be close to the steady state $\theta(\xi, \infty)$. We estimate $\theta(\xi, \infty)$ by noting that it satisfies the equation

$$\nabla^2 \theta(\xi, \infty) + \delta F(\theta(\xi, \infty) = 0.$$

If we expand both $\theta(\xi, \infty)$ and $F(\theta(\xi, \infty))$ in terms of the eigenfunctions $\{u_k(\xi)\}$, we have

$$\theta(\xi, \infty) = \sum_{k=1}^{\infty} a_k\, u_k(\xi)$$

and

$$F(\theta(\xi, \infty) = 1 + \sum_{k=1}^{\infty} b_k\, u_k(\xi),$$

where both a_k and b_k are unknown. However, substitution into the governing equation will give

$$\sum_{k=1}^{\infty} (a_k \lambda_k^2 - \delta b_k)\, u_k(\xi) = \delta.$$

Here, δ is a given constant and we can obtain its eigenfunction expansion

$$\delta = \sum_{k=1}^{\infty} c_k\, u_k(\xi),$$

where c_k is known. Therefore, we have

$$a_k = \frac{\delta b_k + c_k}{\lambda_k^2},$$

where

$$b_k = \left[\exp\left(\frac{\theta(\xi, \infty)}{\alpha + \theta(\xi, \infty)}\right) - 1\right] \cdot u_k(\xi).$$

We know the shape of the exponential function since we have some *a priori* bounds on $\theta(\xi, \infty)$ from the previous sections. We also know that $u_1(\xi)$ is positive while all other $u_k(\xi)$ for $k \geqslant 2$ change sign in $0 \leqslant \xi < 1$. Thus we expect that b_1 is dominant. This, together with the fact that $\lambda_1 < \lambda_2 < \lambda_3 \ldots$, implies that $a_1 u_1(\xi)$ will be the dominant term in the eigenfunction expansion for $\theta(\xi, \infty)$.

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Using the above information, we deduce that for n, $s \to \infty$, we have $\theta_n(\xi, \infty) = O(\mathrm{M}u_1(\xi))$ for some positive constant M. Thus we have

$$F(\theta_n(\xi, \bar{s}) = O\left[\exp\left(\frac{Mu_1(\xi)}{\alpha + Mu_1(\xi)}\right)\right]$$
$$= O\left[1 + \exp\left(\frac{Mu_1(0)}{\alpha + Mu_1(0)}\right)\frac{u_1(\xi)}{u_1(0)}\right].$$

Using this estimate, and the orthogonality of the eigenfunctions, (15) then becomes

$$O\left[\delta \sum_{k=2}^{\infty} \frac{u_k(r)}{\lambda_k^2}(1 - \exp[-\lambda_k^2 T])\, 4\pi \int_0^1 u_k(\xi)\, \xi^2\, d\xi\right]. \tag{16}$$

The series is convergent, so that (15) is $O(\delta)$. Using the same estimate for $T < s < t$, we have, for $n, s \to \infty$,

$$u_1(\xi) \cdot F(\theta_n(\xi, s)) = O\left\{4\pi \int_0^1 \left[1 + \exp\left(\frac{Mu_1(0)}{\alpha + Mu_1(0)}\right)\frac{u_1(\xi)}{u_1(0)}\right] u_1(\xi)\, \xi^2\, d\xi\right\}$$
$$= O\left\{\frac{1}{u_1(0)}\left[1 + \exp\left(\frac{Mu_1(0)}{\alpha + Mu_1(0)}\right)\right]\right\}. \tag{17}$$

Thus, if we are interested in the case when M is large, (17) is dominant. We therefore neglect the second and the third integrals on the right of (14) in approximating θ_{n+1}, and write

$$\theta_{n+1} \sim \delta u_1(r) \int_T^t \exp[-\lambda_1^2(t - s)]\, u_1(\xi) \cdot F(\theta_n(\xi, s))\, ds.$$

