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ORDER-CAUCHY COMPLETIONS OF RINGS AND VECTOR LATTICES OF CONTINUOUS FUNCTIONS

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Introduction. This paper studies sequential order convergence and the associated completion in vector lattices of continuous functions. Such a completion for lattices $C(X)$ is related to certain topological properties of the space X and to ring properties of $C(X)$. The appropriate topological condition on the space X equivalent to this type of completeness for the lattice $C(X)$ was first identified, for compact spaces X, in [**6**]. This condition is that every dense cozero set S in X should be C^*-embedded in X (that is, all bounded continuous functions on S extend to X). We call Tychonoff spaces X with this property *quasi-F spaces* (since they generalize the F-spaces of [**12**]).

In Section 1, the notion of a completion with respect to sequential order convergence is first described in the setting of a commutative lattice group G. A sequence $\{g_n\}$ in G is said to be o-*Cauchy* if there exists a decreasing sequence $\{u_n\}$ with $\wedge\, u_n = 0$ in G and $|g_n - g_{n+p}| \leqq u_n$ for all n, p. If there exist such a sequence $\{u_n\}$ and a $g \in G$ with $|g_n - g| \leqq u_n$, then $\{g_n\}$ o-*converges to* g. G is o-*Cauchy complete* if each o-Cauchy sequence o-converges to some $g \in G$. F. Papangelou [**23**], starting from some earlier work by Everett [**7**], showed that a "one-step" Cantor-type construction gives the appropriate o-Cauchy completion of any commutative lattice group G. In Section 1, we give an abstract characterization of this completion and show how it applies to vector lattices and to certain lattice-ordered rings which satisfy a mild continuity condition for the multiplication. These rings include all rings of functions and, more generally, all archimedean subdirect sums of totally ordered rings (also called "f-rings").

In Section 2, the discussion of Section 1 is specialized to the lattice-ordered algebra $C(X)$ of all continuous real-valued functions on a Tychonoff space X. In Theorem 2.1, the o-Cauchy completion of $C(X)$ is described as the algebra of all $C(X)$-bounded continuous functions defined on some countable intersection of dense cozero sets in βX (the domain depending on the function). A similar description of the Dedekind completion of $C(X)$ (using dense open sets in βX) has been given in [**9**], but the proof in [**9**] does not transfer to the o-Cauchy completion. Our

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proof seems to require a new extension result for z-embedded subsets, which we give in Lemma 2.5.

In Section 3, the description of the o-Cauchy completion of $C(X)$ and of the subalgebra $C^*(X)$ of bounded functions is made more explicit. It is described in terms of the uniform completion of certain algebras of functions defined on dense cozero subsets of X or of βX (see Corollary 3.5). It is shown in Theorem 3.7 that for any Tychonoff space X, $C(X)$ is o-Cauchy complete if and only if X is a quasi-F space (as defined above). This generalizes the same result for compact X given in [**6**]. For every space X, the o-Cauchy completion of $C^*(X)$ takes the form $C(K(X))$ for a certain compact space $K(X)$ (which is necessarily a quasi-F-space). We show how to construct $K(X)$ as the inverse limit space of $\{\beta S : S \text{ is a dense cozero set in } X\}$, as well as two other equivalent inverse limit constructions. This construction is analogous to the inverse limit construction of Gleason's minimal projective cover given in [**9**]. The o-Cauchy completion of $C(X)$ is shown to coincide with the $C(X)$-bounded elements in the uniform completion of $Q_{cl}(X)$, the classical ring of quotients of $C(X)$ (i.e., the ring of formal fractions f/g where g is a nondivisor of zero in $C(X)$). An example is given of a $C(X)$ whose o-Cauchy completion is not a $C(Y)$.

In Section 4, the above mentioned space $K(X)$, for compact X, is characterized by the property of being a quasi-F space admitting a continuous irreducible surjection onto X which is minimal in a certain natural sense. Accordingly, we call $K(X)$ the *minimal quasi-F cover* of X. This is similar to the description of Gleason's minimal projective cover $G(X)$ for a compact X [**11**] as being the only extremally disconnected space admitting a continuous irreducible surjection onto X. Nevertheless, we lack a completely intrinsic characterization of this sort for $K(X)$. Recall also that $C(G(X))$ is the Dedekind completion of $C(X)$, in analogy with the fact mentioned above that $C(K(X))$ is the o-Cauchy completion of $C(X)$. We show that, for an arbitrary X, the o-Cauchy completion of $C(X)$ coincides with the Dedekind completion if and only if $K(X) = G(\beta X)$, and this is true whenever every dense open subset of X contains a dense cozero set. This latter condition holds, in particular, if X is perfectly normal or satisfies the countable chain condition.

In Section 5, we study quasi-F spaces per se and characterize them in terms of the ring $C(X)$. For example, it is shown in Theorem 5.1 that X is a quasi-F space if and only if every finitely generated ideal in $C(X)$ containing a nondivisor of zero is principal, or if and only if every ideal containing a nondivisor of zero is an l-ideal ("solid," or "absolutely convex"). If βX is zero-dimensional, then X is a quasi-F space if and only if every nondivisor of zero in $C(X)$ is a multiple of its absolute value, but the sufficiency can fail if X is not strongly zero-dimensional. A σ-compact space is a quasi-F-space if and only if each of its dense Baire sets is C^*-

embedded. First countable quasi-F spaces are discrete. Every Tychonoff space is a closed subspace of some quasi-F space. We conclude with some results on products of quasi-F spaces.

1. Order-Cauchy completion of l-groups and l-algebras. This section is an exposition of the fundamentals of completeness and completion by order-Cauchy sequences in l-groups (and in vector lattices, l-rings, and l-algebras). Most of the material presented here is due to Everett [**7**] and Papangelou [**23**] in the case of l-groups. We give an abstract characterization of their completion and at the same time offer slight extensions to l-rings and l-algebras.

By *l-group* is meant a commutative lattice-ordered group (see e.g., [**8**] or [**2**]). The group operation is written $+$, the partial order is denoted by $\leqq$; a $\vee$ b, respectively $a \wedge b$, denote the least upper bound and greatest lower bound of a and b; $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a^+ + a^-$.

If $\{g_n\}$ is a sequence in a lattice G then $g_n\downarrow$ means $\{g_n\}$ is decreasing, i.e., $g_1 \geqq g_2 \geqq \ldots$; if in addition $\wedge g_n = g$ then we write $g_n \downarrow g$. Increasing sequences are handled similarly. Recall that a sublattice G of a lattice H is called *σ-regular* if the embedding $G \hookrightarrow H$ preserves all existing countable suprema and infima in G. For l-groups G and H, this just means that if $g_n \downarrow 0$ in G, then $g_n \downarrow 0$ in H.

1.1. *Definitions.* Suppose G is an l-group, $\{g_n\}$ is a sequence in G, and $g \in G$.

(a) The sequence $\{g_n\}$ *order-converges* (or o-*converges*) *to* g, written $g_n \xrightarrow{o} g$ or o-*lim* $g_n = g$, if $|g_n - g| \leqq u_n$ for $n = 1, 2, 3, \ldots$, for some $u_n \downarrow 0$ in G. (Such limits are unique.)

(b) The sequence $\{g_n\}$ is *order-Cauchy* (or o-*Cauchy*) if, for some $u_n \downarrow 0$ in G, $|g_n - g_{n+p}| \leqq u_n$ for all n, p.

(c) G is called *order-Cauchy complete* (or o-*Cauchy complete*) if each o-Cauchy sequence in G o-converges to a limit in G.

Remark. More explicit (and overly cumbersome) terminology would perhaps be *sequentially* o-*Cauchy complete*, to distinguish from the corresponding notion for nets. However, this paper is concerned exclusively with sequences, so dropping "sequentially" introduces no ambiguity here. Moreover, for nets, the phrase "o-net complete" could be used without confusion. We take pains to avoid terminology which might get confused with the quite different notions of Dedekind σ-complete or Dedekind complete lattices. The phrase "o-complete" is used in [**23**, p. 87]; "order Cauchy complete" is used in [**24**, p. 230].

We are interested in constructing a minimal "completion" of l-groups G with respect to o-Cauchy sequences. Our applications to follow are concerned with richer structure (i.e., $G = C(X)$), and it is pertinent to

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ascertain what algebraic structure is preserved by this completion process. Accordingly, we take the following as our definition of completion.

1.2. *Definition*. Let $\mathscr{L}$ denote any subcategory of l-groups (e.g., l-groups, vector lattices, l-rings, l-algebras). For G in $\mathscr{L}$, an o-*Cauchy completion of* G (in $\mathscr{L}$) is an H in $\mathscr{L}$ together with an $\mathscr{L}$-embedding $G \hookrightarrow H$ satisfying:

(a) H is o-Cauchy complete;

(b) G is σ-regular in H; and

(c) for each $h \in H$ there exist sequences $\{g_n\}$, $\{u_n\}$ in G with $u_n \downarrow 0$ and $|g_n - h| \leqq u_n$, $n = 1, 2, \ldots$.

Such an H is called *essentially unique* (in $\mathscr{L}$) if, for every H' which is an o-Cauchy completion of G in $\mathscr{L}$, there is an $\mathscr{L}$-isomorphism from H onto H' which restricts to the identity on G.

We record below (1.3 and 1.5) two lemmas due to Papangelou which are used in the construction of a completion, in the proof of its uniqueness, and in subsequent material.

1.3. LEMMA. [**23**, 2.10] *If $\{g_n\}$ is a sequence in an l-group G, then $\{g_n\}$ is o-Cauchy if and only if there exist sequences $\{u_n\}$, $\{v_n\}$ in G such that $u_n \leqq g_n \leqq v_n$ for all n, $\{u_n\}$ is increasing, $\{v_n\}$ is decreasing, and $\wedge (v_n - u_n) = 0$ in G.*

The following is immediate.

1.4. COROLLARY. *The l-group G is o-Cauchy complete if and only if for every increasing sequence $\{u_n\}$ in G sitting below a decreasing sequence $\{v_n\}$ with $\wedge (v_n - u_n) = 0$, there exists $g \in G$ with $u_n \leqq g \leqq v_n$ for all n (and hence $g = \vee u_n = \wedge v_n$).*

Given l-groups G and H and an l-group embedding $G \hookrightarrow H$, let G_1^H consist of all $h \in H$ for which there exist sequences $\{g_n\}$, $\{u_n\}$ in G such that $u_n \downarrow 0$ in G and $|g_n - h| \leqq u_n$.

1.5. LEMMA. [**23**, 3.3]. *Suppose G is σ-regular in H and $\{v_n\}$ is a decreasing sequence in G_1^H with $v_n \downarrow v$ for some $v \in H$. Then there exists a decreasing sequence $\{u_n\}$ in G with $u_n \geqq v_n$ for all n and $u_n \downarrow v$. The corresponding statement for increasing sequences also holds.*

By "l-ring" we mean an l-group G with a multiplication making G into a ring satisfying $xy \geqq 0$ whenever $x \geqq 0$ and $y \geqq 0$ in G (see [**8**] or [**2**]). In order to construct an o-Cauchy completion for l-rings G, it seems necessary to assume some kind of order continuity for the multiplication in G, for example:

(*) If $u_n \downarrow 0$ in G and $h \geqq 0$ then $hu_n \downarrow 0$ and $u_n h \downarrow 0$.

(See 1.8 and 1.10 for comments on (*)).

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1.6. LEMMA. *Suppose G is a l-ring satisfying* (*).

(a) *If* $g_n \xrightarrow{o} g$ *and* $h_n \xrightarrow{o} h$, *then* $g_n h_n \xrightarrow{o} gh$.

(b) *If H is an l-ring and G is embedded as a σ-regular sub-l-ring of H, then G_1^H is a σ-regular sub-l-ring of H, and G_1^H satisfies* (*).

Proof. (a) follows immediately from (*), the fact that o-convergent sequences are bounded (1.3), and the computation

$$(\dagger) \qquad |g_n h_n - gh| \leqq |g_n||h_n - h| + |g_n - g||h|.$$

If $G \hookrightarrow H$ as in (b), pick $g, h \in G_1^H$ with $|g_n - g| \leqq u_n \downarrow 0$, $|h_n - h| \leqq v_n \downarrow 0$ for some $\{g_n\}$, $\{u_n\}$, $\{h_n\}$, $\{v_n\}$ in G. Clearly G_1^H is a sub-l-group of H, and $|h| \leqq f$ for some $f \in G$ (e.g., by 1.5). By ($\dagger$), $|g_n h_n - gh| \leqq |g_n|v_n + u_n f$, whence by (*) $g_n h_n \xrightarrow{o} gh \in G_1^H$. Thus G_1^H is a sub-l-ring of H and is σ-regular by 1.5. Finally, if $v_n \downarrow 0$ in G_1^H and $h \geqq 0$ in G_1^H then again by 1.5, $h \leqq f$ for some $f \in G$, and there is $\{u_n\}$ in G with $u_n \geqq v_n$ and $u_n \downarrow 0$. Thus $v_n h \leqq u_n f \downarrow 0$ and $hv_n \leqq f\, u_n \downarrow 0$ by the assumption (*) for G; hence G_1^H has (*). This proves the lemma.

We can now state the main theorem of this section.

1.7. THEOREM. *Suppose G is an l-group (resp. vector lattice, l-ring satisfying* (*), *l-algebra satisfying* (*)). *Then G has an essentially unique o-Cauchy completion H among l-groups (resp. vector lattices, l-rings, l-algebras). Moreover, H is minimal in the sense that if H' is an o-Cauchy complete l-group and $\phi: G \to H'$ is a σ-regular l-group embedding, then there is a unique order-preserving $\tilde{\phi}: H \to H'$ extending ϕ, and $\tilde{\phi}(H) = \phi(G)_1^{H'}$. The map $\tilde{\phi}$ is necessarily an l-group embedding, and if in addition $\phi: G \to H$ is an embedding of vector lattices (l-rings, l-algebras), then so is $\tilde{\phi}$.*

Proof. (Outline) The construction of the completion follows the Cantor process. For an l-group G, let G_1 denote the set of all equivalence classes $[\{f_n\}]$ of o-Cauchy sequences, where $\{f_n\}$ and $\{g_n\}$ are equivalent if $f_n - g_n \xrightarrow{o} 0$. Identify each $g \in G$ with $[\{g, g, \ldots\}]$ in G_1. Everett [**7**] made the construction of G_1, showed that G_1 is an l-group under the operations

$$[\{f_n\}] + [\{g_n\}] = [\{f_n + g_n\}] \text{ and } [\{f_n\}] \vee [\{g_n\}] = [\{f_n \vee g_n\}]$$

(so that the identification $G \subset G_1$ is an l-group embedding), and proved (b) and (c) of Definition 1.2 (for $H = G_1$). However, the proof that G_1 is o-Cauchy complete (i.e. (a) of Definition 1.2) turned out to be rather more subtle, and Everett established it only in special cases. Papangelou [**23**] resolved the issue by showing that G_1 is o-Cauchy complete for every l-group G. Thus G has an o-Cauchy completion among l-groups – namely G_1.

If G is a vector lattice, clearly $\lambda[\{f_n\}] = [\{\lambda f_n\}]$ defines the proper scalar multiplication in G_1, whence G has an o-Cauchy completion in vector lattices.

For an l-ring G, in order to extend the multiplication termwise we need to show that if $\{f_n\}$, $\{f_n'\}$, $\{g_n\}$, $\{g_n'\}$ are o-Cauchy sequences with $f_n - f_n' \xrightarrow{o} 0$ and $g_n - g_n' \xrightarrow{o} 0$ then $\{f_n g_n\}$ and $\{f_n' g_n'\}$ are o-Cauchy and $f_n g_n - f_n' g_n' \xrightarrow{o} 0$. Since o-Cauchy sequences are bounded (1.3), the computations (as in the proof of 1.6)

$$|f_n g_n - f_{n+p} g_{n+p}| \leqq |f_n||g_n - g_{n+p}| + |f_n - f_{n+p}||g_{n+p}|$$

and

$$|f_n g_n - f_n' g_n'| \leqq |f_n||g_n - g_n'| + |f_n - f_n'||g_n'|$$

show that the assumption (*) is sufficient. This shows that if G is an l-ring satisfying (*), then G has an o-Cauchy completion in l-rings. Since an l-algebra is a vector lattice and l-ring, and the distributive laws clearly extend to G_1 whenever the operations extend termwise, an o-Cauchy completion for l-algebras satisfying (*) is established.

To prove the minimality assertion, hence also the essential uniqueness of H, let $\phi: G \rightarrow H'$ be a σ-regular l-group embedding with H' o-Cauchy complete. A map $\tilde{\phi}: H \rightarrow H'$ is unambiguously defined by

$$\tilde{\phi}(h) = \text{o-lim } \phi(g_n),$$

where $|g_n - h| \leqq u_n \downarrow 0$ for some $\{g_n\}$, $\{u_n\}$ in G (Definition 1.2(c)). Clearly $\tilde{\phi}|G = \phi$, $\tilde{\phi}$ is one to one (use monotonic sequences given by 1.3), $\tilde{\phi}$ preserves $+$ and $\vee$, and $\tilde{\phi}(H) = \phi(G)_1{}^{H'}$ (again using monotonic sequences). To see that $\tilde{\phi}$ is the only order-preserving extension of ϕ, suppose $\Psi: H \rightarrow H'$ is order-preserving with $\Psi(g) = \phi(g)$ for all $g \in G$. If $h \in H$, then $a_n \uparrow h$ and $b_n \downarrow h$ for some $\{a_n\}$ and $\{b_n\}$ in G (1.2(c) and 1.3) so

$$\phi(a_n) = \Psi(a_n) \leqq \Psi(h) \leqq \Psi(b_n) = \phi(b_n) \text{ and}$$
$$\Psi(h) = \text{o-lim } \phi(a_n) = \tilde{\phi}(h).$$

Thus $\Psi = \tilde{\phi}$.

Now clearly $\tilde{\phi}$ preserves scalar multiplication if G is a vector lattice and $\phi: G \rightarrow H'$ is a vector lattice embedding. If G is an l-ring satisfying (*) and $\phi: G \rightarrow H'$ is an l-ring embedding, then $G_1{}^H = H$ and $\phi(G)_1{}^{H'} = \tilde{\phi}(H)$ are l-rings satisfying (*) (1.6(b)). Applying 1.6(a) to H and $\tilde{\phi}(H)$ yields that $\tilde{\phi}$ preserves the multiplication in H. This proves the theorem.

An l-ring is called an *f-ring* if $g \wedge h = 0$ and $f \geqq 0$ imply $fg \wedge h = gf \wedge h = 0$, or equivalently if it is a subdirect sum of totally ordered rings [**8**]. The following is due independently to Bernau [**1**, p. 622] and Johnson [**17**].

1.8. LEMMA. *Every Archimedean f-ring G satisfies property* (*).

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1.9. COROLLARY. *Every lattice-ordered ring (respectively, lattice-ordered-algebra) of real-valued functions on some set (with pointwise operations) has an essentially unique* o*-Cauchy completion in l-rings (respectively, in l-algebras).*

1.10. *Remarks.* Not every f-ring satisfies (*). For example, let $G = \mathbf{R}[x]$ (the ring of polynomials over $\mathbf{R}$), where $a_0 + a_1x + \ldots + a_nx^n \geqq 0$ if and only if $a_n \geqq 0$. If $u_n = 1/n$ and $h = x$ then $u_n \downarrow 0$ but $hu_n \geqq 1 > 0$. Thus G does not have (*).

This example also shows that condition (*), although sufficient for the existence of an o-Cauchy completion of an l-ring (1.7), is not necessary. For, $G = \mathbf{R}[x]$ is totally ordered and in fact is o-Cauchy complete.

To see this, suppose $\{f_n\}$ and $\{g_n\}$ are sequences of elements of $\mathbf{R}[x]$ such that

(i) $0 \leqq f_m \leqq f_{m+1} \leqq g_{n+1} \leqq g_n$ for $n, m = 1, 2, \ldots$, and

(ii) $\wedge\ (g_n - f_n) = 0.$

By (i) and (ii) there is a positive integer N such that $a_n = g_n - f_n$ is a constant polynomial for $n \geqq N$, and $a_n \to 0$ in the usual topology of $\mathbf{R}$. Hence there is a real number r such that

$$f_m \leqq f_N - f_N(0) + r \leqq g_n, \text{ for } n, m = 1, 2, \ldots .$$

Thus $\mathbf{R}[x]$ is o-Cauchy complete by 1.4.

We do not know of any necessary and sufficient condition on an l-ring to guarantee that multiplication is preserved under the embedding described in Theorem 1.7.

1.11. *Example. The Baire Classes.* If G is an l-group and H is its o-Cauchy completion (as an l-group) then by condition (c) of Definition 1.2 G must be sequentially dense in H with respect to convergence relative to G. This condition can not be relaxed to require only that

(c′) each $h \in H$ is the o-limit in H of a sequence $\{g_n\}$ from G, i.e., $|h - g_n| \leqq u_n$ for some $u_n \downarrow 0$ in H.

For an example, let $G = \mathrm{Ba}_1([0, 1])$ and $H = \mathrm{Ba}_2([0, 1])$, the first and second Baire classes of functions on $[0, 1]$. That is, G consists of all sequential pointwise limits of continuous functions from $[0, 1]$ to $\mathbf{R}$, and H consists of all sequential pointwise limits from G. One easily checks that $u_n \downarrow 0$ in G or in H if and only if $u_n(t) \downarrow 0$ for all $t \in [0, 1]$. In particular, G is σ-regular in H. Also, each $h \in H$ is the o-limit in H of a sequence $\{g_n\}$ from G, for if $g_n(t) \to h(t)$ for all $t \in [0, 1]$, then $|g_n - h| \leqq u_n$ where

$$u_n(t) = \sup_{i \geqq n} g_i(t) - \inf_{i \geqq n} g_i(t).$$

Clearly $u_n \in H$ if $g_n \in G$ for all n, and $u_n(t) \downarrow 0$ for all t. Thus $g_n \xrightarrow{o} h$ in H. It is a known fact that G and H (in fact all Baire classes) are closed with respect to two-sided monotone sequential limits; that is, if $g_n(t) \uparrow f(t)$ and $h_n(t) \downarrow f(t)$ for all t and $\{g_n\}$, $\{h_n\}$ are in G then $f \in G$, and similarly for H (see [**15**, 31.4.52, p. 401]). Thus G and H are o-Cauchy complete by Lemma 1.4, and in particular G is its own o-Cauchy completion. Thus the embedding $G \hookrightarrow H$ satisfies conditions (a), (b), and (c′) (as amended above) of Definition 1.2, and G $\neq H$. Thus the amended definition does not yield the same notion of completion.

2. The o-Cauchy completion of $C(X)$**.** We now specialize the discussion of § 1 to the l-algebra $C(X)$ of all continuous real-valued functions on the Tychonoff space X (equipped with pointwise operations). The sub-l-algebra of bounded functions is denoted $C^*(X)$. In this section and the next we describe the o-Cauchy completion of $C(X)$ (see 1.9) in several ways as l-algebras of functions, and for compact X we obtain in fact a $C(K)$ for a certain compact space K.

