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MULTIPLICATION IDEALS, MULTIPLICATION RINGS, AND THE RING R(X)

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1. