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Factor Model Comparisons with Conditioning Information

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Abstract

We develop methods for testing factor models when the weights in portfolios of factors and test assets can vary with lagged information. We derive and evaluate consistent standard errors and finite sample bias adjustments for unconditional maximum squared Sharpe ratios and their differences. Bias adjustment using a second-order approximation performs well. We derive optimal zero-beta rates for models with dynamically trading portfolios. Factor models' Sharpe ratios are larger but standard test asset portfolios' maximum Sharpe ratios are larger still when there is dynamic trading. As a result, most of the popular factor models are rejected.

I. Introduction

Classical tests of asset pricing models examine portfolio efficiency by comparing squared Sharpe ratios (e.g., Sharpe (1988)), the ratios of expected excess return to standard deviation. The tests of Gibbons, Ross, and Shanken (GRS (1989)) compare the maximum squared Sharpe ratio, $S^2(r,f)$, of a portfolio formed from test assets r and factors f, to that of the portfolio of factors, $S^2(f)$. The difference between $S^2(r,f)$ and $S^2(f)$ is a quadratic form in the factor model's alphas for the test assets. If the two Sharpe ratios are equal, the test assets have 0 alphas in the factor model and the factor portfolio is mean–variance efficient. Dividing the difference of the squared Sharpe ratios by $[1 + S^2(f)]$ leads to a test statistic with an exact F-distribution when normality is assumed.

Barillas and Shanken (BS (2017)) further point out that when comparing two models' factors the model with the higher $S^2(f)$ produces smaller pricing errors for a

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given set of test asset returns. Barillas, Kan, Robotti, and Shanken (BKRS (2020)) provide asymptotic standard errors for inference about squared Sharpe ratios and their differences. Kan, Wang, and Zheng (2019) provide exact and asymptotic standard errors for out-of-sample Sharpe ratios. Fama and French (2018) present an empirical evaluation of alternative factor models using squared Sharpe ratios.

The previous articles restrict their analyses in two important dimensions. First, the portfolios of the factors and test assets whose Sharpe ratios are compared must have fixed weights over time. This is not realistic, as most portfolio weights vary over time. Second, these studies measure returns in excess of a short-term Treasury bill rate, pinning the zero-beta rate to the average bill rate. This is not realistic either, and many studies find zero-beta rates larger than the average bill rate. This article removes both of these unrealistic restrictions.

We allow stylized "dynamic trading" in a discrete-time model, using monthly returns and lagged instruments. As the lagged instruments vary over the months the portfolio weights change month-to-month as functions of the lagged information variables. We derive estimates of an optimal zero-beta rate for squared Sharpe ratio differences in this setting. We derive asymptotic standard errors, recognizing that the zero-beta rate is estimated with sampling error.

We develop a simple approach to testing factor models, with or without dynamic trading. Maximized Sharpe ratios are biased in finite samples, so we provide methods for bias adjustment when there is dynamic trading. Bias-adjusted squared Sharpe ratios or their differences, divided by asymptotic standard errors, form a "t-ratio." The absolute t-ratio is asymptotically distributed as a $\chi(1)$ variable and simulations show that it may be evaluated using standard rules of thumb (e.g., approximate significance at the 5% level if the t-ratio is larger than 2).

We illustrate applications with tests on the capital asset pricing model (CAPM; Sharpe (1964)), the Fama and French (1996) 3-factor model (FF3), the Fama and French (2016) 5-factor model (FF5), a 6-factor model that appends a momentum factor (FF6), the 4-factor investment model (Q4) of Hou, Xue, and Zhang (2015), and a 5-factor extension (Q5) from Hou, Mo, Xue, and Zhang (2021) that adds an expected growth factor. In the Supplementary Material, we provide results for mimicking portfolios when factors are not traded, and evidence for the Chen, Roll, and Ross (1986) nontraded factors, consumption growth and a broker-dealer leverage factor from Adrian, Ettula, and Muir (2014).

We find that dynamic trading improves the maximum Sharpe ratios of portfolios of the models' factors. The Q4 and Q5 model factors are the least affected. The larger impact of dynamic trading is to jack up the maximum Sharpe ratios of the most popular portfolio designs (courtesy of Kenneth French), when treated as the test assets. All of the factor models listed previously are rejected in these portfolios, implying that no portfolio of the model factors is minimum-variance efficient, even with dynamic trading. Direct comparisons reveal that the Q4 model outperforms the FF5 model with or without dynamic trading, and Q5 beats FF6. The evidence for pricing improvement by adding a momentum factor to the FF3 model is strong, but the evidence for adding momentum to the FF5 model appears weak.

When the zero-beta rate is estimated using the factor models and the standard test portfolios, its value is larger than the historical average of a U.S. Treasury bill return. The CAPM produces a larger zero-beta rate than the other factor models. The zero-beta rate can reflect borrowing costs or the premiums of missing factors in the models. The results are consistent with missing factors in the CAPM.

The rest of the article is organized as follows: Section II presents an overview of Sharpe ratio comparisons. Section III develops our asymptotic results and bias correction for squared Sharpe ratios. The data are described in Section IV. Section V presents simulation results. Section VI presents empirical tests and comparisons of factor pricing models. Section VII concludes the article. An appendix provides additional material, and the Supplementary Material presents the longer proofs and more.

II. Testing Asset Pricing Models with Squared Sharpe Ratios

Asset pricing models identify a portfolio that is minimum-variance efficient.[1] In a multiple-factor model, this is a combination of the model's K traded factors, f, or their mimicking portfolios. Classical tests reject the model if the factor portfolio lies significantly inside a sample minimum variance boundary of factors and test assets (see Gibbons (1982), Stambaugh (1982), MacKinlay (1987), or GRS (1989).)

Let the N test assets' excess returns be r. Classical tests compare a maximum squared Sharpe ratio formed from the test assets and the factors, $S^2(r,f)$, to one using only the factors, $S^2(f)$. If the two squared Sharpe ratios are equal, the factor portfolio is efficient and the vector of the f-model's average pricing errors or alphas (α) for the test assets is 0. The alphas are the regression intercepts when the test assets' excess returns, r, are regressed over time on those of the benchmark factors, f.

The difference between $S^2(r,f)$ and $S^2(f)$ is the squared Sharpe ratio of the optimal orthogonal portfolio (OOP), a quadratic form in the f-model's alphas for the test assets:

$$\alpha' \Sigma(u)^{-1} \alpha = S^2(r,f) - S^2(f), \qquad (1)$$

where $\Sigma(u)$ is the covariance matrix of the residuals of the test asset excess returns, r, regressed over time, on the f-model's factors. The OOP has the maximum mispricing using the f-model's factors,[2] equation (1) is called a "law of conservation of squared Sharpe ratios" by Ferson (2019). The maximum squared Sharpe ratio in all of the assets, $S^2(r,f)$ is equal to that in the tested factor portfolios, $S^2(f)$, plus that of the OOP.

A quadratic form in the alphas using the covariance matrix of the alphas in place of $\Sigma(u)$ is equal to $[S^2(r,f) - S^2(f)]/[1 + S^2(f)]$. Multiplying by a degrees-of-freedom adjustment $[(T - N - K)/N]$ produces the exact F test of GRS (1989). This

[1]The capital asset pricing model (CAPM; Sharpe (1964)) implies that the market portfolio should be mean–variance efficient. Multiple-beta asset pricing models such as Merton (1973) imply that a combination of the factor portfolios is minimum-variance efficient (Chamberlain (1983), Grinblatt and Titman (1987)). The consumption CAPM implies that a maximum correlation portfolio for consumption growth is minimum-variance efficient (Breeden (1979)). A stochastic discount factor model implies that a maximum correlation portfolio for the stochastic discount factor is minimum-variance efficient (Hansen and Richard (1987)).

[2]The OOP maximizes its squared alpha divided by its residual variance, thus Arg $\text{Min}_x\, x'\Sigma(u)x$ for a given $x'\alpha$, with solution $x = (\lambda/2)\Sigma(u)^{-1}\alpha$, where λ is the Lagrange multiplier. The maximized squared Sharpe ratio is $(x'\alpha)^2/x'\Sigma(u)x = \alpha'\Sigma(u)^{-1}\alpha$.

article shows how to conduct tests for factor-model efficiency that accommodate dynamic trading.

BS (2017) consider the comparison of models with different factors, say f_1 and f_2, which might overlap. A model is better when the quadratic form in the test assets' alphas using its factors is smaller. For example, model 1 is better than model 2 if $S^2(r,f_1,f_2) - S^2(f_1) < S^2(r,f_1,f_2) - S^2(f_2)$. Equivalently, model 1 is better if the maximum squared Sharpe ratio of its factors is larger: $S^2(f_1) > S^2(f_2)$. This article shows how to conduct such direct factor model comparisons in terms of squared Sharpe ratios when there is dynamic trading.

Tests with Dynamic Trading

Consider a stochastic discount factor model:

$$\text{(2)} \qquad \mathrm{E}(m_{t+1}R_{t+1}|Z_t) = \mathbf{1},$$

where R represents the gross (one plus the rate of) return on the test assets, m is the stochastic discount factor (SDF), Z_t is the conditioning information, and $\mathbf{1}$ is an n-vector of ones. Linear factor models assume that the SDF is linear in the factors. When the SDF is linear in the factors, a linear combination of the factor portfolios is minimum variance efficient (Ferson (1995), Cochrane (1996)). This motivates tests comparing the squared Sharpe ratios of factor portfolios with those of test assets. This article shows how to conduct such tests when there is dynamic trading. The conditioning information Z_t is measured using lagged instruments, and portfolio weights may vary as a function of these instruments.[3] We use monthly data so the weights can vary monthly.

In principle, one could use conditional moments for the tests, as in early studies of conditional asset pricing (e.g., Hansen and Hodrick (1983), Gibbons and Ferson (1985), and Harvey (1989)). These tests impose strong restrictions on the forms of the conditional first and second moments of the factors and test asset returns. Instead, we follow Ferson and Siegel (2009), who derive implications of equation (2) for the *unconditional* moments of dynamic trading portfolios. Thus, the tests are based on unconditional squared Sharpe ratios, and the model comparison logic of BS (2017) applies using unconditional squared Sharpe ratios. As shown by Bekaert and Liu (2004) and Ferson and Siegel (2003), such tests are inherently robust to misspecification of the conditional moments used in their construction.