Some terminology: For $f: X \to \mathbf{R}$, the *cozero* set of f is $\operatorname{coz} f = \{x \mid f(x) \neq 0\}$ and the *zero set* is $Z(f) = X - \operatorname{coz} f$. In a topological space X, a cozero set is a set $\operatorname{coz} f$ for some $f \in C(X)$. For X compact Hausdorff (or just normal), the cozero sets are exactly the open F_σ's.

The method of construction employed here is quite similar to the method of [**9**, § 2.4 and § 4.1]. We first recall the generalities. Suppose $\mathscr{F}$ is a filter base of dense subsets of a topological space X, i.e., $\mathscr{F}$ is a family of dense, nonempty subsets of X closed under finite intersections. Consider the set of all functions $f \in C(S)$ for some $S \in \mathscr{F}$, and identify $f \in C(S)$ with $g \in C(T)$ if and only if $f = g$ on $S \cap T$. Denote the set of all equivalence classes by $C[\mathscr{F}]$, and let $C^*[\mathscr{F}]$ denote all the equivalence classes containing bounded functions. Alternatively, observe that $\{C(S): S \in \mathscr{F}\}$ or $\{C^*(S): S \in \mathscr{F}\}$ form directed systems, where $S \supset T$ in $\mathscr{F}$ yields the bonding homomorphism $f \to f|T$ for $f \in C(S)$ (or $f \in C^*(S)$). Then $C[\mathscr{F}]$ and $C^*[\mathscr{F}]$ are the direct limits $\varinjlim\{C(S): S \in \mathscr{F}\}$ and $\varinjlim\{C^*(S): S \in \mathscr{F}\}$. One easily checks that $C[\mathscr{F}]$ and $C^*[\mathscr{F}]$ are l-algebras under the operations canonically induced by the $C(S)$. Furthermore, each $C(S)$ or $C^*(S)$ for $S \in \mathscr{F}$ is isomorphically embedded as an l-algebra into $C[\mathscr{F}]$ or $C^*[\mathscr{F}]$, since each S is dense. In particular, if $X \in \mathscr{F}$, then $C(X)$ and $C^*(X)$ are sub-l-algebras of $C[\mathscr{F}]$ and $C^*[\mathscr{F}]$.

As a notational convenience, we shall write $f \in C[\mathscr{F}]$ if $f \in C(S)$ for some $S \in \mathscr{F}$, thus ignoring the distinction between equivalence classes and representatives. In this case, we write $S = \operatorname{dom} f$.

If $\mathscr{F}$ is a filter base of dense sets in βX and $\mathscr{F}$ contains all the dense cozero sets of βX, then there is a natural embedding of $C(X)$ into $C[\mathscr{F}]$, as follows. Each $f \in C(X)$ has a unique Stone-Čech extension $\beta f: \beta X \to$

$\mathbf{R} \cup \{\infty\}$ (the one-point compactification of $\mathbf{R}$), and if $\operatorname{fin}(f) = (\beta f)^{-1}(\mathbf{R})$ then

$$\operatorname{fin}(f) \in \mathscr{F} \text{ and } (\beta f)|\operatorname{fin}(f) \in C(\operatorname{fin}(f)).$$

This provides the canonical l-algebra embedding $C(X) \to C[\mathscr{F}]$. Moreover, this embedding induces an embedding $C^*(X) \to C^*[\mathscr{F}]$.

If $\mathscr{F}$ is a filter base of dense sets in βX containing all the dense cozero sets in βX, there is an l-algebra intermediate between $C^*[\mathscr{F}]$ and $C[\mathscr{F}]$ which is central to our subject. This is defined to be

$$C^\#[\mathscr{F}, X] = \{h \in C[\mathscr{F}]: |h| \leqq f \text{ for some } f \in C(X)\},$$

where we have assumed $C(X) \subset C[\mathscr{F}]$ by the above canonical embedding. Thus,

$$C^*[\mathscr{F}] \subset C^\#[\mathscr{F}, X] \subset C[\mathscr{F}],$$

and in case X is compact, then $C^*[\mathscr{F}] = C^\#[\mathscr{F}, X]$. In the following, the dependence of $C^\#[\mathscr{F}, X]$ on the space X will be implicitly understood, and we shall for convenience suppress explicit mention of X and write simply $C^\#[\mathscr{F}]$.

The results of [**9**] deal primarily with the case where $\mathscr{F}$ is taken to be either the family of dense open sets or the family of dense G_δ sets in X (for the latter, X is assumed compact, and closure under finite intersections follows from the Baire category theorem). It turns out that the structure required for the present purposes is obtained by taking for $\mathscr{F}$ either the family of dense cozero sets or the family of countable intersections of dense cozero sets. These families are denoted $\mathscr{C}(X)$ and $\mathscr{C}_\delta(X)$, respectively. Some of the results here are exactly analogous to the corresponding results in [**9**], but the proofs are different, apparently of necessity.

The main result of this section now follows. It is analogous to the representation of the Dedekind-MacNeille completion (by cuts) of $C(X)$ as $C^\#[\mathscr{G}_\delta]$, where $\mathscr{G}_\delta$ is the class of all dense G_δ-sets in βX (see [**9**, 4.11 and 4.6]).

2.1. THEOREM *The* o-*Cauchy completion of* $C(X)$ (*as an* l-*algebra*) *is* $C^\#[\mathscr{C}_\delta(\beta X)]$.

Some preliminary facts are needed for the proof. Recall that a subgroup G of an l-group H is called *order-convex* if $0 \leqq h \leqq g$ and $g \in G$ imply $h \in G$.

2.2. LEMMA. *Any order-convex sub-*l*-group* G *of an* o-*Cauchy complete* l*-group* H *is itself* o-*Cauchy complete.*

Proof. Using 1.4, suppose $\{u_n\}$ and $\{v_n\}$ are sequences in G with $\{u_n\}$ increasing, $\{v_n\}$ decreasing, $u_n \leqq v_n$ for all n, and $(v_n - u_n) \downarrow 0$ relative

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to G. Since G is order-convex, $\wedge(v_n - u_n) = 0$ relative to H. Since H is o-Cauchy complete, 1.4 produces $h \in H$ satisfying $u_n \leqq h \leqq v_n$ for all n. Since G is order-convex, $h \in G$ and G is o-Cauchy complete.

2.3. A subspace S of a space X is called *z-embedded* if whenever Z is a zero-set in S, then $Z = Z' \cap S$ for some zero-set Z' in X. Since every $S \in \mathscr{C}_\delta(\beta X)$ is a Baire set in βX and is therefore Lindelöf [**4**, p. 77], and a Lindelöf subspace is always z-embedded [**4**, p. 79], each $S \in \mathscr{C}_\delta(\beta X)$ is z-embedded in every superspace.

The following approximation property characterizes z-embedded subspaces. See [**14**] or [**3**] for a proof.

2.4. LEMMA. *S is z-embedded in X if and only if given $h \in C(S)$ and $\epsilon > 0$ there exist a cozero set T in X with $S \subset T$ and $g \in C(T)$ such that $|h(x) - g(x)| < \epsilon$ for $x \in S$.*

2.5. LEMMA. *Suppose S is z-embedded in X, $h \in C(S)$, and there exists $f \in C(X)$ such that $|h(x)| \leqq f(x)$ for all $x \in S$. Then there exist sequences $\{u_n\}$ and $\{v_n\}$ in $C(X)$ such that $u_1 \leqq u_2 \leqq \ldots \leqq v_2 \leqq v_1$, and for each $x \in S$,*

$$h(x) = \sup u_n(x) = \inf v_n(x).$$

Proof. We first construct an increasing sequence $\{u_n'\}$ in $C(X)$ such that $u_n'(x) \to h(x)$ for all $x \in S$. Assume with no loss in generality that $f \geqq 1$ and that $h \geqq 0$ on S (by substituting $h + f|S$). Define $h^* \in C^*(S)$ by $h^*(x) = h(x)/f(x)$ for $x \in S$. For each n, apply 2.4 for $\epsilon = 1/n$ and obtain cozero sets T_n and $g_n \in C^*(T_n)$ such that $S \subset T_n$ and $|g_n(x) - h^*(x)| < 1/n$ for $x \in S$. We can assume $0 \leqq g_n \leqq h^*$ on S. Now $T_n = \bigcup_k Z_k^n$ for zero-sets Z_k^n of X (as with any cozero set). Since disjoint zero sets are always completely separated (i.e. separated by continuous functions: see [**13**, 1.15]), there exist $w_k^n \in C(X)$ such that $w_k^n = 1$ on Z_k^n, $w_k^n = 0$ on $X - T_n$, and $0 \leqq w_k^n \leqq 1$. Extend $w_k^n g_n$ over X (and retain the same name) by assigning value 0 on $X - T_n$, so that $w_k^n g_n \in C(X)$. For $x \in T_n$, we have

$$\vee_k w_k^n(x) g_n(x) = g_n(x),$$

and hence for $x \in S$, we have

$$\vee_{k,n} w_k^n(x) g_n(x) = h^*(x).$$

Now set

$$u_m'' = \vee_{k,n \leqq m} w_k^n g_n, \; m = 1, 2, 3, \ldots,$$

so that $u_m''(x) \uparrow h^*(x)$ for $x \in S$ and $u_m'' \in C(X)$. Define $u_m' = u_m'' f \in C(X)$, so that $u_m'(x) \uparrow h(x)$ for $x \in S$.

By applying this result to $-h$, we obtain $\{v_n'\}$ in $C(X)$ with $v_n'(x) \downarrow h(x)$ for $x \in S$. Now define inductively

$$u_1 = u_1', u_{n+1} = (u_{n+1}' \wedge v_{n+1}') \vee u_n \text{ and } v_n = u_n \vee v_n', \text{ for } n = 1, 2, 3, \ldots,$$

obtaining $u_1 \leqq u_2 \leqq \ldots \leqq v_2 \leqq v_1$ in $C(X)$ with $u_n(x) = u_n'(x)$ and $v_n(x) = v_n'(x)$ for all $x \in S$. Thus $u_n(x) \uparrow h(x)$ and $v_n(x) \downarrow h(x)$ for $x \in S$, as required. This proves 2.5.

We are now prepared to prove the main result.

Proof of 2.1. We need to show that the l-algebras $G = C(X)$ and $H = C^\#[\mathscr{C}_\delta(\beta X)]$ satisfy the three conditions of Definition 1.2.

To prove conditions (a) and (b) of Definition 1.2, we will need the following:

(†) If $S \in \mathscr{C}_\delta(\beta X)$, $w_n \downarrow 0$ in $C(S)$, and $T = \{x \in S: w_n(x) \to 0\}$, then $T \in \mathscr{C}_\delta(\beta X)$.

For, $T = \bigcap_k T_k$, where $T_k = \bigcup_n\{x \in S: w_n(x) < 1/k\}$, $k = 1, 2, \ldots,$ whence each T_k is a cozero set in S. Furthermore, each T_k is dense in S, for if $x_0 \in S - \bar{T}_k$ for some x_0 and k, then by complete regularity of S there is $h \in C(S)$ such that $h \equiv 0$ on T_k, $h(x_0) > 0$, and $0 \leqq h \leqq 1/k$ on S. Then $h \leqq w_n$ for all n, contradicting $\wedge\, w_n = 0$ in $C(S)$. Thus T_k is dense in S, hence also in βX. Since S is z-embedded in βX (see 2.3), $T_k = S \cap S_k$ for some cozero set S_k in βX, and each S_k is dense in βX. Thus

$$S_k \in \mathscr{C}(\beta X), \bigcap_k S_k \in \mathscr{C}_\delta(\beta X), \text{ and } T = \bigcap_k T_k = S \cap \bigcap_k S_k \in \mathscr{C}_\delta(\beta X),$$

proving (†).

To show $C^\#[\mathscr{C}_\delta(\beta X)]$ is o-Cauchy complete, it suffices (by 2.2) to show $C[\mathscr{C}_\delta(\beta X)]$ is o-Cauchy complete. We use the criterion of Corollary 1.4. So suppose $u_1 \leqq u_2 \leqq \ldots \leqq v_2 \leqq v_1$ in $C[\mathscr{C}_\delta(\beta X)]$ with $(v_n - u_n) \downarrow 0$. Let $w_n = v_n - u_n$ for $n = 1, 2, \ldots,$ and let $S = \bigcap_n \operatorname{dom} w_n$. Then $S \in \mathscr{C}_\delta(\beta X)$, and regarding each w_n as an element of $C(S)$, we have $w_n \downarrow 0$ in $C(S)$. If

$$T = \{x \in S: w_n(x) \to 0\},$$

then $T \in \mathscr{C}_\delta(\beta X)$ by (†) above, and

$$\sup u_n(x) = \inf v_n(x) \text{ for } x \in T.$$

Denoting this common limit by $g(x)$, we have $g \in C(T)$ since g is both upper semicontinuous and lower semicontinuous on T, whence $g \in C[\mathscr{C}_\delta(\beta X)]$ and $u_n \leqq g \leqq v_n$ for all n. By 1.4, $C[\mathscr{C}_\delta(\beta X)]$ (hence also

$C^{\#}[\mathscr{C}_\delta(\beta X)]$) is o-Cauchy complete. This proves condition (a) of Definition 1.2.

To prove condition (b) of Definition 1.2, i.e., that the embedding $C(X) \subset C[\mathscr{C}_\delta(\beta X)]$ is σ-regular, let $u_n \downarrow 0$ in $C(X)$. If fin $(u_n) = (\beta u_n)^{-1}(\mathbf{R})$, where $\beta u_n: \beta X \to \mathbf{R} \cup \{\infty\}$ is the Stone-Čech extension, then

$$S \equiv \bigcap_n \operatorname{fin}(u_n) \in \mathscr{C}_\delta(\beta X) \text{ and } X \subset S.$$

If $w_n = (\beta u_n)|S$ for each n, then $w_n \downarrow 0$ in $C(S)$, so that

$$T \equiv \{x \in S: w_n(x) \to 0\} \in \mathscr{C}_\delta(\beta X)$$

(by (†) above). If $h \in C[\mathscr{C}_\delta(\beta X)]$ and $0 \leqq h \leqq u_n$ relative to $C[\mathscr{C}_\delta(\beta X)]$, then $0 \leqq h \leqq w_n$ relative to $C[\mathscr{C}_\delta(\beta X)]$ and

$$0 \leqq h(x) \leqq w_n(x) \to 0 \text{ for } x \in T \cap \operatorname{dom} h.$$

Thus $h = 0$ on $T \cap \operatorname{dom} h \in \mathscr{C}_\delta(\beta X)$, which means that $h = 0$ in $C[\mathscr{C}_\delta(\beta X)]$. Thus $u_n \downarrow 0$ in $C[\mathscr{C}_\delta(\beta X)]$, and $C(X)$ is σ-regular in $C[\mathscr{C}_\delta(\beta X)]$ (hence clearly $C(X)$ is σ-regular in $C^{\#}[\mathscr{C}_\delta(\beta X)]$).

To prove (c) of Definition 1.2, suppose $h \in C^{\#}[\mathscr{C}_\delta(\beta X)]$, so that $|h| \leqq f$ for some $f \in C(X)$. If $Y = (\beta f)^{-1}(\mathbf{R})$ and $S = Y \cap \operatorname{dom} h$, then $S \in \mathscr{C}_\delta(\beta X)$ and S is z-embedded in Y (2.3). Regarding h and f as elements of $C(S)$ and $C(Y)$, respectively, apply 2.5 to obtain $u_n \uparrow$ and $v_v \downarrow$ in $C(Y)$ such that

$$\sup u_n(x) = h(x) = \inf v_n(x) \text{ for } x \in S.$$

Then $u_n(x) - v_n(x) \to 0$ for $x \in S$, so $(v_n - u_n) \downarrow 0$ in $C(Y)$, S being dense in Y. If $w_n \in C(X)$ is defined by $w_n = (v_n - u_n)|X$, then $w_n \downarrow 0$ in $C(X)$ (because if $w \in C(X)$ satisfies $0 \leqq w \leqq w_n$ for all n, then $0 \leqq (\beta w)|Y \leqq (v_n - u_n)$ for all n). Define now $g_n \in C(X)$ by $g_n = v_n|X$. Then $|g_n - h| \leqq w_n$ relative to $C[\mathscr{C}_\delta(\beta X)]$, and $w_n \downarrow 0$ in $C(X)$. This proves (c) of Definition 1.2 for $G = C(X)$ and $H = C^{\#}[\mathscr{C}_\delta(\beta X)]$, and finishes the proof of 2.1.

2.6. Corollary. *The o-Cauchy completion of $C^*(X)$ is $C^*[\mathscr{C}_\delta(\beta X)]$. In particular, for compact X, the o-Cauchy completion of $C(X)$ is $C^*[\mathscr{C}_\delta(X)]$.*

Proof. The case of compact X (i.e., $X = \beta X$) is immediate from 2.1. For arbitrary X, apply this to $C(\beta X)$, which is isomorphic to $C^*(X)$.

3. More on the o-Cauchy completion of $C(X)$**.** In order to amplify the description of the o-Cauchy completion of $C(X)$ given in 2.1, we need to study the relationship between the l-algebras $C[\mathscr{F}]$ for various filter-bases $\mathscr{F}$ of dense sets in X or in βX. We will be specifically concerned with $\mathscr{C}(X)$, $\mathscr{C}(\beta X)$, and $\mathscr{C}_\delta(\beta X)$ (where $\mathscr{C}$ denotes dense cozero sets and $\mathscr{C}_\delta$-sets are countable intersections of $\mathscr{C}$-sets).

Observe first that $C[\mathscr{C}(\beta X)]$ is embedded as a sub-l-algebra of $C[\mathscr{C}(X)]$ by restriction: if $S \in \mathscr{C}(\beta X)$ and $f \in C(S)$ then $S \cap X \in \mathscr{C}(X)$ and $f|(S \cap X) \in C[\mathscr{C}(X)]$. By abuse of notation we write $C[\mathscr{C}(\beta X)] \subset C[\mathscr{C}(X)]$. This relation is in fact an equality, as the following lemma will show. The essence of this result is contained in [**9**, 3.8]. Recall that a subspace S of X is *C^*-embedded* if every $f \in C^*(S)$ extends to some $\tilde{f} \in C^*(X)$.

3.1. LEMMA. *Let X be a C^*-embedded subspace of a Tychonoff space Y. Every continuous function on a cozero subset of X extends continuously to a cozero subset of Y.*

Proof. Let $f \in C(S)$ where $S = \operatorname{coz} g$ and $g \in C(X)$. Define $h \in C^*(X)$ by $h = g/(1 + f^2 + g^2)$ on S and $h = 0$ on $X - S$, and extend to $\tilde{h} \in C^*(Y)$. Define $k \in C^*(X)$ by $k = fh$ on S and $k = 0$ on $X - S$, and extend to $\tilde{k} \in C^*(Y)$. Then $\tilde{f} = \tilde{k}/\tilde{h}$ on $\operatorname{coz} \tilde{h}$ is the desired extension.

3.2. COROLLARY.

$$C[\mathscr{C}(\beta X)] = C[\mathscr{C}(X)].$$
$$C^{\#}[\mathscr{C}(\beta X)] = C^{\#}[\mathscr{C}(X)].$$
$$C^*[\mathscr{C}(\beta X)] = C^*[\mathscr{C}(X)].$$

Proof. Since X is C^*-embedded in βX, 3.1 shows that the embedding $C[\mathscr{C}(\beta X)] \to C[\mathscr{C}(X)]$ is onto. Clearly the spaces $C^{\#}$ (resp. C^*) correspond under this embedding.

If $\mathscr{F}$ is a filter base of dense sets in X, $C[\mathscr{F}]$ has a natural metric topology; the topology of *uniform convergence*, in which a sequence $\{f_n\}$ converges to f if and only if for each $\epsilon > 0$, eventually $|f_n - f| \leqq \epsilon.1$ in the lattice $C[\mathscr{F}]$. Following [**9**, 4.1], a metric for this topology is given by

$$(f, g) \to \sup \frac{|f - g|}{1 + |f - g|}$$

where the sup is taken over $\operatorname{dom} f \cap \operatorname{dom} g \in \mathscr{F}$. We use the term *uniform* in reference to this topology.

The following lemma is [**9**, 4.5].

3.3 LEMMA. *If $\mathscr{F}$ is closed under countable intersections then $C[\mathscr{F}]$ is uniformly complete.*

3.4. PROPOSITION. *$C[\mathscr{C}(\beta X)]$ is uniformly dense in $C[\mathscr{C}_\delta(\beta X)]$, so that $C[\mathscr{C}_\delta(\beta X)]$ is the uniform completion of $C[\mathscr{C}(X)]$ (or of $C[\mathscr{C}(\beta X)]$).*

Proof. Since each $S \in \mathscr{C}_\delta(\beta X)$ is z-embedded (2.3), the first part is immediate from 2.4. Since $C[\mathscr{C}(\beta X)] = C[\mathscr{C}(X)]$ (3.2) and $\mathscr{C}_\delta(\beta X)$ is closed under countable intersections, the last statement of 3.4 follows from 3.3.

3.5. COROLLARY. *The* o-*Cauchy completion of* $C(X)$ *(i.e.,* $C^{\#}[\mathscr{C}_\delta(\beta X)]$*) is the uniform completion of* $C^{\#}[\mathscr{C}(X)] = C^{\#}[\mathscr{C}(\beta X)]$. *The* o-*Cauchy completion of* $C^*(X)$ *(i.e.,* $C^*[\mathscr{C}_\delta(\beta X)]$*) is the uniform completion of* $C^*[\mathscr{C}(X)] = C^*[\mathscr{C}(\beta X)]$.

Remark. The analogues of 3.2 and 3.4 for dense open sets are proved in [**9**]. Let $\mathscr{G}(X)$ denote the dense open sets in X, and $\mathscr{G}_\delta(\beta X)$ denote the dense G_δ-sets in βX. The analogue of 3.2 follows from [**9**, 3.8]:

$$C[\mathscr{G}(\beta X)] = C[\mathscr{G}(X)],\ C^{\#}[\mathscr{G}(\beta X)] = C^{\#}[\mathscr{G}(X)] \text{ and } C^*[\mathscr{G}(\beta X)] = C^*[\mathscr{G}(X)].$$

The analogue of 3.4 is slightly stronger: for any filter base $\mathscr{F}$ of dense sets such that $\mathscr{F} \supset \mathscr{G}(X)$, $C[\mathscr{G}(X)]$ is uniformly dense in $C[\mathscr{F}]$ (see [**9**, 4.4]). The cozero set result 3.4 is not proved in [**9**], evidently because it depends on 2.4 which was not known then.

We are now in a position to characterize those X such that $C(X)$ is o-Cauchy complete. Recall that X is an *F-space* if each of its cozero sets is C^*-embedded (see [**13**, 14.25]).

3.6. *Definition.* A Tychonoff space is called a *quasi-F-space* if each dense cozero set is C^*embedded.

The following result was originally proved in [**6**] for the case of compact X by a rather more direct argument. An extensive description of quasi-F spaces is given subsequently in Section 5.