Now suppose, for $t > T$, we have $u_1(\xi) \cdot F(\theta_n(\xi, s)) \geqslant K_n$ for some n, where K_n is independent of s. Then, for $t \gg T$, we have

$$\theta_{n+1} \geqslant \frac{K_n \delta}{\lambda_1^2} u_1(r). \tag{18}$$

Using the above representation for θ_{n+1}, we can proceed to consider $u_1(\xi) \cdot F(\theta_{n+1})$. Supposing we have $u_1(\xi) \cdot F(\theta_{n+1}) \geqslant K_{n+1}$, which is independent of s. Clearly, by repeating the above, we can generate a sequence of members K_i, $i = n, n+1, \ldots$. We now compare K_n with K_{n+1}. If, for a fixed δ, we have $K_{n+1} \geqslant K_n$, then the sequence $\{K_i\}$ is monotone increasing. Since we know the solution θ is bounded, $\{K_i\}$ tends to a limit. If the limit $K_\infty = O(e^\alpha)$, then the solution of the initial value problem is supercritical.

For the case of the sphere, we have

$$u_1(r) = \frac{\sin \pi r}{\sqrt{(2\pi)}\, r} \quad \text{and} \quad \lambda_1 = \pi.$$

Following the above considerations, we have to evaluate $u_1 \cdot F(\theta_{n+1})$. We have

$$u_1(r) \cdot F(\theta_{n+1}) = 2\sqrt{2\pi} \int_0^1 r^{-1} \sin \pi r \exp\left\{\frac{K_n \delta r^{-1} \sin \pi r}{\alpha \pi^2 \sqrt{(2\pi)} + K_n \delta r^{-1} \sin \pi r}\right\} r^2\, dr.$$

To render the integral tractable, a number of approximations must be made. We approximate $r^{-1} \sin \pi r$ by $\pi \cos(\pi r/2)$, and make the change of variables $y = \cos(\pi r/2)$ to obtain

$$u_1 \cdot F(\theta_{n+1}) \approx 4\sqrt{2\pi} \int_0^1 \frac{y r^2(y)}{(1-y^2)^{\frac{1}{2}}} \exp\left\{\frac{K_n \delta y}{\alpha \pi \sqrt{(2\pi)} + K_n \delta y}\right\} dy.$$

Using a little numerical experimentation, we see that $r^2(y)/(1-y^2)^{\frac{1}{2}}$ is well approximated by $(1-y)$. Thus we have

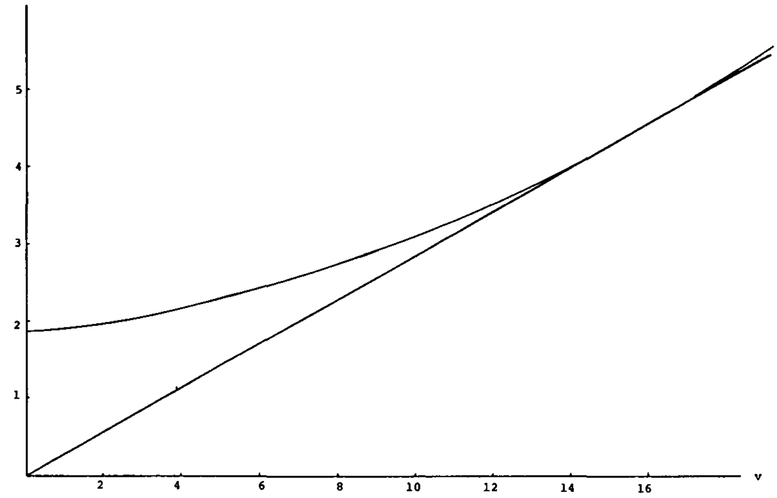
$$\begin{aligned} u_1 \cdot F(\theta_{n+1}) &\approx 4\sqrt{2\pi} \int_0^1 y(1-y) \exp\left\{\frac{K_n \delta y}{\alpha \pi \sqrt{(2\pi)} + K_n \delta y}\right\} dy \\ &\geqslant 4\sqrt{2\pi} \int_0^1 y(1-y) \exp\left\{\frac{K_n \delta y}{\alpha \pi \sqrt{(2\pi)} + K_n \delta}\right\} dy \\ &= \frac{4\sqrt{2\pi}}{A^3} \{(A-2)e^A + (A+2)\} \equiv K_{n+1}, \end{aligned}$$