Ferson and Siegel (2009) use an implication of equation (2) for unconditional moments:

$$\text{(3)} \qquad E[m_{t+1}w'(Z_t)R_{t+1}] = 1 \quad \forall w(Z_t) : w'(Z_t)\mathbf{1} = 1.$$

As an implication of equation (2), equation (3) is less general, but not by much.[4]

[3]The chosen Z is not likely to be the full information set used by market participants, but equation (2) may be arrived at by applying iterated expectations to a version of the equation conditioned on a finer public information set. We must restrict ourselves to models that are testable, in the sense that m_{t+1} depends only on the observable data and parameters that we can estimate.

[4]Equation (2) is equivalent to the unconditional expectation $E\{m_{t+1}R_{t+1}f(Z_t)\} = E\{1\,f(Z_t)\}$ holding for all bounded integral functions, $f(.)$. Equation (3) restricts to portfolio weight functions that sum to 1.0.

The tests examine unconditional efficiency (UE), which we now define. A set of dynamically trading portfolio returns determines a mean–standard deviation frontier, as shown by Hansen and Richard (1987), depicting the unconditional means versus the unconditional standard deviations. A portfolio R_p is defined to be UE with respect to the information Z_t when it is on this frontier. Thus, equation (4) is satisfied (equivalently, there exists constants γ_0 and γ_1 such that equation (5) is satisfied) for all $x(Z_t)$ such that the weights sum to 1 almost surely and the relevant unconditional moments exist and are finite:

(4) $$\text{var}\left(R_{p,t+1}\right) \leq \text{var}[x'(Z_t)R_{t+1}] \text{ if } E\left(R_{p,t+1}\right) = E[x'(Z_t)R_{t+1}],$$

(5) $$E[x'(Z_t)R_{t+1}] = \gamma_0 + \gamma_1 Cov\left[x'(Z_t)R_{t+1}, R_{p,t+1}\right].$$

Equation (4) states that $R_{p,t+1}$ is on the minimum variance boundary. Equation (5), shown by Hansen and Richard (1987), states that the expected return–covariance relation from Fama (1973) and Roll (1977) holds for the unconditional moments of dynamic portfolios. In equation (5), γ_0 and γ_1 are fixed scalars that do not depend on the functions $x(\cdot)$ or the realizations of Z_t.

UE portfolios have proven useful in asset-pricing tests, in forming hedging portfolios and other portfolio management problems (e.g., Abhyankar, Basu, and Stremme (2012), Chiang (2015), and Ehsani and Linnainmaa (2020)). Siegel (2021) provides a review. We develop asymptotic standard errors and bias correction for the squared Sharpe ratios of UE portfolios.

Ferson and Siegel (2009) show that, given an SDF such that equation (3) is satisfied, a portfolio with maximum correlation to m with respect to Z must be UE. The maximum correlation is among all portfolios of the test assets that may trade dynamically using the information, Z.[5] This result provides the foundation for Sharpe ratio difference tests with dynamic trading. To implement the tests with dynamic trading, we use closed-form solutions for UE portfolio weights, which exist for three relevant cases.[6]

Ferson and Siegel (2015) generalize the law of conservation of squared Sharpe ratios (1) for dynamic trading. The dynamic OOP trades as a function of the lagged

Thus, e.g., with equation (3) it is not possible to expand the scale of the risky investments without borrowing or lending at the risk-free rate.

[5]Formally, a portfolio R_P is maximum correlation for a random variable, m, with respect to Z, if:

$$\rho^2(R_P, m) \geq \rho^2[w'(Z)R, m] \;\; \forall w(Z) : w'(Z)\mathbf{1} = 1,$$

where $\rho^2(.,.)$ is the squared unconditional correlation coefficient and we restrict to functions $w(.)$ for which the correlation exists. Ferson, Siegel, and Xu (2006) present closed-form solutions for maximum correlation portfolios with respect to Z.

[6]In the first case, there is a fixed risk-free rate, and in the second case, there is no risk-free asset. Solutions are provided by Ferson and Siegel (2001). In the third case, there is a conditionally time-varying risk-free asset whose return $R_{ft+1} = R_f(Z_t)$ is measureable and known as part of the information set Z_t so that $\text{var}\{R_{ft+1}|Z_t\} = 0$, but which is unconditionally risky in the sense that $\text{var}\{R_{ft+1}\} > 0$. The solution is provided by Ferson and Siegel (2015) and Penaranda (2016). The same solution can be obtained using the expression from Ferson and Siegel (2001) for no risk-free asset, applied to the expanded return vector (R_{t+1}, R_{ft+1}) (see Section 3 of the Supplementary Material). Penaranda (2016) also describes residual efficiency, where the expected conditional variance is minimized for a given unconditional mean return.

information to maximize the square of its unconditional alpha on a benchmark factor model divided by its unconditional variance of return.[7] A version of equation (1) holds, with the unconditional squared Sharpe ratio of the dynamic OOP on the left-hand side. The maximum unconditional squared Sharpe ratio of portfolios of the test assets and the factors that trade with the information, $S_{UE}{}^2(r,f)$, replaces $S^2(r,f)$.

The maximum squared Sharpe ratio of the factor benchmark with dynamic trading depends on the model. With dynamic trading using lagged instruments Z, there are three ways to specify a linear factor model for the SDF (suppressing the time subscripts):

(i) $m = a + b'f$, with fixed (a,b),
(ii) $m = a + b(Z)'f$, with fixed a, and
(iii) $m = a(Z) + b(Z)'f$.

In cases (ii) and (iii), the SDF is conditionally linear in the factors. There can be an error term uncorrelated with the test assets in each of these equations for m.

Case (i). If $m = a + b'f$ with fixed (a,b) parameters then $b'f$ is maximum correlation to m and therefore is UE in the test asset returns r. Asking the model to price the excess returns of its own factors, $E(mf) = 0$ identifies $b = -a\,E(ff')^{-1}E(f)$ as a fixed minimum variance efficient portfolio weight for the factors. This motivates testing the hypothesis that $S^2_{\text{fix}}(f) = S^2_{\text{UE}}(r)$.

Case (ii). The portfolio $b(Z)'f$ is maximum correlation to m when a is a constant, and therefore it must be UE. $E(mf|Z) = 0$ identifies $b(Z) = -a\,E(ff'|Z)^{-1}\,E(f|Z)$ as UE weights for the factors. This motivates testing the hypothesis that $S^2_{\text{UE}}(f) = S^2_{\text{UE}}(r)$.

Case (iii). We call this a "fully conditional model." The maximum correlation portfolio to $a(Z) + b(Z)'f$, with respect to Z, should be UE in r. $E(mf|Z) = 0$ identifies $b(Z) = -a(Z)\,E(ff'|Z)^{-1}\,E(f|Z)$ and $b(Z)'f$ is conditionally minimum variance in f but need not be UE. We can find $a(Z)$ assuming a conditionally risk-free rate that is known given Z_t from $E\{mR_{ft}|Z_t\} = 1$, as $a(Z) = (1/R_{ft})/[1 - E(f|Z)'E(ff'|Z)^{-1}\,E(f|Z)]$. Using this to construct the maximum correlation portfolio to m, we can test the hypothesis that is UE by comparing squared Sharpe ratios.

III. Main Results

A. Asymptotic Variances

The asymptotic variance of an estimator $\hat{\theta}$, that depends on the conditional mean vector and covariance matrix estimators, such as a squared Sharpe ratio, may be characterized in terms of two canonical matrices C $(L \times N)$ and D $(N \times N)$ that capture the sensitivity of the estimator to uncertainty in the conditional means and in

[7]There may be time-varying conditional alphas in this setting, but an unconditional regression is used to define the unconditional alphas.

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the conditional covariance respectively. We first present a general theorem, and then derive special cases. The proofs are provided in the Supplementary Material.

Our analysis is based on the following assumptions. The conditional expected returns are described by a linear regression:

(6) $$R_t = \delta' Z_{t-1} + \varepsilon_t.$$

The return vector for N assets at time t is R_t and the L lagged instruments (including a constant) are Z_{t-1} for $t = 1, \ldots, T$. The coefficient δ is an $L \times N$ matrix of regression coefficients. The conditional mean returns at time t are $\delta' Z_{t-1}$. The regression errors ε_t are independent and identically distributed (IID) $N \times 1$ vectors with mean zero and nonsingular conditional covariance matrix $V = E\left[(R_t - \delta' Z_{t-1})(R_t - \delta' Z_{t-1})' \middle| Z_{t-1}\right] = E\left(\varepsilon_t \varepsilon_t' \middle| Z_{t-1}\right)$.

We estimate the OLS regression (6) and express

(7) $$\hat{\delta} - \delta = A^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} Z_{t-1}\varepsilon_t'\right) = O_p\left(1/\sqrt{T}\right),$$

where the $L \times L$ matrix $A = E(Z'Z/T)$ is assumed to be nonsingular. We define the estimated covariance matrix as $\hat{V} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(R_t - \hat{\delta}' Z_{t-1}\right)\left(R_t - \hat{\delta}' Z_{t-1}\right)' = \frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_t'$, using the estimated residuals, $\hat{\varepsilon}_t \equiv R_t - \hat{\mu}_t = R_t - \hat{\delta}' Z_{t-1}$.