3.7. THEOREM. *For an arbitrary Tychonoff space* X, $C(X)$ *is* o-*Cauchy complete if and only if* X *is a quasi-F space.*

Proof. Suppose first that $C(X)$ is o-Cauchy complete. Then by 2.1,

(1) $\quad C(X) = C^{\#}[\mathscr{C}_\delta(\beta X)]$.

Let S be a dense cozero set in X and $h \in C^*(S)$. By 3.1, there exist a cozero set S_1 of βX such that $S_1 \supset S$ and $h_1 \in C^*(S_1)$ such that $h_1|S = h$. By (1), there exists $h_2 \in C(X)$ such that $h_1 = h_2$ on $S_1 \cap X$. Thus $h_2 = h$ on S, and S is C^*-embedded in X.

Conversely, suppose X is a qusi-F space. We claim that

(2) $\quad C(X) = C^{\#}[\mathscr{C}(\beta X)]$.

Indeed, let $h \in C^{\#}[\mathscr{C}(\beta X)]$. Then for some $S \in \mathscr{C}(\beta X)$ and $f \in C(X)$, $h \in C(S)$ and $|h| \leqq f$ on $S \cap X$. But $S \cap X$ is a dense cozero set of X, hence C^*-embedded, and thus there exists $h_1 \in C(X)$ such that $h_1 = h$ on $S \cap X$. But then $h_1 = h$ as elements of $C^{\#}[\mathscr{C}(\beta X)]$, proving (2). Thus by 3.5, $C(X)$ is uniformly dense in $C^{\#}[\mathscr{C}_\delta(\beta X)]$, and since $C(X)$ is uniformly complete, (1) is true. Thus by 2.1, $C(X)$ is o-Cauchy complete. This proves 3.7.

We now proceed to represent the o-Cauchy completion of $C^*(X)$ as a $C(K)$ for a certain compact space K. The construction is motivated by the construction of the Dedekind-MacNeille completion (by cuts) of $C^*(X)$ as a $C(G)$ given in [**9**], where G is Gleason's "minimal projective cover" of βX. In [**9**, 4.11 and 6.9], G is given as the inverse limit space $\varprojlim\{\beta S: S \in \mathscr{G}(X)\}$, where, as before, $\mathscr{G}(X)$ denotes the dense open sets of X. Our approach is to replace dense open sets by dense cozero sets. The present methods differ in one important respect from those in [**9**]: The approximation theorem 3.4, which is the pivotal technical result, was not available to [**9**].

We recall some generalities about direct and inverse limits. Let $\{K_a\}$ be any inverse system of compact Hausdorff spaces with respect to surjections $\pi_a{}^b: K_b \to K_a$ for $a \leqq b$. Then $\{C(K_a)\}$ is a direct system of l-algebras with respect to the embeddings $f_a \to f_b = f_a \circ \pi_a{}^b$, $a \leqq b$. The inverse limit space $K = \varprojlim_a K_a$ is a compact Hausdorff space and the direct limit $A = \varinjlim_a C(K_a)$ is an l-algebra.

3.8. THEOREM [**9**, 6.8]. *The l-algebra $A = \varinjlim_a C(K_a)$ is isomorphic with a uniformly dense sub-l-algebra of $C(K)$, where $K = \varprojlim_a K_a$, and K is the maximal ideal space of A.*

For a Tychonoff space X, we consider the directed systems $\mathscr{C}(X)$ of dense cozero sets in X and $\mathscr{C}_\delta(\beta X)$ of dense countable intersections of cozero sets in βX. The system $\{\beta S: S \in \mathscr{C}(X)\}$ is an inverse system of compact spaces with respect to the surjections $\pi_T{}^S: \beta S \to \beta T$ which extend the inclusions $S \subset T$. Similarly, $\{\beta S: S \in \mathscr{C}_\delta(\beta X)\}$ is an inverse system of compact spaces. We now define the inverse limit spaces

$$K(X) = \varprojlim\{\beta S: S \in \mathscr{C}(X)\} \text{ and}$$
$$K_\delta(X) = \varprojlim\{\beta S: S \in \mathscr{C}_\delta(\beta X)\}.$$

Since $K(X)$ is a certain subset of $\prod\{\beta S: S \in \mathscr{C}(X)\}$, there exists a natural, continuous surjection and projection $\pi_X: K(X) \to \beta X$. This induces a natural embedding of $C(\beta X)$ into $C(K(X))$ by $f \to f \circ \pi_X$. Since $C^*(X)$ is isomorphic with $C(\beta X)$, $C^*(X)$ is naturally embedded as a sub-l-algebra of $C(K(X))$.

3.9. THEOREM. (a) *The spaces $K(X)$, $K(\beta X)$, and $K_\delta(X)$ are all homeomorphic and are quasi-F spaces.*

(b) *The natural embedding $C^*(X) \to C(K(X))$ is a realization of the o-Cauchy completion of $C^*(X)$ as the space $C(K(X))$.*

Proof. $C^*[\mathscr{C}(X)] = \varinjlim\{C^*(S): S \in \mathscr{C}(X)\} \approx \varinjlim\{C(\beta S): S \in \mathscr{C}(X)\}$, so by 3.8 $K(X)$ is the maximal ideal space of $C^*[\mathscr{C}(X)]$. Likewise $K(\beta X)$ is the maximal ideal space of $C^*[\mathscr{C}(\beta X)]$, and $K_\delta(X)$ is the maximal ideal space of $C^*[\mathscr{C}_\delta(\beta X)]$. Since $C^*[\mathscr{C}(\beta X)] = C^*[\mathscr{C}(X)]$ (3.2), $K(X)$ and

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and $K(\beta X)$ are homeomorphic. Also by 3.8, the uniform completion of

$$C^*[\mathscr{C}(\beta X)] = \varinjlim\{C(\beta S): S \in \mathscr{C}(\beta X)\}$$

is $C(K(\beta X))$. But by 3.5, this uniform completion is also $C^*[\mathscr{C}_\delta(\beta X)]$. Thus $C(K(\beta X))$ and $C^*[\mathscr{C}_\delta(\beta X)]$ are isomorphic, so their maximal ideal spaces $K(\beta X)$ and $K_\delta(X)$ are homeomorphic. This proves the first assertion in (a).

It follows from the above that $C(K(X))$ is naturally isomorphic with $C^*[\mathscr{C}_\delta(\beta X)]$ by an isomorphism which preserves the natural embeddings $C^*(X) \to C(K(X))$ and $C^*(X) \to C^*[\mathscr{C}_\delta(\beta X)]$. Since, by 2.6, the latter is a realization of the o-Cauchy completion of $C^*(X)$, (b) is proved. Since $C^*[\mathscr{C}_\delta(\beta X)]$, and therefore $C(K(X))$, is o-Cauchy complete (2.6), $K(X)$ is a quasi-F space by 3.7. This proves 3.9.

3.10. *Remarks.* (i) We shall subsequently call $K(X)$ the *minimal quasi-F cover* of X (for compact X); see Theorem 4.3.

(ii) We indicate the analogy of 3.9 with results in [**9**]. Let $\mathscr{G}(X)$ denote the family of dense open sets in X and $\mathscr{G}_\delta(\beta X)$ denote the family of dense G_δ-sets in βX. Define

$$G(X) = \varprojlim\{\beta S: S \in \mathscr{G}(X)\} \text{ and } G_\delta(X) = \varprojlim\{\beta S: S \in \mathscr{G}_\delta(\beta X)\}.$$

The natural projection $g_X: G(X) \to \beta X$ induces an embedding $f \to f \circ g_X$ of $C^*(X) = C(\beta X)$ into $C(G(X))$. Then, parallel to 3.9, we have:

(a) $G(X)$, $G(\beta X)$, and $G_\delta(X)$ are all homemorphic and are extremally disconnected.

(b) The natural embedding $C^*(X) \to C(G(X))$ is a realization of the Dedekind-MacNeille completion (by cuts) of $C^*(X)$.

For proofs, see [**9**, 4.11 and 6.9].

Our notation is chosen to reflect the fact that the pair $(G(X), g_X)$ is Gleason's "minimal projective cover" of βX [**10**]. We present a quasi-F analogue in the next section.

3.11. It is shown in [**9**, 2.6] that the classical ring of quotients of $C(X)$, denoted $Q_{cl}(X)$, can be concretely realized as $C[\mathscr{C}(X)]$. This is the ring of formal fractions f/g where g is a nondivisor of zero in $C(X)$; see [**27**, Chapter 1, § 19]. $Q_{cl}^*(X)$ denotes the bounded elements of $Q_{cl}(X)$. The following corollary supplements [**9**, 4.6].

COROLLARY. *Let X be a Tychonoff space.*

(i) *The uniform completion of $Q_{cl}(X)$ is $C[\mathscr{C}_\delta(\beta X)]$.*

(ii) *The uniform completion of $Q_{cl}^*(X)$ is isomorphic to $C(K(X))$ and to the o-Cauchy completion of $C^*(X)$.*

Proof. Apply 3.4, 3.5, and 3.9.

3.12. *Example*. In contrast to 3.9, if X fails to be compact, the o-Cauchy completion of $C(X)$ need not be a $C(Y)$. In [**21**], a space X is called a *weak* cb-*space* if each locally bounded lower semi-continuous real-valued function on X is bounded above by some element of $C(X)$, and it is shown that the Dedekind-MacNeille completion of $C(X)$ is a $C(Y)$ if and only if X is a weak cb-space. Thus it suffices to find an X which is not weak cb, for which the o-Cauchy completion of $C(X)$ is Dedekind complete. It is shown in Proposition 4.6 that this is true if every dense open subset of X contains a dense cozero set in X. We will exhibit next a space with this latter property that is not a weak cb-space.

Suppose Y is any Tychonoff space that is not a weak cb-space; see [**21**, p. 238]. Let $\alpha N = N \cup \{\omega\}$ denote the one-point compactification of the countable discrete space N, let $X = Y \times \alpha N$, and let $E = \{(y, n): y \in Y, n \in N\}$. Equip X with the topology obtained from the product topology by adding every subset of E. It is not difficult to verify that any dense subset of X contains the dense cozero set E. Moreover every continuous (resp. lower semi-continuous) real-valued function on $(X - E)$ has a continuous (resp. lower semi-continuous) extension over X, and it follows that X is not a weak cb-space.

4. The quasi-F cover.

Recall from 3.9 that for compact X, the o-Cauchy completion of $C(X)$ is $C(K(X))$, where $K(X)$ is the quasi-F space given by either $\varprojlim\{\beta S: S \in \mathscr{C}(X)\}$ or $\varprojlim\{\beta S: S \in \mathscr{C}_\delta(X)\}$. We examine the properties of the pair $(K(X), \pi_X)$, which we shall call the *minimal quasi-F cover* of X, where $\pi_X: K(X) \to X$ is the canonical projection (see 4.3).

Recall that a map $\pi: X \to Y$ is *irreducible* if X is the only closed subspace of X whose image under π is all of Y. A subset G of an l-group H is *order-dense* if for each nonzero $h \geqq 0$ in H there exists a nonzero $g \in G$ with $0 \leqq g \leqq h$. The following lemma appears in [**26**, p. 17]. For the sake of completeness we include a proof.

4.1. LEMMA. *If X and Y are compact then a map $\pi: X \to Y$ is irreducible if and only if the dual embedding $\pi^0: C(Y) \to C(X)$ has an order-dense image in $C(X)$.*

Proof. Suppose $\pi: X \to Y$ is irreducible, and $h \geqq 0$ in $C(X)$ with h not identically 0. Then for some $\epsilon > 0$, $F \equiv h^{-1}([0, \epsilon]) \neq X$, so $\pi[F] \neq Y$. Define $g \in C(Y)$ so that $g = 0$ on $\pi(F)$, $0 \leqq g \leqq \epsilon$, and g is not identically 0. Then $0 \leqq g \circ \pi \leqq h$, so $\pi^0(C(Y))$ is order-dense in $C(X)$.