where

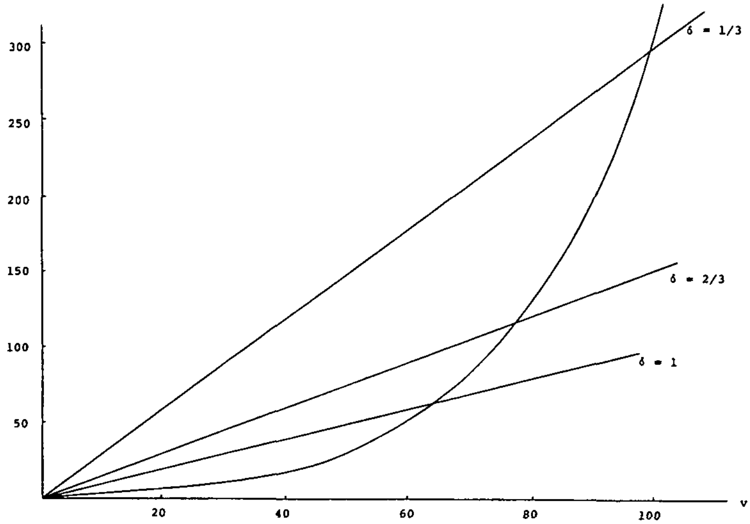
$$A = \frac{\alpha v}{\alpha \pi \sqrt{(2\pi)} + v} \quad \text{and} \quad v = K_n \delta.$$

In Figure 1 we have plotted K_{n+1} against v for $\alpha = 20$. It is clear that a comparison of K_n with K_{n+1} becomes a comparison of the straight line v/δ with K_{n+1}.

When δ is sufficiently small, the straight line intersects K_{n+1} at one point, where K_{n+1} is $O(1)$. When δ is increased beyond a certain value, say $\tilde{\delta}$, the straight line intersects K_{n+1} at three points. When δ is further increased to be greater than $\bar{\delta}$, say, the number of intersections is reduced to one, where K_{n+1} is $O(e^\alpha)$. This result is interpreted as follows. When the parameter δ is greater than or equal to $\bar{\delta}$, the steady-state solution of (1) and (2) is supercritical, regardless of the initial data. Thus $\bar{\delta}$ is a critical or threshold value for the parameter. For δ between $\tilde{\delta}$ and $\bar{\delta}$, let the coordinates of the middle intersection point of v/δ and K_{n+1} be denoted by (v^*, K^*). If, for a given δ, there is a K_n such that $\delta K_n \geqslant v^*$, then the steady state solution of (1) and (2) is super-critical. As an illustration, we have obtained a few numbers graphically for $\alpha = 20$:



(a)



(b)

Fig. 1. Plots of K_{n+1} against v for the sphere, with $\alpha = 20$, (a) $\bar{\delta} = 3.53$, (b) $\delta = 1/3$, $2/3$, 1 and (c) $\bar{\delta} = 1.5 \times 10^{-3}$

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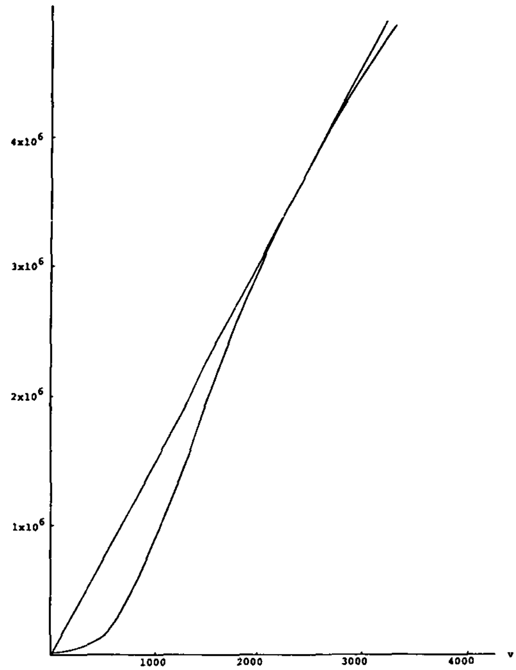

(c)

Fig. 1. (*continued*)

δ:	$\frac{1}{3}$	$\frac{2}{3}$	1
$v^*(\delta)$:	99	77	64

with $\tilde{\delta} = 1.5 \times 10^{-3}$ and $\bar{\delta} = \delta_{cr} = 3.53$.

We note that the critical value δ_{cr} obtained by Parks [5] is 3.51. Clearly, similar results can be obtained for other values of δ, and also for different values of α.

With the information obtained in the above, we are in a position to answer the questions set out at the beginning of Section 3. For fixed α and $\delta > \tilde{\delta}(\alpha)$, to see whether a given initial $\theta(x,0)$ must necessarily lead to a super-critical steady state solution, we calculate the inner product $u_1(\xi) \cdot F(\theta_0(\xi))$. If the number so obtained is not less than v^*/δ, the super-critical steady state will result.

For the case of the infinite cylinder, we have

$$u_1(r) = \frac{1}{\sqrt{\pi}} \frac{J_0(\lambda_1 r)}{J_1(\lambda_1)}, \quad \lambda_1 = 2.405 \quad \text{and} \quad J_1(\lambda_1) = 0.5191.$$

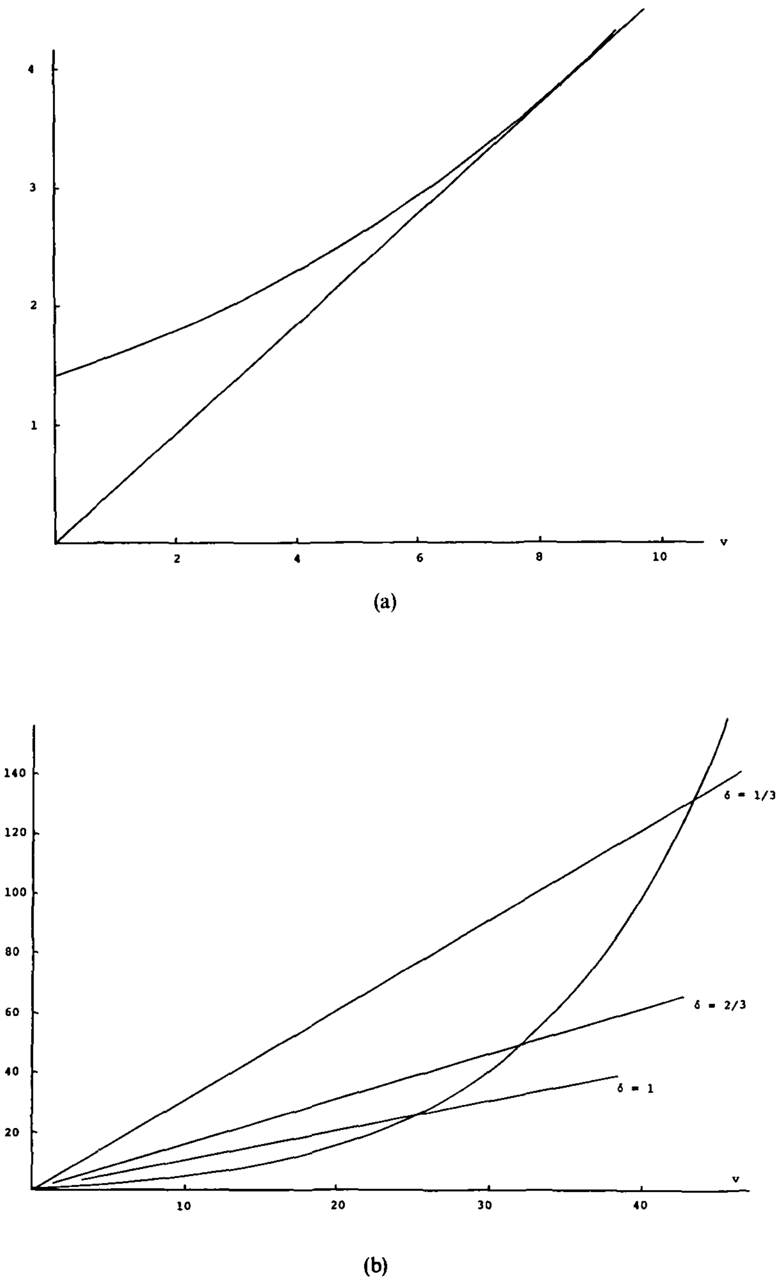

Fig. 2. Plots of K_{n+1} against v for the cylinder, with $\alpha = 20$, (a) $\bar{\delta} = 2.18$, (b) $\delta = 1/3$, $2/3$, 1 and (c) $\bar{\delta} = 8.2 \times 10^{-5}$.

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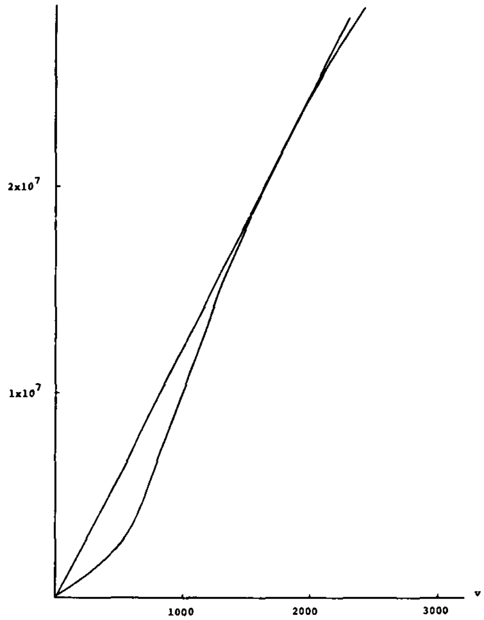