Theorem 1. Consider a scalar estimator of the form:

$\hat{\theta} = \theta + \sum_{t=1}^{T} C_t'(\hat{\mu}_t - \mu_t) + tr\left[D\left(\hat{V} - V\right)\right] + O_p(1/T)$, where $C_1, \ldots, C_T$ are $N \times 1$ vectors that depend on $\{Z_t\}$ and D is a fixed $N \times N$ matrix. Defining the $L \times N$ matrix $C \equiv A^{-1}\sum_{t=1}^{T} Z_{t-1}C_t'$, where $A = E(Z'Z/T)$. The unconditional asymptotic variance of $\hat{\theta}$ is estimated as

(8) $$AVAR\left(\hat{\theta}\right) \cong tr\left(\hat{C}\hat{V}\hat{C}'A\right) - \left[tr\left(\hat{D}\hat{V}\right)\right]^2 + \frac{1}{T}\sum_{t=1}^{T}\left(2\bar{Z}'\hat{C}\hat{\varepsilon}_t + \hat{\varepsilon}'_t\hat{D}\hat{\varepsilon}_t\right)\left(\hat{\varepsilon}'_t\hat{D}\hat{\varepsilon}_t\right),$$

where $\hat{C}$ and $\hat{D}$ are consistent estimates for the canonical matrices C and D, respectively, and $\bar{Z} \equiv \frac{1}{T}\sum_{t=1}^{T} Z_{t-1}$. Let $\psi_t = Z_{t-1}' C\varepsilon_t + \varepsilon_t' D\varepsilon_t$. The unconditional variance of $\hat{\theta}$ may be estimated as the sample variance of the ψ_t:

(9) $$\sqrt{T}\left\{(1/T)\Sigma_t\psi_t^2 - [(1/T)\Sigma_t\psi_t]^2\right\},$$

where consistent estimates of C and D can replace the true parameter values in ψ_t.

Squared Sharpe ratios are a natural special case of Theorem 1, as their sampling errors depend on the sampling errors in a vector of mean returns and a covariance matrix. The asymptotic variance of a function of two estimators may be found by using estimates $\hat{C}$ and $\hat{D}$ of the canonical matrices associated with each estimator. For example, the asymptotic variance of the difference between two squared Sharpe ratios, $\hat{\theta} = \hat{S}^2(r)$ and $\hat{\theta}^* = \hat{S}^2(r_p)$ has canonical matrices equal to the differences in the two canonical matrices, $\hat{C} - C^*$ *and* $\hat{D} - D^*$. Using the differences allows for dependence. For example, the $(C - C^*)'V(C - C^*)A$ in the first term of (8) accounts for the covariance due to common dependence on the estimation error in the conditional means. The third term of (8) involves higher comoments. The asymptotic variance of the ratio $\hat{\theta}/\hat{\theta}^*$ has canonical matrices $\frac{\hat{\theta}^*\hat{C}-\hat{\theta}\hat{C}^*}{\hat{\theta}^{*2}}$ and $\frac{\hat{\theta}^*\hat{D}-\hat{\theta}\hat{D}^*}{\hat{\theta}^{*2}}$. The asymptotic variance of $\frac{\hat{\theta}-\hat{\theta}^*}{1+\hat{\theta}^*}$ has canonical matrices $[(1+\hat{\theta}^*)\hat{C}-(1+\hat{\theta})\hat{C}^*]/(1+\hat{\theta}^*)^2$ and $[(1+\hat{\theta}^*)\hat{D}-(1+\hat{\theta})\hat{D}^*]/(1+\hat{\theta}^*)^2$.

Our main results assume a constant covariance matrix, V, but stock returns are known to be conditionally heteroscedastic through time. The Supplementary Material presents an extension of the Theorem 1 that allows for heteroscedasticity. But the results will depend on the particular model used for the time-varying covariances, and we leave the full development of this case for future research.

Tests such as case (i) above require comparing Sharpe ratios for fixed-weight portfolios with those that use the information. Theorem 1 can be applied to such cases, as described in the Appendix as Corollary A1.

B. Distributions of the Test Statistics

Our suggestion is to test models with a "*t*-ratio," dividing the bias-adjusted squared Sharpe ratio or difference by its asymptotic standard error. We first argue that the estimated squared Sharpe ratios are consistent and asymptotically normal. The argument uses the generalized method of moments (Hansen (1982)), similar to BKRS (2019) for the fixed-weight case.

Define the moment conditions $g_{1t} = Vec([r_t - \delta' Z_{t-1}]Z_{t-1}')$ and $g_{2t} = Vech(\varepsilon_t\varepsilon_t' - V)$. Stack them into $g_t = (g_{1t}', g_{2t}')'$ and let $g = (1/T)\Sigma_t g_t$. For any quadratic form $E(g)'W\,E(g)$, the GMM estimates for the parameters $\varphi = (Vec(\delta)', Vech(V)')'$ are found by setting $E(g) = 0$, as the problem is exactly identified. The GMM estimates are the OLS estimates, and they are the same when a consistent estimate for the asymptotic covariance matrix of g is used for the weighting matrix, W. Hansen's (1982) Theorems 2.1 and 3.1 show that the estimates of φ are consistent and asymptotically normal on the assumptions that $\{g_t\}$ is strictly stationary and ergodic, g has continuous derivatives with respect to the parameters in a neighborhood of the true values, and the parameters lie in a compact set. Given these results we argue that the estimated squared Sharpe ratios less their true values are asymptotically normal. We can express the squared Sharpe ratio or difference as a function $\theta = \theta(\delta, V)$ and take the first-order derivatives to find $Avar(\theta) \approx (\partial\theta/\partial\varphi)\ Avar(\varphi)\ (\partial\theta/\partial\varphi)'$. Our Theorem 1 presents asymptotically equivalent expressions for standard errors.

A quadratic form in the estimated squared Sharpe ratio differences, using a consistent estimator of their variance as the inverse matrix in the form, is asymptotically distributed as a chi-squared random variable. Since the squared Sharpe ratio or difference is a scalar, the square of the *t*-ratio that we propose is asymptotically distributed as $\chi^2(1)$. The absolute value of the *t*-ratio is therefore asymptotically distributed as a $\chi(1)$, or half-normal random variable. In most applications positive Sharpe ratio differences are examined, so our simulations evaluate the absolute *t*-ratios relative to a chi distribution with one degree of freedom. The critical values of this density are similar to the standard rules of thumb used with *t*-ratios. For example, the critical value for 5% significance is 1.96 and for 10% significance it is 2.33.

We use *T*-consistent estimates for a finite number of assets, *N*. It is therefore important to understand how the estimators perform when confronted in practice with relatively large values of *N*. Factor model comparisons are typically conducted using portfolios, so we present simulations using up to 99 portfolios.

C. Asymptotic Variances for UE Portfolio Squared Sharpe Ratios

We start with results from Ferson and Siegel ((2001), Theorem 3), who derive the portfolio weights of the UE portfolio when there is no risk-free asset. The portfolio has target unconditional mean μ_p and the weight is

(10) $$w_t' \equiv \frac{\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{\mu_p - \alpha_2}{\alpha_3}\mu_t'\left(\Lambda_t - \frac{\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}\right),$$

where: $\Lambda_t \equiv \left(\mu_t\mu_t' + V\right)^{-1}$, and the efficient set constants α_1, α_2, and α_3 are defined in Ferson and Siegel (2001) and in our Supplementary Material. The portfolio unconditional variance is $\sigma_p^2 = \left(\alpha_1 + \frac{\alpha_2^2}{\alpha_3}\right) - \frac{2\alpha_2}{\alpha_3}\mu_p + \frac{1-\alpha_3}{\alpha_3}\mu_p^2$. The estimators based on $\hat{\mu}_t$ and $\hat{V}$ are denoted $\hat{\Lambda}_t$, $\hat{w}_t$, $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\alpha}_3$. Our goal is to estimate the asymptotic variance of the estimated maximized squared Sharpe ratio of the portfolio.

Corollary 1. The asymptotic variance of the squared Sharpe ratio of the UE portfolio with the weights in equation (10), when μ_{p} corresponds to the zero-beta rate φ, may be obtained using Theorem 1 with canonical matrices:[8]

$$C = \frac{-[\alpha_2 - \phi(1-\alpha_3)]^2 C_{\alpha_1} + 2(\alpha_1 - \phi\alpha_2)[\alpha_2 - \phi(1-\alpha_3)]C_{\alpha_2} + (\alpha_1 - \phi\alpha_2)^2 C_{\alpha_3}}{\left[\alpha_1(1-\alpha_3) - \alpha_2^2\right]^2} \text{ and},$$
$$D = \frac{-[\alpha_2 - \phi(1-\alpha_3)]^2 D_{\alpha_1} + 2(\alpha_1 - \phi\alpha_2)[\alpha_2 - \phi(1-\alpha_3)]D_{\alpha_2} + (\alpha_1 - \phi\alpha_2)^2 D_{\alpha_3}}{\left[\alpha_1(1-\alpha_3) - \alpha_2^2\right]^2},$$

(11)

where the component matrices $C\alpha_1$, $D\alpha_1$, $C\alpha_2$, $D\alpha_2$, $C\alpha_3$, and $D\alpha_3$ and their consistent estimates are provided with the proof in the Supplementary Material.

Corollary 1 may be used to compare models with dynamic trading by using the differences in the portfolios' canonical matrices. For example, in the nested tests where model 1 adds factors to those of model 2, the matrices C_1 and D_1 are formed using the larger model 1 factors, and the model 2 matrices are computed by filling the positions for the additional factors with zeros.

[8]When we say that μ_p corresponds to the zero-beta rate φ, we mean that a tangent line drawn to the minimum standard deviation boundary at the portfolio with mean μ_{p} intersects the *y*-axis at the point φ. For a given boundary, only one of the two parameters (μ_p, φ) need be specified. The variance of a minimum variance efficient portfolio is a quadratic function of its mean: ${\sigma_p}^2 = a - 2b\mu_p + c{\mu_p}^2$. The slope of the line from a zero-beta rate, φ, to this boundary, $\partial\mu_p/\partial\sigma_p = (\mu_p - \varphi)/\sigma_p$. Thus, the target mean is related to the zero-beta rate as $\mu_p = (a - b\varphi)/(b - c\varphi)$.

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D. Zero-Beta Rates

The analyses of BS (2017), Fama and French (2018), BKRS (2020), and others assume that the average short-term Treasury rate is the zero-beta rate. However, models and evidence often imply zero-beta rates in excess of a short-term Treasury return (e.g., Black, Jensen, and Scholes (1972), Frazzini and Pedersen (2014), and Lewellen, Nagel, and Shanken (2010)). The difference between the zero-beta rate and the Treasury rate can reflect the costs of borrowing against the underlying assets (Black (1972), Frazzini and Pedersen (2014)). Because the zero-beta rate is the expected return on assets that have zero covariance with the factors in a model, if there are missing priced factors the zero-beta portfolio may embed a premium because of its correlation with the missing priced factors.

We consider three alternative treatments for the zero-beta rate. The first is to work with returns in excess of a Treasury bill, as in most of the recent work on Sharpe ratio comparisons. The mean value of the Treasury bill return is the expected zero-beta rate. The second approach is to allow independent dynamic trading in the Treasury bill; viewed as a time-varying risk-free asset return, which is conditionally known at the beginning of the period. This approach is implemented using the solutions provided by Ferson and Siegel (2015). Our third approach is to estimate the zero-beta rate while assuming that no risk-free asset exists.

We estimate a zero-beta parameter following Kandel (1986), who worked in a normal, maximum likelihood setting. Kandel derived the zero-beta rate that maximizes the log likelihood ratio, the difference between the log likelihood function that imposes the null hypothesis and the log likelihood for the unrestricted data. The null hypothesis is $S^2(r,f) = S^2(f)$, so the optimal zero-beta rate minimizes the difference between the two maximum squared Sharpe ratios. The following proposition provides the solution, exploiting the fact that the maximum squared Sharpe ratio of any portfolio is a quadratic function of the zero-beta rate: $\theta(\varphi) = a - 2b\varphi + c\varphi^2$, where the coefficients (a, b, and c) depend on the situation.

Proposition 1. The zero-beta rate φ that minimizes the difference between the maximum squared Sharpe ratio of a portfolio with squared Sharpe ratio $\theta(\varphi) = a - 2b\varphi + c\varphi^2$ and that of another with maximum squared Sharpe ratio $\theta^*(\varphi) = a^* - 2b^*\varphi + c^*\varphi^2$, is found by minimizing $(a^* - a) - 2(b - b^*)\varphi + (c - c^*)\varphi^2$ and the solution is

$$\varphi = (b - b^*)/(c - c^*). \tag{12}$$

The zero-beta rate in the case of a normalized difference, like the test of GRS (1989), should minimize: $\frac{1+\theta(\varphi)}{1+\theta^*(\varphi)} = \frac{1+a-2b\varphi+c\varphi^2}{1+a^*-2b^*\varphi+c^*\varphi^2}$. The first-order condition delivers the quadratic expression:

$$(b^* - b) + (ab^* - a^*b) + (-2(a+1)c^* + 2(a^*+1)c)\varphi + (-3bc^* + 3b^*c)\varphi^2 = 0.$$

There are two solutions to this equation, one representing the local maximum and one the minimum. The consistent estimates are found by substituting consistent estimates of the (a,b,c) parameters into equation (12) or the solution to the quadratic equation.

When the optimal zero-beta rate from Proposition 1 is used in the comparison of squared Sharpe ratios, we accommodate the estimation error in the common zero-beta rate as another special case of Theorem 1, given in Corollary 2.

Corollary 2. Two maximum squared Sharpe ratios can be written as $\theta = a - 2b\phi + c\phi^2$ and $\theta^* = a^* - 2b^*\phi + c^*\phi^2$. By Proposition 1, the optimum zero-beta rate is $\phi = \frac{b-b^*}{c-c^*}$. The difference between two squared Sharpe ratios is $\theta - \theta^*$. Assume the estimator for a, $\hat{a}$, can be expanded as in the Theorem 1, with canonical matrices C_a and D_a. Make the same assumption for $\hat{b}$, $\hat{c}$, $\hat{a}^*$, $\hat{b}^*$, and $\hat{c}^*$. Then, the estimated value of the difference between two squared Sharpe ratios with a common estimated optimal zero-beta rate can be written as in the Theorem 1, with canonical matrices:

$$C = C_a - C_{a^*} - 2\phi(C_b - C_{b^*}) - 2(b - b^*)C_\phi + 2\phi(C_c - C_{c^*}) + 2\phi(c - c^*)C_\phi, \tag{13}$$

$$D = D_a - D_{a^*} - 2\phi(D_b - D_{b^*}) - 2(b - b^*)D_\phi + 2\phi(D_c - D_{c^*}) + 2\phi(c - c^*)D_\phi, \tag{14}$$

where

$$C_\phi = \frac{1}{c - c^*}C_b - \frac{1}{c - c^*}C_{b^*} - \frac{b - b^*}{(c - c^*)^2}C_c + \frac{b - b^*}{(c - c^*)^2}C_{c^*},$$

$$D_\phi = \frac{1}{c - c^*}D_b - \frac{1}{c - c^*}D_{b^*} - \frac{b - b^*}{(c - c^*)^2}D_c + \frac{b - b^*}{(c - c^*)^2}D_{c^*}.$$

The expressions are evaluated at consistent estimates of *(a,b,c)* and *(a*,b*,c*)*. The Supplementary Material presents the matrices for cases where one of the portfolios is a maximum correlation portfolio with respect to *Z*, as in models with nontraded factors.

E. Bias Correction

Jobson and Korkie (1980) provide a bias adjustment for sample squared Sharpe ratios based on the assumption that returns are normally distributed and the optimal portfolios have fixed weights. This adjustment is used by Ferson and Siegel (2003) and BKRS (2020). The bias-corrected estimate, improving the biased estimate $\hat{\theta}$ is given by

$$\hat{\theta}^* = \hat{\theta}\left(\frac{T - N - 2}{T}\right) - \frac{N}{T}, \tag{15}$$

where N is the number of assets and T is the number of time-series observations.

Ferson and Siegel (2003) find that the bias adjustment in (15) does not control the bias very well for dynamic portfolios, so we evaluate alternative bias adjustments. One is based on the noncentral chi-square distribution, assuming that the sample covariance matrix is at its probability limit. A second is based on the noncentral F distribution, which considers estimation error in the covariance matrix. A third is based on the odd-month, even-month approach of Jegadeesh, Noh, Pukthuanthong, Roll, and Wang (2019), which attempts to control for correlated errors. A fourth, and it turns out the best approach, is based on a second-order Taylor expansion.

Our preferred bias adjustment is based on the exact expectation of the second-order Taylor series expansion of the estimate minus its true value. The results are T-consistent when consistently estimated values are substituted for the unknown parameters. The bias correction is expressed in a proposition, where we define the squared Sharpe ratio in terms of the coefficients $(\alpha_1, \alpha_2, \alpha_3)$ as in Proposition 1.

Proposition 2. The approximate bias of the estimated maximized squared Sharpe ratio at zero-beta rate φ: $\hat{S}_\varphi^2 = \frac{\hat{\alpha}_1 - 2\varphi\hat{\alpha}_2 + \varphi^2\hat{\alpha}_3}{\hat{\alpha}_1\hat{\alpha}_3 - \hat{\alpha}_2^2} - 1$ with respect to the true (but unknown) maximized squared Sharpe ratio $S_\varphi^2 = \frac{\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2} - 1$ may be expressed using the expectation of its second-order Taylor Series expansion:

(16)
$$\text{BIAS} = E\left(\hat{S}_\varphi^2 - S_\varphi^2\right) \cong \left(\sum_{i=1}^{3} E(\hat{\alpha}_i - \alpha_i)\frac{\partial S_\varphi^2}{\partial \alpha_i}\right) + \left(\sum_{i,j=1}^{3} \frac{E\left[(\hat{\alpha}_i - \alpha_i)(\hat{\alpha}_j - \alpha_j)\right]}{2}\frac{\partial^2 S_\varphi^2}{\partial \alpha_i \partial \alpha_j}\right).$$

As in Siegel and Woodgate (2007a), (2007b)), we use the method of statistical differentials to find Taylor-series approximations to expectations of random variables. The partial derivatives in Proposition 2 are presented in the Supplementary Material. Note that the $\hat{\alpha}_i$ are not necessarily unbiased, so their expectations are necessary in the expression (16).

IV. The Data

We follow Cooper and Maio (2019) in the selection of six lagged information variables. These are a short-term Treasury bill rate, a value spread (Cohen, Polk, and Vuolteenaho (2003)), a measure of stock return dispersion following Stivers and Sun (2010), net equity expansion following Boudoukh, Michaely, Richardson, and Roberts (2007), and the investment to capital ratio following Cochrane (1991). We also present some results using a set of "classical" lagged instruments following Fama and French (1989) and a more "modern" set of instruments following Goyal, Welch, and Zafirov (2021).

The lagged conditioning variables in much of the literature are highly persistent. We assume that they are stationary, but if they are unit root processes, the distribution of the sample squared Sharpe ratios will be nonstandard (e.g., Phillips (2014)). We address this by following the suggestion of Ferson, Sarkissian, and Simin (2003). We subtract a 12-month trailing average from each of the lagged instruments to stochastically detrend them when the first-order autocorrelation exceeds 0.95.[9] All of the autocorrelations of the stochastically detrended series

[9]One exception is the investment-to-capital ratio, which is available quarterly. Cooper and Maio (2019) fill in for monthly data, assuming a constant value for the months within a quarter. This produces

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are well below values that Ferson, Sarkissian, and Simin find raise concern over spurious regression bias.

Standard test portfolio returns are monthly data from Kenneth French's data library at Dartmouth. Individual stocks sorted in two or more dimensions to form cross sections of portfolio returns. We use the 25 size × value portfolios, the 25 investment × profitability portfolios, 32 size × investment × profitability triple-sorted portfolios, and 49 industry portfolios.

We compare a number of popular factor models. The factors include the CRSP value-weighted stock market index, the Fama–French (1996) 3-factors (FF3), and the Fama and French (2015) 5-factors (FF5). We also examine the 4-factor (Q4) model of Hou et al. (2015), using data on the investment factors from Lu Zhang, and the Q5 model of Hou et al. (2021), using data on investment growth. In the Supplementary Material, we examine nontraded factors, following Chen et al. (1986).