Conversely, suppose π^0 has order-dense image. If $F \subset X$ is closed and $F \neq X$, define $f \in C(X)$ so that $f \geqq 0$, $f = 0$ on F, but f does not vanish identically. Then there exists $g \in C(Y)$, not identically 0, so that $0 \leqq g \circ \pi \leqq f$. Then $g = 0$ on $\pi(F)$, and $\pi(F) \neq Y$. This proves the lemma.

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4.2. *Definition.* A *minimal quasi-F cover* for a compact space X is a pair (K, π) such that:

(a) K is a compact quasi-F space;

(b) $\pi: K \rightarrow X$ is a continuous irreducible surjection;

(c) if (K_1, π_1) is a pair satisfying (a) and (b) then there exists a continuous surjection $\tau: K_1 \rightarrow K$ such that $\pi_1 = \pi \circ \tau$.

4.3. THEOREM. *If X is compact, then $(K(X), \pi_X)$ is a minimal quasi-F cover which is unique in the sense that if (K, π) is a minimal quasi-F cover, then there exists a unique homeomorphism $\tau: K \rightarrow K(X)$ such that $\pi = \pi_X \circ \tau$.*

Proof. $K(X)$ satisfies 4.2(a) by 3.9(a). Every l-group is order dense in its o-Cauchy completion (1.2 and 1.3), and $\pi_X{}^0: C(X) \rightarrow C(K(X))$ is a realization of the o-Cauchy completion of $C(X)$ (3.9), so π_X is irreducible by 4.1. This proves (b) of 4.2 for $(K(X), \pi_X)$.

Now assume (K_1, π_1) satisfies (a) and (b) of 4.2. Then the embedding $\pi_1{}^0: C(X) \rightarrow C(K_1)$ is σ-regular (being order dense by 4.1), and $C(K_1)$ is o-Cauchy complete (3.7). By the minimality assertion in Theorem 1.7, there exists a unique l-algebra embedding $\tilde{\phi}: C(K(X)) \rightarrow C(K_1)$ such that $\pi_1{}^0 = \tilde{\phi} \circ \pi_X{}^0$. Dualizing, we obtain a unique continuous surjection $\tau: K_1 \rightarrow K(X)$ such that $\pi_1 = \pi_X \circ \tau$. This proves (c) of 4.2, and so $(K(X), \pi_X)$ is a minimal quasi-F cover.

Specializing the above uniqueness of τ to the special case $(K_1, \pi_1) = (K(X), \pi_X)$, we deduce that if $\rho: K(X) \rightarrow K(X)$ is a continuous surjection satisfying $\pi_X = \pi_X \circ \rho$, then ρ is the identity homeomorphism on $K(X)$.

Now assume (K, π) is a minimal quasi-F cover. Then there exist continuous surjections $\tau: K \rightarrow K(X)$ and $\sigma: K(X) \rightarrow K$ such that $\pi = \pi_X \circ \tau$ and $\pi_X = \pi \circ \sigma$. Thus $\pi_X = \pi_X \circ \rho$ if $\rho = \tau \circ \sigma$. By the above uniqueness assertion, ρ is the identity homeomorphism on $K(X)$. Thus τ is one-to-one, i.e., τ is a homeomorphism. The uniqueness of τ follows as in the second paragraph above. This proves Theorem 4.3.

4.4. *Questions.* (1) What is an intrinsic characterization of the minimal quasi-F cover $(K(X), \pi_X)$? Recall that if G is extremally disconnected and $g: G \rightarrow X$ is an irreducible surjection, then (G, g) is the Gleason minimal projective cover $(G(X), g_X)$ (see Remark 3.10). What intrinsic (topological?) property of a surjection $\pi: K \rightarrow X$, where K is a quasi-F space, guarantees that (K, π) is the minimal quasi-F cover $(K(X), \pi_X)$?

(2) Gleason showed [**11**] that in the category of compact spaces and continuous maps, the projectives are exactly the extremally disconnected spaces (hence the term "projective cover" for $G(X)$). Is there a reasonable sense in which quasi-F spaces are projective?

We now consider the relationship between the o-Cauchy completion of $C(X)$ and the Dedekind-MacNeille completion by cuts. For any l-group G, the o-Cauchy completion and the Dedekind-MacNeille completion by cuts (see [**2**]) are the same if and only if the o-Cauchy completion (as given in Theorem 1.7) is Dedekind complete.

4.5. PROPOSITION. *Let X by a Tychonoff space. The following are equivalent:*

(1) *The o-Cauchy completion of $C(X)$ is Dedekind complete.*
(2) *The o-Cauchy completion of $C^*(X)$ is Dedekind complete.*
(3) *$K(X)$ is extremally disconnected.*
(4) *The minimal quasi-F cover of βX is the same as Gleason's minimal projective cover of βX.*

Proof. By 2.1 and 2.6, the o-Cauchy completion of $C(X)$ (resp. $C^*(X)$) is $C^\#[\mathscr{C}_\delta(\beta X)]$ (resp. $C^*[\mathscr{C}_\delta(\beta X)]$). By [**9**, 4.6 and 4.11], the Dedekind completion of $C(X)$ (resp. $C^*(X)$) is $C^\#[\mathscr{G}_\delta(\beta X)]$ (resp. $C^*[\mathscr{G}_\delta(\beta X)]$), where as before, $\mathscr{G}_\delta(\beta X)$ denotes the family of dense G_δ-sets in βX. Thus (1) is equivalent to

$$(1')\quad C^\#[\mathscr{C}_\delta(\beta X)] = C^\#[\mathscr{G}_\delta(\beta X)],$$

and (2) is equivalent to

$$(2')\quad C^*[\mathscr{C}_\delta(\beta X)] = C^*[\mathscr{G}_\delta(\beta X)].$$

Clearly $(1') \Rightarrow (2')$. We shall show that $(2')$ implies $C[\mathscr{C}_\delta(\beta X)] = C[\mathscr{G}_\delta(\beta X)]$, i.e., every continuous function on a dense G_δ set in βX can be defined on a dense countable intersection of cozero sets in βX. Clearly this implies $(1')$.

Let $h \in C(V)$ for a dense G_δ set V in βX. Define $h' = 1/(1 + h^2) \in C^*(V)$. Then $h\,h' \in C^*(V)$, and by assumption $(2')$ there exist $S_1, S_2 \in \mathscr{C}_\delta(\beta X)$ so that h' can be defined on S_1 and $h\,h'$ can be defined on S_2. Since $h' \neq 0$, $h = h\,h'/h'$ can be defined on $S_1 \cap S_2 \in \mathscr{C}_\delta(\beta X)$. This proves that (1) and (2) are equivalent.

By 3.9, the o-Cauchy completion of $C^*(X)$ is $C(K(X))$, which (by the Stone-Nakano theorem [**9**, 4.12]) is Dedekind complete if and only if $K(X)$ is extremally disconnected. Thus (2) and (3) are equivalent.

The Gleason space $(G(X), g_X)$ is characterized by the conditions that $G(X)$ is an extremally disconnected compact space and $g_X: G(X) \to \beta X$ is an irreducible surjection (see [**11**]). By 3.9(a),

$$(K(X), \pi_X) \approx (K(\beta X), \pi_{\beta X}),$$

so by 4.3 $\pi_X \approx \pi_{\beta X}$ is irreducible. Thus (3) and (4) are equivalent. This proves 4.5.

The relationship between the o-Cauchy completion and the Dedekind completion of $C(X)$ is tied up with the relationship between dense open subsets and dense cozero subsets of X.

4.6. PROPOSITION. *If X is a Tychonoff space and every dense open set of X contains a dense cozero set of X, then $K(X)$ is extremally disconnected, and the $\circ$-Cauchy completion of $C(X)$ is Dedekind complete.*

Proof. If every dense open set contains a dense cozero set, then $C[\mathscr{C}(X)] = C[\mathscr{G}(X)]$. By 3.4, $C[\mathscr{C}_\delta(\beta X)]$ is the uniform completion of $C[\mathscr{C}(X)]$. By [**9**, 2.6 and 4.6], $C[\mathscr{G}_\delta(\beta X)]$ is the uniform completion of $C[\mathscr{G}(X)]$. Thus $C[\mathscr{C}_\delta(\beta X)] = C[\mathscr{G}_\delta(\beta X)]$, and as in the proof of 4.5(1), the $\circ$-Cauchy completion of $C(X)$ is Dedekind complete. Thus $K(X)$ is extremally disconnected by 4.5(3).

Recall from [**5**] that a space X is called *weakly Lindelöf* if each of its open covers contain a countable subfamily whose union is dense in X.

4.7. COROLLARY. *If X satisfies any one of the conditions:*
(1) *X is perfectly normal (in particular if X is metrizable);*
(2) *X has the countable chain condition;*
(3) *every dense (open) subset of X is weakly Lindelöf;*
then $K(X)$ is extremally disconnected and the $\circ$-Cauchy completion of $C(X)$ is the Dedekind-MacNeille completion.

Proof. The sufficiency of (1) is immediate from 4.6. For (2), given a dense open set U in X, pick a maximal disjoint family of cozero sets contained in U. Their union is the desired cozero subset of U. For (3), cover such a U by cozero sets contained in U, and then extract a countable subset with dense union. (Indeed, (2) implies (3).)

4.8. COROLLARY. *If X is a quasi-F space in which every dense open subset contains a dense cozero set (in particular, if any of the conditions of 4.7 hold), then X is extremally disconnected.*

4.8 also follows immediately from the definition of quasi-F-spaces, since X is extremally disconnected whenever every dense open set is C^*-embedded.

It follows from 5.5 below that any metrizable quasi-F space is discrete.

4.9. *Question.* As in the proof of 4.6, the equality $C[\mathscr{C}(X)] = C[\mathscr{G}(X)]$ implies that $K(X)$ is extremally disconnected. Is it possible for a space X to satisfy the equivalent conditions of 4.5, so that $K(X)$ is extremally disconnected, yet $C[\mathscr{C}(X)] \neq C[\mathscr{G}(X)]$? This asks whether the inclusion $C[\mathscr{C}(X)] \subset C[\mathscr{G}(X)]$ can be both uniformly dense and proper. In the language of [**9**], this asks whether the classical ring of quotients of a $C(X)$ can be both uniformly dense and properly contained in its maximal ring of quotients.

5. Characterizations of quasi-F-spaces. In this section, quasi-F-spaces are characterized in a number of ways both topologically and in terms of the ring of continuous real-valued functions on the space. These characterizations are used in a number of ways; in particular to study when a finite product of quasi-F-spaces is a quasi-F-space.

Recall that an element r of a commutative ring A is called *regular* if $ra = 0$ for $a \in A$ implies that $a = 0$. An ideal of A is called *regular* if it contains a regular element. Note that an $r \in C(X)$ is regular if and only if coz (r) is dense in X. The ideal generated by $a \in A$ is denoted (a).

If A and A' are lattice-ordered and $\phi: A \rightarrow A'$ is a ring homomorphism that preserves the partial ordering on A, then we call the kernel of ϕ an *order-convex ideal* of A. If ϕ also preserves the lattice operations of A, we call its kernel an *l-ideal* of A. It is well-known that a ring ideal I is order-convex (resp. an l-ideal) if and only if $0 \leqq a \leqq b$ (resp. $|a| \leqq |b|$) and $b \in I$ imply that $a \in I$ [**8**]. (In [**13**] our order-convex ideals are called convex ideals, and our l-ideals are called absolutely convex ideals. In [**20**], our l-ideals are called solid ideals.)

The main theorem of this section follows.