(c)

Fig. 2. (*continued*)

Again, we have to evaluate

$$u_1(r)\cdot F(\theta_{n+1})=\frac{2\sqrt{\pi}}{J_1(\lambda_1)}\int_0^1 J_0(\lambda_1 r)\exp\left(\frac{K_n\,\delta J_0(\lambda_1 r)}{\alpha\sqrt{(\pi)}\,J_1(\lambda_1)\,\lambda_1^2+K_n\,\delta J_0(\lambda_1 r)}\right) r\,dr.$$

We approximate $J_0(\lambda_1 r)$ by $\cos(\pi r/2)$, and make the change of variables $y=\cos(\pi r/2)$ to obtain

$$u_1\cdot F(\theta_{n+1})=\frac{4}{\sqrt{(\pi)}\,J_1(\lambda_1)}\int_0^1 y\exp\left(\frac{\alpha K_n\,\delta y}{\alpha\sqrt{(\pi)}\,J_1(\lambda_1)\,\lambda_1^2+K_n\,\delta y}\right)\frac{r(y)}{\sin(\pi/2)\,r\,(y)}\,dy.$$

Using $(\pi r/2)\geqslant\sin(\pi r/2)$, we have

$$\begin{aligned}u_1\cdot F(\theta_{n+1}) &> \frac{8}{\pi\sqrt{(\pi)}\,J_1(\lambda_1)}\int_0^1 y\,e^{By}\,dy\\ &= \frac{8}{\pi\sqrt{(\pi)}\,J_1(\lambda_1)\,B^2}\left[e^B(B-1)+1\right]\equiv K_{n+1},\end{aligned}$$

where

$$B=\frac{\alpha v}{\alpha\sqrt{(\pi)}\,J_1(\lambda_1)\,\lambda_1^2+v},\quad v=K_n\,\delta.$$

In Figure 2 we have plotted K_{n+1} against v for $\alpha = 20$. The critical values of δ and some typical values of δ and $v^*(\delta)$ are tabulated as follows. The critical value δ_{cr} obtained by Parks [5] is 2.11.