The summary statistics for the factors and conditioning information are presented in Table 1. Returns are measured in excess of a 3-month Treasury bill. The first-order autocorrelations of the factors are 0.33 or below. Even with these small values, Ehsani and Linnainmaa (2020) find that dynamic trading using lagged factors as the information can materially increase Sharpe ratios. This is consistent with the logic of Campbell (1996) and Cooper and Maio (2019).

V. Simulations

We use the parametric bootstrap to evaluate the corrections for finite sample bias in estimated Sharpe ratios, the accuracy of our asymptotic standard errors, and the finite sample distributions of the "*t*-ratios." We model the *Z*s in the simulations as a first-order autoregressive process to capture their persistence.

A. Simulation Methods

Assume that the portfolio returns and the factors in the model follow equations (17) and (18):

(17) $$R_t = \delta' Z_{t-1} + \varepsilon_t,$$

(18) $$f_t = b_F' Z_{t-1} + \varepsilon_{ft},$$

and the conditioning information (without the constant) Z follows an AR(1) process:

(19) $$Z_t = \delta_{0z} + \delta'_{1z} Z_{t-1} + \varepsilon_{zt}.$$

We estimate the coefficients in the original data and calculate the regression residuals. We keep the coefficients as "true" parameters in our simulations. At each date in a simulation trial, we randomly choose a calendar date from the real data, and select the regression residuals from equations (17)–(19) as a vector for that date to

high autocorrelation which the 12-month moving average does not correct. But a 6-month moving average works for this series.

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TABLE 1

Summary Statistics of Factors and Lagged Information

Table 1 contains summary statistics for our sample of monthly traded factors from the French data library (Feb. 1959 to Dec. 2020), the *Q* factors from Hou, Xue, and Zhang (2015), and the nontraded factors. AR(1) is the first-order autocorrelation (after stochastic detrending when needed for the lagged instruments). Squared SR is the squared Sharpe ratio, where the zero-beta rate is the average Treasury bill rate, equal to 0.39% per month. The *R*-square is obtained by regressing market excess return on the lagged instruments. Returns, yields, and yield spreads are measured as percent per month.

Traded Factors	Mean	Std. Dev.	AR(1)	Squared SR
Market-risk free	0.57	4.43	0.066	0.016
SMB	0.2	2.96	0.064	0.005
HML	0.26	2.81	0.179	0.008
RMW	0.23	2.09	0.149	0.012
CMA	0.24	1.92	0.121	0.015
Momentum	0.61	4.06	0.047	0.022
Investment	0.29	1.77	0.099	0.027
Profitability	0.44	2.4	0.117	0.034
Investment growth	0.7	1.87	0.102	0.141
Nontraded Factors				
Consumption growth	0.26	0.82	0.01	NA
Broker–dealer leverage	0.09	6.71	0.09	NA
Expected inflation	0.13	0.11	0.95	NA
Change expected inflation	0.00	0.03	0.19	NA
Industry production	0.10	0.47	0.33	NA
Real interest rate	0.32	0.25	0.98	NA
Unexpected inflation	0.00	0.12	0.18	NA
Lagged Instruments	Mean	Std. Dev.	AR(1)	R^2 (%)
Short-term *T*-bill rate	−0.03	0.87	0.82	1.12
Investment capital ratio	0.00	0.00	0.86	0.05
Value spread	0.00	0.08	0.73	0.18
Stock return dispersion	2.64	1.24	0.64	0.04
Net equity expansion	0.00	0.01	0.83	0.16
Relative bill rate	0.00	0.80	0.80	0.64

preserve the correlations across the shocks. We take the sample average value of each conditioning variable in the data as its starting value, and build up the simulated instruments series recursively using the residuals and the regression coefficients of equation (19). We discard the first 500 simulated samples to wash out the initial conditions.

B. Simulation Results

1. Evaluating Bias Adjustments

The estimators are consistent, converging in probability to the true values as the number of time-series observations, T, grows. We evaluate finite sample bias relative to a "true" squared Sharpe ratio where the number of time series observations is $T \times 1{,}000$, and $T = 743$ or 587. Informal experiments suggest that $1{,}000 \times T$ is larger than required to estimate the probability limits. In each of 5,000 artificial samples with the same length as the original data samples, we estimate the bias--adjusted squared Sharpe ratios. The expected adjusted squared Sharpe ratio in finite samples is computed as the average value across the 5,000 trials.

Panel A of Table 2 evaluates bias adjustment for fixed-weight portfolios of test assets and factors. The unadjusted squared Sharpe ratios are larger than the true values, reflecting the finite sample bias. The Jobson and Korkie (1980) adjustment works well when there is no dynamic trading. After adjustment, the remaining bias is 5% or less of the true value. The bottom rows show results for $N = 25$, 49, and

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99 portfolios. The $N = 25$ portfolios are the 5 × 5 size × book/market sorts, the $N = 49$ are industry portfolios and the $N = 99$ combine the first two sets with 25 investment × profitability portfolios. The bias shows no obvious relation to N.

Panel B of Table 2 looks at UE portfolio squared Sharpe ratios. Comparing the left-hand columns of Panels A and B of Table 2 fixed-weight portfolios, on the "true" maximum squared Sharpe ratios. The impact is large for R_m, the FF6 squared Sharpe ratio increases by more than 50% with dynamic trading, and the Q5 by about 11%. The impact of dynamic trading on the standard portfolio designs is larger still, and the maximum squared Sharpe ratios of the FF25 and the 49 industry portfolios more than double.

Panel B of Table 2 addresses the various adjustments for finite sample bias with dynamic trading. The unadjusted squared Sharpe ratios can have a large bias, increasing when there are more assets or factors. The Q5 model, which delivers the highest Sharpe ratios among the factor models, has the smallest bias. While all of the bias correction methods reduce the finite sample bias, the direct expansion approach is clearly the most accurate. The remaining bias in the adjusted ratios is less than 3.2% of the true value in each case, and the percentage bias shows no obvious relation to N. The direct expansion bias adjustment works as well on the UE portfolios as the JK (1980) adjustment does on the fixed-weight portfolios. While not shown in the table, we also find that the direct expansion method works well on fixed-weight portfolios, but its calculation is not as simple as the JK adjustment in such cases.

TABLE 2

Accuracy of Bias Adjustments for Squared Sharpe Ratios

The "true" squared Sharpe ratios shown in Table 2 are from simulations with a large number (1,000 × 743) of time series observations. The values are stated in percent (multiplied by 100). The average values across 5,000 simulation trials are shown for five alternative bias-adjustment methods. The number of time series observations in the finite samples is 743. The JK uses the results of Jobson and Korkie (1980) and are based on a non-central F distribution. The four adjustments for dynamic portfolios are the chi-square, non-central F, odd–even, and direct expansion. The adjustments are applied to the squared Sharpe ratios of fixed weight portfolios in Panel A and to efficient with respect to Z portfolios in Panel B. The six lagged instruments that comprise the vector Z are described in the text and Table 1. The $N = 25$ portfolios are the 5 × 5 size × book/market sorts, the $N = 49$ are industry portfolios, and the $N = 99$ combine the first two sets with 25 investment × profitability portfolios from Kenneth French.

Panel A. Fixed-Weight Factor Portfolios

	TRUE	JK	% Diff
R_m	1.66	1.74	−5%
S_{fix}(FF3)	3.33	3.46	−4%
S_{fix}(FF5)	8.46	8.59	−2%
S_{fix}(FF6)	11.65	11.91	−2%
S_{fix}(Q5)	31.28	31.28	0%
$S_{fix}(r)$ $N = 25$	16.35	16.55	−1%
$S_{fix}(r)$ $N = 49$	32.21	32.60	−1%
$S_{fix}(r)$ $N = 99$	77.19	78.23	−1%

Panel B. Efficient-with-Respect to Z Portfolios (% Indicates Differences from True)

	TRUE	No-Adj	Chi-Square	Non-Central F	Odd–Even	Direct Expansion
S_{UE}(FF3)	6.02	31.9%	24.3%	13.6%	12.6%	3.2%
S_{UE}(FF5)	13.38	26.7%	20.4%	13.2%	11.9%	1.7%
S_{UE}(FF6)	17.78	25.3%	19.3%	8.5%	7.1%	2.0%
S_{UE}(Q5)	34.81	10.6%	7.6%	2.6%	1.6%	1.0%
$S_{UE}(r)$ $N = 25$	39.63	52.8%	38.2%	6.1%	0.4%	2.5%
$S_{UE}(r)$ $N = 49$	90.2	47.1%	9.2%	−0.6%	8.7%	1.5%
$S_{UE}(r)$ $N = 99$	184.07	58.9%	23.1%	1.5%	22.9%	2.7%

2. Asymptotic Standard Errors

We examine the accuracy of our asymptotic standard errors for squared Sharpe ratios and their differences. The standard deviations of the ratios estimated in finite samples, and bias adjusted with the second-order expansion method, are taken across 1,000 simulation trials. These empirical standard errors measure the sampling variability in the estimates. Good asymptotic standard errors should predict the empirical standard errors.

The average value of the asymptotic standard errors across the simulation trials is shown as average asymptotic in Table 3. Panel A describes results for the levels of squared Sharpe ratios. The asymptotic standard errors present an expected bias averaging about 7% of the empirical overall, but as high as 11% (fixed-weight FF3 model). The asymptotics tend to understate the sampling variability. In the case with

TABLE 3

Accuracy of Asymptotic Standard Deviations

A parametric bootstrap generates 1,000 simulation trials, each with 743 observations. Squared Sharpe ratios and squared Sharpe ratio differences are estimated in Table 3, and the asymptotic standard deviations are calculated using the propositions and Theorem 1. The first columns (empirical) are the standard deviations of the estimates across the 1,000 simulation trials. The average asymptotic are the averages of the estimated asymptotic standard deviations. Fix(r) or fix(f) refers to a mean–variance efficient portfolio that ignores the conditioning information and uses fixed weights. UE is efficient with respect to Z. The lagged instruments are described in the data section. The average return of a 3-month Treasury bill is taken to be the zero-beta rate. The $N = 25$ portfolios are the 5 × 5 size × book/market sorts, the $N = 49$ are industry portfolios, and the $N = 99$ combine the first two sets with 25 investment × profitability portfolios from Kenneth French.