5.1. THEOREM. *If X is a Tychonoff space, then the following are equivalent:*

(a) *X is a quasi-F space.*

(b) *Every dense z-embedded subspace of X is C^*-embedded.*

(c) *Whenever f and r are elements of $C(X)$ such that $|f| \leqq |r|$ and r is regular, then f is a multiple of r.*

(d) *Every regular ideal of $C(X)$ is order-convex.*

(e) *Every regular ideal of $C(X)$ is an l-ideal.*

(f) *Every finitely generated regular ideal of $C(X)$ (with generators $f_1, \ldots, f_n$) is principal (with generator $|f_1| + \ldots + |f_n|$).*

(f′) *Every regular ideal of $C(X)$ with two nonnegative generators is principal.*

(g) *$C(X)$ is o-Cauchy complete as a vector lattice.*

(h) *βX is a quasi-F-space.*

Furthermore, an equivalent condition is obtained if $C(X)$ is replaced by $C^*(X)$ in any of the preceding conditions.

Proof. We will show that (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (f′) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (f′), (a) $\Leftrightarrow$ (g), and (a) $\Leftrightarrow$ (h). The last assertion follows from the fact that $C^*(X)$ and $C(\beta X)$ are isomorphic.

(a) $\Rightarrow$ (c): Assume (a) and $|f| \leqq |r|$ as in (c). For $x \in$ coz r, define $k(x) = f(x)/r(x)$. Then $k \in C^*(\text{coz } r)$, so by (a), k has an extension $\bar{k} \in C^*(X)$. Clearly $f = \bar{k}r$, proving (c).

(c) $\Rightarrow$ (d): Assume (c), and let $I \subset C(X)$ be an ideal containing the the regular element r. Suppose $0 \leqq f \leqq g$ and $g \in I$. Then $0 \leqq f \leqq$

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$g \leqq g + r^2$, and $g + r^2$ is clearly regular. By (c), $f = k(g + r^2)$ so $f \in I$ and I is order-convex, proving (d).

(d) $\Rightarrow$ (f'): Assume (d), and let $f_1, f_2 \geqq 0$ in $C(X)$ be such that (f_1, f_2) is a regular ideal. Then coz $f_1 \cup$ cos f_2 is dense, so $f_1 + f_2$ is a regular element. Since $0 \leqq f_i \leqq f_1 + f_2$, (d) implies $(f_1, f_2) \subset (f_1 + f_2)$. Obviously $(f_1 + f_2) \subset (f_1, f_2)$, so ($f'$) is true.

(f') $\Rightarrow$ (b): Assume (f'), and let $E \subset X$ be dense and z-embedded. It suffices to show that each two disjoint zerosets Z_1, Z_2 of E are contained in disjoint zerosets of X, and then E would be C^*-embedded in X, as is shown in [**13**, pp. 17–18]. Since E is z-embedded, there are f_1, $f_2 \geqq 0$ in $C^*(X)$ such that $Z(f_i) \cap E = Z_i$, $i = 1, 2$. The assumption $Z_1 \cap Z_2 = \emptyset$ implies $E \subset$ coz r, where $r = f_1 + f_2$, and the density of E forces r to be regular. Therefore, by (f'), $(f_1, f_2) =$ (d) where $E \subset$ coz $r \subset$ coz d so d is regular. Thus there exist g_1, g_2, a_1, a_2 in $C(X)$ such that

(1) $\quad f_i = g_i d$, $i = 1, 2$, and

(2) $\quad d = a_1 f_1 + a_2 f_2$.

Then $(a_1 g_1 + a_2 g_2 - 1)d = 0$, and since d is regular, $a_1 g_1 + a_2 g_2 = 1$ so

$$Z(g_1) \cap Z(g_2) = \emptyset.$$

But from (1) we infer

$$Z_i = Z(f_i) \cap E = Z(g_i) \cap E.$$

Thus Z_1 and Z_2 are contained in the disjoint zero sets $Z(g_1)$ and $Z(g_2)$ of X. This proves (b).

(b) $\Rightarrow$ (a): Every cozero set in any space is z-embedded [**3**, p. 42].

(a) $\Rightarrow$ (e): Assume (a), and let $I \subset C(X)$ be an ideal containing a regular element r. By the above proofs of (a) $\Rightarrow$ (c) $\Rightarrow$ (d), I is order-convex. To show I is an l-ideal, it suffices to show $|f| \in I$ whenever $f \in I$ [**13**, p. 67].

Assume $r \geqq 0$ (by substituting r^2 if necessary), and let

$$Z_1 = \{x \in \text{coz}\, r: f(x) \geqq r(x)\} \text{ and}$$
$$Z_2 = \{x \in \text{coz}\, r: f(x) \leqq -r(x)\}.$$

Then Z_1 and Z_2 are disjoint zero sets of coz r, so there exists $e_0 \in C^*(\text{coz}\, r)$ such that $e_0[Z_1] = 1$, $e_0[Z_2] = -1$, and $|e_0| \leqq 1$ [**13**, p. 17]. Since by (a) coz r is C^*-embedded, $e_0 = e|\text{coz}\, r$ for some $e \in C^*(X)$. Now define for $x \in$ coz r.

$$g_0(x) = \frac{|f(x)| - e(x)f(x)}{r(x)}.$$

Clearly $g_0(x) = 0$ for $x \in Z_1 \cup Z_2$, and for $x \in \text{coz}\, r - (Z_1 \cup Z_2)$, $|f(x)| < r(x)$. So $|g_0(x)| < 2$. Thus $g_0 \in C^*(\text{coz}\, r)$, and by extending g_0

we obtain $g \in C^*(X)$ such that $gr = |f| - ef$, i.e., $|f| = ef + gr \in I$. This proves (e).

(e) $\Rightarrow$ (f): Assume (e), and suppose $I = (f_1, \ldots, f_n)$ is regular. Let $g = |f_1| + \ldots + |f_n|$; then g is regular, and $|f_i| \leqq g$ for $i = 1, \ldots, n$ implies by (e) that $I \subset (g)$. But again by (e), I is an l-ideal, so $|f_i| \in I$ for $i = 1, \ldots, n$. Thus $I = (g)$, proving (f).

(f) $\Rightarrow$ (f′) is obvious.

(a) $\Rightarrow$ (g) is a restatement of Theorem 3.7.

(a) $\Rightarrow$ (h) is obvious from the topological definition of quasi-F spaces, and (h) $\Rightarrow$ (a) is immediate from Lemma 3.1.

This completes the proof of 5.1.

Before stating our next result, we will remind the reader of a definition. Suppose X is a topological space. The members of the σ-field of subsets of X generated by the cozero sets of X are called *Baire sets*.

5.2. COROLLARY. *Consider the following properties of a Tychonoff space X:*

(a) *Every dense Baire set in X is C^*-embedded.*

(b) *X is a quasi-F-space.*

(c) *Every dense Lindelöf subspace of X is C^*-embedded.*

Then (a) *implies* (b), (b) *implies* (c), *and if X is σ-compact then* (a), (b), *and* (c) *are equivalent.*

Proof. Clearly (a) implies (b).

A Lindelöf space is z-embedded in any space containing it by [**4**, p. 79]. So (b) implies (c) by Theorem 5.1(b).

If X is σ-compact, then every one of its Baire sets is a Lindelöf space, [**4**, 9.10], so (c) implies (a). This proves the corollary.

We call a space X *strongly zero-dimensional* if βX has a base for its topology consisting of sets that are closed (and open). In [**16**], L. Heider showed that X is strongly zero-dimensional if and only if each of its zero sets is a countable intersection of open and closed sets. The first part of the following lemma follows immediately from Heider's characterization. The second part is also noted in [**16**] and in [**13**, Chapter 16].

5.3. LEMMA. *Suppose X is strongly zero-dimensional.*

(a) *Every z-embedded subspace of X is strongly zero-dimensional.*

(b) *If Z_1 and Z_2 are disjoint zero-sets of X, then there is an open and closed set U in X such that $Z_1 \subset U$ and $Z_2 \subset X \setminus U$.*

5.4. THEOREM. *Consider the following conditions on a Tychonoff space X:*

(a) *X is a quasi-F-space.*

(b) *If $f \in C(X)$ is regular, then there is a $k \in C(X)$ such that $f = k|f|$.*

(c) *If $f \in C(X)$ is regular, then* pos f *and* neg f *are completely separated.*

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Then (a) *implies* (b), (b) *and* (c) *are equivalent, and if X is strongly zero-dimensional, then* (a), (b), *and* (c) *are equivalent.*

Proof. That (a) implies (b) follows from Theorem 5.1(c). Obviously (b) and (c) are equivalent.

Assume that X is strongly zero-dimensional, that (c) holds, and suppose $V = \operatorname{coz} f$ is a dense cozero set of X. To show that V is C^*-embedded in X, it suffices to show that disjoint zero sets Z_1, Z_2 of V are completely separated in X by [**13**, pp. 17–18]. By Lemma 5.3, V is strongly zero-dimensional, so there is an open and closed subset U of V such that $Z_1 \subset U$ and $Z_2 \subset V \backslash U$. Let $g(x) = |f(x)|$ if $x \in U$, let $g(x) = -|f(x)|$ if $x \in V \backslash U$, and let $g(x) = 0$ if $x \in Z(f)$. It is easy to verify that $g \in C(X)$, $\operatorname{pos} g = U$, and $\operatorname{neg} g = V \backslash U$. It follows from (c) that U and $V \backslash U$ are completely separated in X; a fortiori Z_1 and Z_2 are completely separated in X, and the proof that (c) implies (a) is complete.

Remark. For a compact (strongly) zero-dimensional X, Theorem 5.4 is proved in [**6**, § 1] by a slightly different argument.

The next proposition generalizes a known property of F-spaces [**13**, Chapter 14].

5.5. Proposition. *If X is a quasi-F-space, and $x \in X$ is a G_δ point, then x is not the limit of a distinct sequence in X. In particular, any quasi-F-space satisfying the first axiom of countability is discrete.*

Proof. If x is a non-isolated G_δ point of the quasi-F-space X, then $V = X - \{x\}$, being a dense cozero set, is C^*-embedded in X. By [**13**, 9N.2], such x is not the limit of a distinct sequence in X.

Next we give an example to show that neither the assumption that X is strongly zero-dimensional in Theorem 5.4 nor the assumption of σ-compactness in Corollary 5.2 can be deleted. First we describe a way of constructing certain kinds of topological spaces.

Let D denote an uncountable discrete space and let $\alpha D = D \cup \{\infty\}$ denote its one-point compactification. Suppose Y is a subspace of a Tychonoff space X, let

$$\Delta = \Delta(Y, X) = (X \times \alpha D) \backslash \{(p, q) : p \notin Y \text{ and } q \neq \infty\},$$

and refine the product topology on Δ by letting any point whose second coordinate is not ∞ be isolated. Then Δ is said to be *the space obtained by attaching a copy of αD to each point of Y.*

5.6. *Example.* (Of a Tychonoff space satisfying (c) of Theorem 5.4 that is not a quasi-F-space.)

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Let $\Delta_1 = \Delta([0, 1), [0, 1])$ be the space obtained by attaching a copy of αD to the closed unit interval $[0, 1]$ at each point of $[0, 1)$. If $f \in C(\Delta_1)$ and $f(t, \infty) = 0$ for some $t \in [0, 1)$, then $\operatorname{Int} Z(f) \neq \emptyset$. Hence if g is a regular element of $C(\Delta_1)$, then $Z(g) = \emptyset$ or $Z(g) = \{(1, \infty)\}$. Since $[0, 1] \times \{\infty\}$ is connected, no such g can change sign on this set and hence (c) of Theorem 5.4 holds. But clearly $\Delta_1 \backslash \{(1, \infty)\}$ is a dense cozero set of Δ that fails to be C^*-embedded, so Δ_1 is not a quasi-F-space.