δ:	$\frac{1}{3}$	$\frac{2}{3}$	1
$v^*(\delta)$:	43.6	32	25.8

with $\tilde{\delta} = 8.2 \times 10^{-5}$ and $\bar{\delta} = \delta_{cr} = 2.18$.

4. The temporal evolution for the system

With the information obtained in Section 3, we now attempt to construct an approximate description of the temporal evolution of θ for the system when the super-critical state is reached. For simplicity, we shall take $\chi(\mathrm{r}, 0) = N$.

We make the following observations about the θ- and χ-equations, taken separately. For χ = constant and $H\chi = \delta > \delta_1$, the solution of the θ-equation will become super-critical if $\theta(r, \theta)$ is sufficiently large. If we construct upper and lower solutions for θ by replacing the nonlinear term $F(\theta) = \exp[\alpha\theta/(\alpha + \theta)]$ with suitable constants, then the resulting linear equation can be solved explicitly and it is seen that the steady state is reached when $t = 3/\lambda_1^2$. During this time interval, χ in fact decreases, but its rate of decrease is slower than $N e^{-t}$. At $t = 3/\lambda_1^2$, χ will still be larger than $N \exp(-3/\lambda_1^2)$. Thus if $HN \exp(-3/\lambda_1^2) = \delta^* > \tilde{\delta}$, and $\theta(r, 0)$ is such that $u_1 \cdot F(\theta) \geqslant v^*(\delta^*)$, we are certain that the system will become super-critical. Let $\delta(t) = H\chi(t)$ and suppose $\theta(r, 0)$ is such that the θ-equation reaches a super-critical state. Then θ_{max} will remain exponentially large until $\delta = \tilde{\delta}$. The duration of this period is estimated as $HN e^{-T} = \tilde{\delta}$; that is, $T = \ln(HN/\tilde{\delta})$. For $HN = O(1)$, we have $T \approx 10$ for both the sphere and cylinder. For $t > T$, we have $H\chi < \tilde{\delta}$ and θ_{max} will change rapidly from the exponentially large value to $O(1)$. From then on, χ will decrease as $O(e^{-\varepsilon t})$. Based on the above considerations, we construct the following approximate description for $\theta(r, t)$:

$$\theta(r, t) = G(r, \xi, t) \cdot \theta(\xi, 0) + H u_1(r) \int_0^t \exp[-\lambda_1^2(t - \tau)]\, u_1(\xi) \cdot \chi F(\theta)\, d\tau,$$

where

$$\begin{aligned} \chi F(\theta) &= N F(\theta(\xi, 0)) \quad \text{for } 0 < t < \frac{2}{\lambda_1^2} \\ &= N e^{-\tau} F(\bar{\theta}) \quad \text{for } \frac{2}{\lambda_1^2} \leqslant t < T \\ &= N e^{-T} e^{-\varepsilon t} F(\tilde{\theta}) \quad \text{for } t \geqslant T. \end{aligned}$$

The functions $\bar{\theta}$ and $\tilde{\theta}$ are obtained as follows:

$$\bar{\theta}(\xi,\tau) = \frac{KHN e^{-\tau}}{\lambda_1^2} u_1(\xi),$$

where K is the largest value of K_{n+1} satisfying the functional equation

$$K_{n+1}(v) = \frac{v}{HN e^{-\tau}}$$

for each fixed τ, and

$$\tilde{\theta}(\xi,\tau) = \frac{KHN e^{-T} e^{-\varepsilon\tau}}{\lambda_1^2} u_1(\xi),$$

where K is the (unique) value of K_{n+1} satisfying the functional equation

$$K_{n+1}(v) = \frac{v}{HN e^{-T} e^{-\varepsilon\tau}}$$

for each fixed τ. The multiple time scale effect of the phenomenon is apparent in the above description for θ.

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