	Empirical (Simulated)	Average FSW Asymptotic	Average BKRS Asymptotic	Difference FSW (% Empirical)	Difference BKRS (% Empirical)
Panel A. Standard Errors for Squared Sharpe Ratio Levels					
R_m	0.28	0.28	0.28	−2%	−1%
S_{fix}(FF3)	0.45	0.40	0.41	−11%	−9%
S_{fix}(FF6)	0.87	0.80	0.81	−9%	−7%
S_{UE}(FF3)	0.60	0.58		−3%	
S_{UE}(FF6)	1.09	1.08		−2%	
$S_{fix}(r)$ $N = 25$	1.02	0.95	0.95	−7%	−7%
$S_{fix}(r)$ $N = 49$	1.25	1.24	1.30	0%	4%
$S_{fix}(r)$ $N = 99$	2.33	2.48	2.54	7%	9%
$S_{UE}(r)$ $N = 25$	1.49	1.38		−8%	
$S_{UE}(r)$ $N = 49$	1.91	1.79		−6%	
$S_{UE}(r)$ $N = 99$	3.50	3.33		−5%	
Panel B. Standard Errors for Squared Sharpe Ratio Differences (N = 25)					
$S_{fix}(r) - R_m$	0.87	0.91	0.93	4%	6%
$S_{fix}(r) - S_{fix}$(FF3)	0.85	0.89	0.91	5%	7%
$S_{fix}(r) - S_{fix}$(FF5)	0.83	0.90	0.91	9%	10%
$S_{UE}(r) - R_m$	1.39	1.41		2%	
$S_{UE}(r) - S_{UE}$(FF3)	1.41	1.40		0%	
$S_{UE}(r) - S_{UE}$(FF5)	1.41	1.43		1%	
$S_{UE}(r) - S_{fix}(r)$	1.06	1.04		−2%	
Panel C. Standard Errors for Squared Sharpe Ratio Differences (N = 99)					
$S_{fix}(r) - R_m$	2.37	2.45	2.50	4%	6%
$S_{fix}(r) - S_{fix}$(FF3)	2.38	2.44	2.49	3%	5%
$S_{fix}(r) - S_{fix}$(FF5)	2.34	2.39	2.43	2%	4%
$S_{UE}(r) - R_m$	3.90	3.64		−7%	
$S_{UE}(r) - S_{UE}$(FF3)	3.76	3.60		−4%	
$S_{UE}(r) - S_{UE}$(FF5)	3.72	3.54		−5%	
$S_{UE}(r) - S_{fix}(r)$	2.90	2.97		2%	
Panel D. Standard Errors for Squared Sharpe Ratio Differences (factors alone)					
S_{fix}(FF5) − S_{fix}(FF3)	0.54	0.48	0.49	−11%	−9%
S_{fix}(FF6) − S_{fix}(FF5)	0.52	0.47	0.47	−10%	−9%
S_{fix}(FF6) − S_{fix}(Q5)	1.06	1.02	1.03	−4%	−2%
S_{UE}(FF5) − S_{UE}(FF3)	0.63	0.68		8%	
S_{UE}(FF6) − S_{UE}(FF5)	0.57	0.55		−3%	
S_{UE}(FF6) − S_{fix}(Q5)	1.15	1.40		21%	

fixed portfolio weights, our asymptotic standard errors (FSW) are very similar to the BKRS standard errors. The FSW standard errors are typically more accurate in the cases with dynamic trading than with fixed weights.

The rows in Table 3 for the fixed-weight efficient portfolios show that Theorem 1 may be used for the standard errors in this case. These simulations do reflect the randomness of the conditioning information.[10] We see no strong evidence that the bias in the standard errors is larger when more portfolios are used.

Panels B–D of Table 3 examine standard errors for Sharpe ratio differences, as used in tests of the models. The average asymptotic values are within 10% of the empirical standard errors in most cases and there is no indication that the performance degrades when more test asset portfolios are used. The accuracy of the standard errors is somewhat worse when combinations of factors are compared directly in Panel B, where the asymptotic standard error overstates the sampling variability.

3. *T*-Ratios

We form "*t*-ratios," dividing the bias-adjusted squared Sharpe ratio or difference by its standard error. Table 4 evaluates the sampling distributions of the absolute *t*-ratios against their asymptotic distribution, a chi distribution with one degree of freedom. The true values of the numerators are from simulations with 743,000 observations. Fractiles of the distribution of the *t*-ratios from the 1,000 simulation trials are shown. $\chi(1)$ are the critical values for the asymptotic distribution, which are very similar to the usual rules of thumb as can be seen in the table.

Panel A of Table 4 presents critical *t*-ratios for the levels of maximum squared Sharpe ratios. Using the standard test portfolios ($N = 25$, 49, or 99) the *t*-ratios appear reasonably well specified in the tails. When small numbers of factors are used, the performance is worse and the empirical critical values are too large. In the worst cases, however, a *t*-ratio of 2 is significant at the 10% level instead of the 5% level. This is reminiscent of results from GRS (1989), where the Wald and Lagrange multiplier tests rejected the CAPM too often in various test portfolio designs. Here, the CAPM displays the largest bias.

Panels B–D of Table 4 present results for squared Sharpe ratio differences. The bias remaining in the bias-adjusted squared Sharpe ratio levels may be offsetting in the differences. The table shows that the distributions of the sample *t*-ratios are close to the $\chi(1)$ in most cases. This supports the validity of a standard rule of thumb where a *t*-ratio of 2 is considered significant at the 5% level in a maximum squared Sharpe ratio difference test between a factor model and the standard test asset portfolios.

Panel E of Table 4 presents results for direct comparisons between the squared Sharpe ratios of two models' factors. In this application the *t*-ratios are the least well specified and are expected to reject the hypothesis of no difference between the two models too often. The fixed-weight cases are worse than the UE cases, but the Q5 model produces better specified *t*-ratios.

[10]To further explore the impact of the volatility of the conditioning information, we shut it off by using the same values of the conditioning variables at each simulation trial. Otherwise, the simulations are the same as before. We find that the average FSW asymptotic and the empirical standard deviations are even closer to each other in this experiment.

TABLE 4

The Empirical Distributions of *t*-Ratios

A parametric bootstrap generates 1,000 simulation trials. Each set of simulated data has 743 observations. In Table 4, squared Sharpe ratios, $S(.)$, and their differences are estimated, bias adjusted using the second-order expansion method, and their asymptotic standard deviations are calculated. Squared *t*-ratios are formed as the squared adjusted Sharpe ratio or difference less its "true" value, divided by its asymptotic standard error. The true values are from simulations with 743 × 1,000 observations. Critical values of the empirical distribution from the 1,000 simulation trials are shown. $\chi(1)$ are the values for a chi distribution with one degree of freedom. Fixed weight portfolios ignore the conditioning information. UE is efficient with respect to Z. The lagged instruments are described in the text. The average return of a 3-month Treasury bill is taken to be the zero-beta rate. The $N = 25$ portfolios are the 5 × 5 size × book/market sorts, the $N = 49$ are industry portfolios and the $N = 99$ combine the first two sets with 25 investment × profitability portfolios from Kenneth French.

	Fixed Weight Portfolios			Dynamic UE Portfolios		
Panel A. Critical Values for t-Ratios of Squared Sharpe Ratio Levels						
Percentile	90%	95%	98%	90%	95%	98%
$\chi(1)$	1.65	1.96	2.33	1.65	1.96	2.33
R_m	2.23	2.96	3.70	2.23	2.96	3.70
S(FF3)	1.76	2.22	2.72	1.59	1.91	2.29
S(FF5)	1.90	2.21	2.79	1.54	1.84	2.16
S(FF6)	1.85	2.24	2.50	1.60	1.88	2.17
S(Q5)	1.78	2.16	2.58	1.34	1.59	1.92
$S(r)$ $N = 25$	1.62	1.97	2.21	1.62	1.97	2.21
$S(r)$ $N = 49$	1.59	1.88	2.28	1.82	2.12	2.49
$S(r)$ $N = 99$	1.65	1.95	2.24	1.85	2.18	2.56
Panel B. Squared Sharpe Ratio Differences (N = 25)						
Percentile	90%	95%	98%	90%	95%	98%
$\chi(1)$	1.65	1.96	2.33	1.65	1.96	2.33
$S(r) - R_m$	1.55	1.88	2.27	1.72	2.05	2.47
$S(r) - S$(FF3)	1.56	1.86	2.21	1.67	1.94	2.28
$S(r) - S$(FF5)	1.51	1.84	2.27	1.59	1.98	2.28
$S_{UE}(r) - S_{fix}(r)$	1.62	1.87	2.26	NA	NA	NA
Panel C. Squared Sharpe Ratio Differences (N = 49)						
Percentile	90%	95%	98%	90%	95%	98%
$\chi(1)$	1.65	1.96	2.33	1.65	1.96	2.33
$S(r) - R_m$	1.57	1.91	2.27	1.77	2.15	2.52
$S(r) - S$(FF3)	1.58	1.91	2.13	1.85	2.16	2.51
$S(r) - S$(FF5)	1.57	1.90	2.22	1.82	2.16	2.58
$S_{UE}(r) - S_{fix}(r)$	1.72	2.17	2.53	NA	NA	NA
Panel D. Squared Sharpe Ratio Differences (N = 99)						
Percentile	90%	95%	98%	90%	95%	98%
$\chi(1)$	1.65	1.96	2.33	1.65	1.96	2.33
$S(r) - R_m$	2.04	2.34	2.67	1.86	2.23	2.50
$S(r) - S$(FF3)	1.68	1.98	2.32	1.84	2.14	2.50
$S(r) - S$(FF5)	1.77	2.10	2.54	1.77	2.19	2.54
$S_{UE}(r) - S_{fix}(r)$	1.65	2.01	2.51	NA	NA	NA
Panel E. Squared Sharpe Ratio Differences for Factors Alone						
	Fixed Weight Portfolios			Dynamic UE Portfolios		
Percentile	90%	95%	98%	90%	95%	98%
$\chi(1)$	1.65	1.96	2.33	1.65	1.96	2.33
S(FF5) − S(FF3)	1.95	2.38	2.96	1.54	1.83	2.42
S(FF6) − S(FF5)	2.08	2.65	3.45	1.84	2.22	2.86
S(FF6) − S(Q5)	1.70	1.98	2.50	1.39	1.73	2.08

VI. Factor Model Tests and Comparisons

Table 5 presents tests of factor model efficiency in various test portfolio designs. Returns are measured in excess of a 3-month Treasury bill rate. Its average value, 0.39% per month, is the zero-beta rate. In Panel A, no conditioning information is used. We include a factor model from Chen et al. (CRR (1986)), where mimicking portfolios replace the nontraded factors for unexpected inflation,

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expected inflation, and industrial production. Most of the bias-adjusted Sharpe ratio differences between test assets and factors are statistically significant (t-ratios of 2.2 or higher) except where the portfolios sort on profitability and the factor models feature investment, profitability or production factors (FF5, FF6, Q4, Q5, and CRR3). Using the value-weighted market index, the CAPM is rejected in all the portfolio designs, and all the models are rejected in the size × value portfolios.