Note also that Δ_1 contains no dense Lindelöf subspace, so the implication (c) implies (b) of Corollary 5.2 need not hold if the hypothesis of σ-compactness is deleted.

Recall from [**13**, Chapter 14] that a Tychonoff space X is called a *P-space* if every zeroset of X is open and from [**19**], that X is called an *almost-P-space* if each of its zerosets has a nonempty interior. Clearly every almost-P-space is a quasi-F-space. If X is any noncompact realcompact space, then $\Delta(X, \beta X)$ is a quasi-F-space that is not an almost-P-space. A space with this latter property is called a *proper-quasi-F-space.*

We note in passing that X is an almost-P-space if and only if each of its Baire sets has nonempty interior (since every Baire set is a union of zero sets). Similarly, X is a P-space if and only if each of its Baire sets is open (hence also closed).

A closed subspace of a quasi-F-space need not be a quasi-F-space. In fact, we have:

5.7. Proposition. *Every Tychonoff space X is homeomorphic to a closed subspace of an almost-P-space.*

Proof. Clearly X is homeomorphic to a closed subspace of the almost-P-space $\Delta(X, X)$.

X is called an *F'-space* if for every $f \in C(X)$, pos f and neg f have disjoint closures. Every normal F'-space is an F-space, but there are F'-spaces that are not F-spaces [**12**, 8.14] and [**5**]. In [**5**, Theorem 1.1] it is shown that X is an F'-space if and only if every cozero set in X is C^*-embedded in its closure. So, every F'-space is a quasi-F-space. Indeed, we have:

5.8. Proposition. *Consider the following properties of a Tychonoff space X:*

(a) *X is an F'-space.*

(b) *The closure of any cozero set of X is a quasi-F-space.*

(c) *X is an F-space.*

(d) *Every closed subset of X is a quasi-F space.*

Properties (a) *and* (b) *are equivalent. If X is normal, then* (a), (b), (c), *and* (d) *are equivalent.*

Proof. Suppose (a) holds, $f \in C(X)$, let $Y = \mathrm{Cl}(\mathrm{coz}\, f)$, and suppose $\mathrm{coz}\, \phi$ is a dense cozero set of Y. Let $g(x) = f(x)\phi(x)$ if $x \in Y$, and let $g(x) = 0$ if $x \in X \backslash Y$. Clearly $g \in C(X)$, and $\mathrm{coz}\, g = (\mathrm{coz}\, f) \cap (\mathrm{coz}\, \phi)$ is dense in Y. By the result in [**5**] cited above, $\mathrm{coz}\, g$ is C^*-embedded in Y. Since $\mathrm{coz}\, g$ is a dense subspace of $\mathrm{coz}\, \phi$, the latter is also C^*-embedded in Y. Thus (a) implies (b).

Suppose that (b) holds and $f \in C(X)$. By assumption $Y = \mathrm{Cl}(\mathrm{coz}\, f)$ is a quasi-F-space, so the function ϕ such that $\phi[\mathrm{pos}\, f] = 1$ and $\phi[\mathrm{neg}\, f] = -1$ has an extension $F \in C^*(Y)$. Hence $\mathrm{Cl}(\mathrm{pos}\, f)$ and $\mathrm{Cl}(\mathrm{neg}\, f)$ are disjoint, and (a) holds. This establishes the equivalence of (a) and (b).

The second assertion is an easy consequence of the fact that every normal F'-space or C^*-embedded subspace of an F-space is an F-space [**13**, 14.26]. This completes the proof of Proposition 5.8.

In [**18**, Example 3], Carl Kohls gives an example of an (extremally disconnected) F-space X with a closed subspace Y that is not an F'-space. We will describe it next and show that Y is not even a quasi-F-space.

5.9. *Example.* (Of an extremally disconnected space X with a closed subspace Y that is not a quasi-F-space.)

Let $\Pi = N \cup D$ denote the subspace of the Stone-Čech compactification βN of the countable discrete space N, comprised of N together with the discrete subspace of $\beta N \backslash N$ of power c constructed in [**13**, 6Q]. Since D is not C^*-embedded in Π, there is a $p \in \beta N \backslash N$ that is in the closure of two disjoint zero sets of D. Let $X = \Pi \cup \{p\}$. As is noted by Kohls, X is extremally disconnected and D is a (dense) cozero set of $Y = D \cup \{p\}$. Hence Y is not a quasi-F-space.

We do not know if there is a Tychonoff space that is not an F'-space such that every closed C^*-embedded subspace is a quasi-F-space.

Next we consider conditions under which the property of being a quasi-F-space is preserved under finite products.

5.10. PROPOSITION. *Suppose X_1 and X_2 are Tychonoff spaces:*

(a) *$X_1 \times X_2$ is an almost-P-space if and only if both X_1 and X_2 are almost-P-spaces.*

(b) *If $X_1 \times X_2$ is a quasi-F-space, then so are X_1 and X_2.*

(c) *If X_1 and X_2 are strongly zero-dimensional and $X_1 \times X_2$ is a quasi-F-space, then X_1 or X_2 is an almost P-space.*

(d) *If $X_1 \times X_2$ is a quasi-F-space and X_2 is a compact proper quasi-F-space, then X_1 is a P-space.*

Proof. If $X_1 \times X_2$ is an almost-P-space, then so are X_1 and X_2 since projection maps are open. Suppose, conversely, that both X_1 and X_2 are almost-P-spaces, $\phi \in C(X_1 \times X_2)$, let

$$\Sigma_n = \{(x_1, x_2) \in X_1 \times X_2 : |\phi(x_1, x_2)| < 1/n\} \text{ for } n = 1, 2, \ldots, \text{ and}$$

$$p = (p_1, p_2) \in Z(\phi) = \bigcap_{n=1}^{\infty} \Sigma_n.$$

Choose zerosets $U_n^{(i)}$ in X_i for $i = 1, 2$, and $n = 1, 2, \ldots$, such that $p \in U_n^{(1)} \times U_n^{(2)} \subset \Sigma_n$, and let

$$U^{(i)} = \bigcap_{n=1}^{\infty} U_n^{(i)}.$$

Since both X_1 and X_2 are almost-P-spaces, Int $U^{(i)}$ is nonempty for $i = 1, 2$, and $Z(\phi)$ contains the nonempty open set Int $U^{(1)} \times$ Int $U^{(2)}$. Thus $X_1 \times X_2$ is an almost-P-space and (a) is established.

The proof of (b) is an exercise.

Assume the hypotheses of (c) and for $i = 1, 2$, let Z_i denote a nonempty zeroset of X_i with empty interior.

Since X_i is strongly zero-dimensional, Z_i is a countable intersection of sets that are both open and closed [**16**], and there is an $f_1 \in C(X_1)$ whose image is $\{1/(2n + 1) : n = 1, 2, \ldots\} \cup \{0\}$, and an $f_2 \in C(X_2)$ whose image is $\{1/2n : n = 1, 2, \ldots\} \cup \{0\}$. Define $\phi \in C(X_1 \times X_2)$ by letting

$$\phi(x_1, x_2) = f_1(x_1) - f_2(x_2).$$

Then ϕ changes sign on every neighborhood of any point of $Z_1 \times Z_2$, and $Z(\phi) = Z_1 \times Z_2$ has empty interior. Thus $X_1 \times X_2$ is not a quasi-F-space by Theorem 5.4, so (c) holds.

Assume the hypothesis of (d) and suppose $T = \operatorname{coz} f$ is a proper dense cozero set of X_2. Then $X_1 \times T$ is a dense cozero set of $X_1 \times X_2 = X_1 \times \beta T$, and so $X_1 \times T$ is C^*-embedded in $X_1 \times \beta T$. By a result given in [**25**, 8.11], X_1 is a P-space.

5.11. Corollary. *The product of two infinite compact spaces is never a proper quasi-F-space.*

Proof. Let X_1 and X_2 be infinite and compact with $X_1 \times X_2$ a quasi-F-space. Then X_1 and X_2 are quasi-F-spaces by 5.10(b), and if one of them is a proper quasi-F-space the other must be a P-space by 5.10(d). But all compact P-spaces are finite [**13**, 4K.3], so X_1 and X_2 are both almost-P-spaces. By 5.10(a), $X_1 \times X_2$ is an almost-P-space.

Since $X = \beta N \backslash N$ is an almost-P-space, so is $X \times X$ by Proposition 5.10(a). Hence the adjective "proper" cannot be deleted in the statement of Corollary 5.11.

We do not know if the requirements that X_1 and X_2 be strongly zero-dimensional in the statement of Proposition 5.10(c), or the requirement that X_2 be compact in the statement of Proposition 5.10(d) are necessary.

In [**22**, Theorem 6.5], S. Negrepontis shows that X is a P-space if and only if $X \times \beta X$ is an F-space. An analog of this result follows.

5.12. COROLLARY. *For any Tychonoff space X, $X \times \beta X$ is a quasi-F-space if and only if X is a P-space or βX is an almost-P-space.*

Proof. If X is a P-space, then $X \times \beta X$ is an F-space by the theorem of Negrepontis cited above. If βX is an almost-P-space, so is X. Thus $X \times \beta X$ is an almost-P-space by Proposition 5.10(a).

Conversely, if $X \times \beta X$ is a quasi-F-space, then so is βX by Proposition 5.10(b). If βX is a proper quasi-F-space, then X is a P-space by Proposition 5.10(c). Otherwise both βX and X are almost-P-spaces, so the necessity holds as well.

The argument given in the next example modifies the one given by L. Gillman in [**10**] to show that the product of a P-space and an F-space need not be an F-space.

5.13. *Example.* (Of a P-space X_1 and an extremally disconnected space X_2 such that $X_1 \times X_2$ is not a quasi-F-space.)

Suppose D is a discrete space of power $\aleph_1$ and $p_1 \notin D$. Let $X_1 = D \cup \{p_i\}$ be topologized so that each point of D is open and deleted neighborhoods of p_1 are co-countable subsets of D. Choose a point $p_2 \in \beta D \backslash D$ such that every neighborhood in βD of p_2 meets D in an uncountable set, and let $X_2 = D \cup \{p_2\}$ carry the relative topology of βD. As is noted in [**10**], X_1 is a P-space, X_2 is extremally disconnected, and there is a $g \geqq 0$ in $C^*(X_2)$ such that coz $g = D$. Then $X_1 \times D$ is a dense cozero set in $X_1 \times X_2$. Define $h: X_1 \times D \rightarrow \mathbf{R}$ by letting

$$h(x_1, x_2) = \begin{cases} g(x_2) \text{ if } x_1 = x_2 \in D \\ -g(x_2) \text{ otherwise.} \end{cases}$$

Clearly $h \in C^*(X_1 \times D)$, so if $X_1 \times X_2$ is a quasi-F-space, then h has a continuous extension $\bar{h}$ over $X_1 \times X_2$. By Theorem 5.1(c), there is a $k \in C(X_1 \times X_2)$ such that

$$k[\text{pos } \bar{h}] = 1 \text{ and } k[\text{neg } \bar{h}] = -1.$$

But $h > 0$ on the diagonal of $D \times D$, $h < 0$ on $\{(p_1, x_2): x_2 \in D\}$, and every neighborhood of (p_1, p_2) meets both of these sets, so k cannot be continuous at (p_1, p_2). This contradiction shows that $X_1 \times X_2$ is not a quasi-F-space.

The problem of determining exactly when a product of two spaces is a quasi-F-space seems to be at least as complicated as the corresponding one for F-spaces. See [**5**].

Added in proof (May 1, 1980). The problem of an intrinsic characterization of the quasi-F-cover $K(X)$ mentioned in Section 4.4 has been solved by F. Dashiell. A paper is in preparation.

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