Given that factor models with fixed weights are rejected in test portfolios with no dynamic trading, it is clear (and we do find) that factor models with fixed weights can be rejected in test portfolios with dynamic trading. Thus, case (i) above for linear factor models of the SDF with constant coefficients can be rejected. In Panel B of Table 5, dynamically trading factor models face dynamically trading test assets. The Sharpe ratio differences are larger than in Panel A, with large t-ratios in almost all cases. This is because dynamic trading increases the Sharpe ratios of the test assets more than factors. Thus, case (ii) for linear factor models of the SDF can be strongly rejected.

In Panel C of Table 5, the fully conditional factor models of case (iii) are tested, and rejected with t-ratios larger than 2.16 in every case. With the time-

TABLE 5

Tests of Factor Model Efficiency

The test statistic in Table 5 is the difference in the bias-adjusted squared Sharpe ratios (not multiplied by 100) for the test assets and the factors versus the factors alone. The t-ratios are in parentheses. The factor model abbreviations and test asset portfolios are described in the text. Monthly Sharpe ratios are computed using the average Treasury bill return of 0.39% per month as the zero-beta rate. The dynamic models trade optimally using the lagged instruments. The sample period is Jan. 1967 to Dec. 2013.

Panel A. Fixed-Weight Models and Assets

	CAPM	FF3	FF5	FF6	Q4	Q5	CRR3
25 size × value portfolios	0.15	0.12	0.10	0.08	0.11	0.10	0.12
	(3.92)	(3.32)	(2.80)	(2.45)	(2.99)	(2.53)	(3.45)
25 investment × profitability	0.07	0.05	0.00	−0.01	0.01	0.01	0.05
	(2.34)	(1.87)	(−0.20)	(−0.67)	(0.34)	(0.38)	(1.43)
32 size × value portfolios × prod	0.10	0.07	0.05	0.05	0.06	0.06	0.07
	(2.85)	(2.27)	(1.55)	(1.45)	(1.82)	(1.56)	(2.32)
49 industry	0.08	0.12	0.17	0.15	0.14	0.12	0.04
	(2.48)	(3.07)	(3.78)	(3.37)	(2.88)	(2.52)	(1.28)

Panel B. Dynamic Models Versus Dynamic Assets

	CAPM	FF3	FF5	FF6	Q4	Q5	CRR3
25 size × value portfolios	0.31	0.28	0.28	0.27	0.29	0.27	0.22
	(5.11)	(4.67)	(4.66)	(4.61)	(4.60)	(4.03)	(3.89)
25 investment × profitability	0.19	0.18	0.12	0.11	0.13	0.14	0.09
	(3.64)	(3.39)	(2.75)	(2.67)	(2.91)	(2.60)	(1.73)
32 size × value portfolios × prod	0.22	0.18	0.15	0.15	0.19	0.19	0.15
	(3.48)	(3.03)	(2.98)	(3.00)	(3.20)	(2.90)	(2.60)
49 industry	0.23	0.26	0.33	0.31	0.30	0.29	0.17
	(3.86)	(4.15)	(4.63)	(4.52)	(4.14)	(3.90)	(2.89)

Panel C. Dynamic Assets Versus Full Conditional Models

	CAPM	FF3	FF5	FF6	Q4	Q5	CRR3
25 size × value portfolios	0.24	0.24	0.24	0.24	0.24	0.24	0.25
	(4.13)	(4.13)	(4.13)	(4.13)	(4.13)	(4.13)	(4.22)
25 investment × profitability	0.11	0.11	0.11	0.11	0.11	0.11	0.13
	(2.17)	(2.17)	(2.17)	(2.17)	(2.18)	(2.18)	(2.49)
32 size × value portfolios × prod	0.19	0.19	0.19	0.19	0.19	0.19	0.18
	(3.16)	(3.16)	(3.16)	(3.16)	(3.15)	(3.15)	(3.08)
49 industry	0.19	0.19	0.19	0.19	0.19	0.19	0.20
	(3.43)	(3.43)	(3.43)	(3.43)	(3.43)	(3.43)	(3.49)

TABLE 6

Tests of Factor Models with an Estimated Zero-Beta Rate

The test statistic in Table 6 is the difference in bias-adjusted squared Sharpe ratios for the test assets and the factors versus the factors alone. The factor model abbreviations and test asset portfolios are described in the text. Monthly Sharpe ratios are computed using the estimated zero-beta rate. The dynamic models trade optimally using the lagged instruments following Cooper and Maio (2019), Jan. 1967 to Dec. 2013.

Panel A. Fixed-Weight Models

	CAPM	FF3	FF5	FF6	Q4	Q5	CRR3
25 size × value portfolios:	0.05	0.06	0.06	0.05	0.06	0.08	0.02
	(1.94)	(1.00)	(0.69)	(0.49)	(0.95)	(0.73)	(0.63)
Zero-beta rate	0.0152	0.0040	0.0039	0.0038	0.0032	0.0028	0.0076
25 investment × profitability:	0.03	0.04	−0.01	−0.02	−0.01	−0.01	0.00
	(1.18)	(1.67)	(−0.15)	(−0.31)	(−0.12)	(−0.14)	(0.12)
Zero-beta rate	0.0156	0.0054	0.0041	0.0039	0.0048	0.0047	0.0070
32 size × value portfolios × prod:	0.03	0.04	0.02	0.02	0.03	0.04	0.00
	(1.39)	(0.66)	(0.26)	(0.21)	(0.55)	(0.39)	(0.00)
Zero-beta rate	0.0116	0.0044	0.0041	0.0039	0.0039	0.0036	0.0054
49 industry:	0.08	0.08	0.08	0.09	0.08	0.09	0.01
	(2.51)	(2.27)	(1.73)	(1.75)	(2.14)	(1.92)	(0.35)
Zero-Beta rate	0.0083	0.0074	0.0066	0.0061	0.0080	0.0072	0.0050

Panel B. Dynamic Models

	CAPM	FF3	FF5	FF6	Q4	Q5	CRR3
25 size × value portfolios:	0.16	0.17	0.16	0.15	0.18	0.19	0.09
	(2.96)	(2.27)	(1.67)	(1.46)	(2.23)	(1.67)	(1.82)
Zero-beta rate	0.0147	0.0041	0.0039	0.0038	0.0032	0.0028	0.0074
25 investment × profitability:	0.08	0.10	0.04	0.03	0.04	0.03	0.03
	(1.76)	(2.08)	(0.61)	(0.40)	(0.70)	(0.42)	(0.62)
Zero-beta rate	0.0155	0.0054	0.0040	0.0039	0.0047	0.0046	0.0069
32 size × value portfolios × prod:	0.12	0.10	0.09	0.08	0.12	0.12	0.06
	(2.17)	(1.55)	(0.90)	(0.84)	(1.68)	(1.22)	(1.13)
Zero-beta rate	0.0114	0.0044	0.0041	0.0040	0.0038	0.0036	0.0053
49 industry:	0.19	0.18	0.19	0.21	0.20	0.21	0.12
	(3.25)	(3.23)	(3.16)	(3.34)	(3.59)	(3.52)	(1.89)
Zero-beta rate	0.0081	0.0074	0.0066	0.0061	0.0080	0.0071	0.0049

varying $a(Z)$ term that does not depend on the factors, the models are less sensitive to the choice of factors, and the Sharpe ratios and test statistics are similar across the factor models.

The strong rejections of the models in Table 5 assume that the zero-beta rate is the average Treasury bill rate of 0.39%. In Table 6, this assumption is removed, and the zero-beta rate is estimated. Panel A uses fixed-weight portfolios. There are far fewer rejections of the models allowing the more general zero-beta rates. T-ratios larger than 2 only appear in the industry portfolio design. The CAPM produces both the largest t-ratio and the largest zero-beta rate among the factor models, with zero-beta rates of 0.8% to 1.5% per month. The zero-beta rate can reflect borrowing costs or the average risk premiums for missing factors. It seems likely that the high zero-beta rates for the CAPM reflect the impact of missing factors. For other models the zero-beta rate is closer to the average bill rate.

In Panel B of Table 6, dynamic trading is used for both factors and test assets. The zero-beta rate estimates are similar to those of Panel A, but we find more frequent rejections of the models as dynamic trading increases the Sharpe ratios for the test assets more than it does for the factors. All the models except CRR3 are rejected in the industry portfolio design.

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Direct Factor Model Comparisons

Table 7 presents direct comparisons of the maximum squared Sharpe ratios for two models' bias-adjusted factors. In the first column, the factors are held with fixed weights over time. The next 3 columns use dynamic trading with different choices of lagged instruments, described in the data section. The results are similar for the different instrument choices.

Table 7 shows that dynamic trading helps the FF3 factors beat the market index, but makes less of a difference for the other model comparisons. The Sharpe ratio performance of the Q models is striking. The Q5 model outperforms every model to which it is compared. This includes a version of FF6, denoted FF6* in the table, where a monthly rebalanced HML factor is used.

The FF5 model beats FF3 in Table 7, but the FF6 model (with a momentum factor) only beats the FF5 when there is dynamic trading using the "modern" instruments. Fama and French (2018) equivocate on whether the FF6 beats the FF5 model. The Q4 model beats FF5 in a fixed-weight comparison, but the two are not significantly different with dynamic trading. Dynamic trading improves the FF5 factors more than the Q4 factors.

The tests in Table 7 are based on returns in excess of the Treasury bill. We conduct, but do not report in the tables, direct model comparisons where we estimate an optimal zero-beta rate. In only one of the direct comparisons is the null hypothesis rejected that the two Sharpe ratios are equal with dynamic trading. Unlike in tests using the standard test asset portfolios, the optimal zero-beta rates vary widely over these model comparisons, but they are not very sensitive to the choice of instruments.

TABLE 7

Direct Factor Model Comparisons

The test statistic in Table 7 is the difference in bias-adjusted squared Sharpe ratios (not multiplied by 100) for the first model less the second model. The factors are held with fixed weights over time (no instruments) or dynamically traded using the various sets of instruments. The *t*-ratios in parentheses are the differences divided by the asymptotic standard errors for the difference. The factor model abbreviations and test asset portfolios are described in the text. FF6* replaces the HML factor in FF6 with a monthly rebalanced version. The sample period is Jan. 1967 to Dec. 2020.

	No Instruments	Classical Instruments	Modern Instruments	Cooper Miao
FF3 – R_m	0.03 (1.73)	0.05 (2.45)	0.06 (2.74)	0.04 (1.96)
FF5 – FF3	0.06 (2.38)	0.08 (2.74)	0.08 (2.44)	0.08 (2.04)
FF6 – FF5	0.03 (1.37)	0.03 (1.46)	0.06 (2.40)	0.04 (1.54)
Q5 – Q4	0.20 (3.75)	0.20 (3.80)	0.21 (3.81)	0.20 (3.52)
Q4 – FF5	0.08 (2.20)	0.03 (0.81)	0.06 (1.36)	0.07 (1.65)
Q5 – FF6	0.25 (4.05)	0.21 (3.23)	0.20 (3.09)	0.23 (3.46)
Q5 – FF6*	0.22 (3.27)	0.19 (2.86)	0.19 (2.81)	0.20 (2.84)
Q5 – CCR3	0.31 (3.71)	0.26 (3.07)	0.32 (3.37)	0.32 (3.40)

VII. Conclusions

This article contributes to the literature on factor model tests and comparisons in three ways. First, we remove the limitation that optimal portfolios must have fixed weights over time, allowing dynamic trading as a function of lagged variables. Second, we develop finite sample bias adjustments for maximum squared Sharpe ratios. Third, we allow general and optimal zero-beta rates in the models.

We develop asymptotic standard errors and propose simple "*t*-ratios" that our simulations show are reliable in most cases for realistic samples of stock portfolio returns and popular factor models. This allows inferences without simulation.

The impact of dynamic trading with the conditioning information is to raise the maximum squared Sharpe ratios, and to a larger extent for popular portfolios sorted on stock characteristics or industries than for portfolios of the models' factors. Most of the popular factor models are thereby rejected, and the factors in the models appear far from efficient even when the factors are traded dynamically. This presents the awkward situation where the factor models' Sharpe ratios appear "too large" from some perspectives, yet "too small" to be efficient in the standard portfolio designs.

In factor model efficiency tests using standard test portfolios, we find that the estimated zero-beta rate is the largest in the CAPM, at about 1.5% per month, while the other factor models produce values closer to the average bill rate of 0.39% per month. This is robust to the choice of lagged instruments. The zero-beta rate is almost twice as large in the industry portfolio design than in the other portfolios.

In direct comparisons of factor models that do not use the test portfolios, we find that the FF3 factors significantly beat the market index and the FF5 model beats FF3, but the FF6 model with momentum only beats the FF5 when there is dynamic trading using a "modern" set of lagged instruments. The Q4 model beats FF5 in a fixed-weight comparison, but the two are not significantly different with dynamic trading. Dynamic trading improves the FF5 factors more than the Q4 factors. However, the Q5 model outperforms every model to which it is compared.

Appendix

A.1. Special Cases Without Conditioning Information

Theorem 1 can be applied to cases with no conditioning information, such as fixed-weight factor models, with the following corollary.

Corollary A1. The asymptotic variance of the maximal estimated squared Sharpe ratio S_φ^2 with fixed weights w and zero-beta rate φ, with mean μ_φ, variance σ_φ^2 and all estimated from the data, may be obtained using the Theorem 1 together with the following canonical matrices:

$$C = \frac{2\left(\mu_\varphi - \varphi\right)A^{-1}}{T\sigma_\varphi^2}\sum_{t=1}^{T} Z_{t-1}\left[1 - \frac{\mu_\varphi - \varphi}{\sigma_\varphi^2}\left(\mu_t' w - \mu_\varphi\right)\right]w'$$
$$= \frac{2\left(\mu_\varphi - \varphi\right)}{\sigma_\varphi^2}\left[A^{-1}\overline{Z} - \frac{\mu_\varphi - \varphi}{\sigma_\varphi^2}\left(\delta w - \mu_\varphi A^{-1}\overline{Z}\right)\right]w'$$
$$D = -\frac{\left(\mu_\varphi - \varphi\right)^2}{\sigma_\varphi^4} ww' = -\frac{S_\varphi^2}{\sigma_\varphi^2} ww'.$$

One example of using Corollary A1 is testing the CAPM using, say, the value-weighted market portfolio. Corollary A1 delivers an asymptotic variance of the squared Sharpe ratio for the market portfolio. The weights vector w is 1.0 on the market return and zero on the other test assets.

A.2. Details for Corollary 2

The estimated squared Sharpe ratio is

$\hat{S}_\varphi^2 = \frac{\hat{\alpha}_2^2 + \hat{\alpha}_1\hat{\alpha}_3 - 2\varphi\hat{\alpha}_2 + \varphi^2(1-\hat{\alpha}_3)}{\hat{\alpha}_1(1-\hat{\alpha}_3) - \hat{\alpha}_2^2}$, which follows from maximizing

$\hat{S}^2 = \frac{\left(\hat{\mu}_p - \varphi\right)^2}{\hat{\sigma}_p^2} = \frac{\left(\hat{\mu}_p - \varphi\right)^2}{\left(\hat{\alpha}_1 + \frac{\hat{\alpha}_2^2}{\hat{\alpha}_3}\right) - \frac{2\hat{\alpha}_2}{\hat{\alpha}_3}\hat{\mu}_p + \frac{1-\hat{\alpha}_3}{\hat{\alpha}_3}\hat{\mu}_p^2}$ with respect to the mean $\hat{\mu}_p$. The asymptotic variance of $\hat{S}_\varphi^2$ follows from Theorem 1 together with the following choices for the canonical matrices C and D:

$$C = \frac{-[\alpha_2 - \varphi(1-\alpha_3)]^2 C_{\alpha_1} + 2(\alpha_1 - \varphi\alpha_2)[\alpha_2 - \varphi(1-\alpha_3)]C_{\alpha_2} + (\alpha_1 - \varphi\alpha_2)^2 C_{\alpha_3}}{\left[\alpha_1(1-\alpha_3) - \alpha_2^2\right]^2}$$
$$D = \frac{-[\alpha_2 - \varphi(1-\alpha_3)]^2 D_{\alpha_1} + 2(\alpha_1 - \varphi\alpha_2)[\alpha_2 - \varphi(1-\alpha_3)]D_{\alpha_2} + (\alpha_1 - \varphi\alpha_2)^2 D_{\alpha_3}}{\left[\alpha_1(1-\alpha_3) - \alpha_2^2\right]^2}.$$

A.3. Mimicking Portfolios and Cross-Sectional Fit

We use the same test assets whose cross section we wish to explain to form the mimicking portfolios. One might think this will artificially improve the explanatory power. We show here it does not do so in the fixed-weight case.

In this section, we redefine f and r to be the $T \times K$ and $T \times N$ matrices of factor and test asset excess returns data. Represent the betas of the test assets on the factors as $\beta = (f'f)^{-1} f'r$. (GLS works, too.) Mimicking portfolios are found by regressing the factors on the test assets. The mimicking excess returns are $f^* = r\,(r'r)^{-1}\, r'f$. The betas of the test assets on the mimicking factor portfolios are

$$\beta* = (f*'f*)^{-1} f*'r = \left[f'r(r'r)^{-1}r'f\right]^{-1} f'r.$$

Thus, the mimicking betas are related to the factor betas by an invertible $K \times K$ rotation: $\beta^* = A\beta$. A cross-sectional regression of returns on the factor betas delivers a coefficient $\lambda' = (\beta\beta')^{-1}\beta r'$ and fitted values $\lambda\beta$. A cross-sectional regression of returns on the mimicking betas delivers a coefficient

$$\lambda*' = (\beta*\beta*')^{-1}\beta*r' = A^{-1}\lambda \text{ and fitted values } \lambda*\beta* = \lambda\beta.$$

With the same fitted values, there is no impact of using mimicking portfolios on the cross-sectional fit of the model.

If portfolios are formed to maximize the Sharpe ratio, there is an upward bias in any finite sample. This bias will artificially inflate the explanatory power, as measured by the fitted Sharpe ratio. One of our contributions, therefore, is bias adjustment for maximum Sharpe ratios. The mimicking portfolios with dynamic trading do not maximize the Sharpe ratio. We see no reason why they would artificially increase the explanatory power in the cross section. Indeed, in Table 5, some of the mimicking portfolios display a negative finite sample bias.

Supplementary Material

To view supplementary material for this article, please visit http://doi.org/10.1017/S002210902400005X.

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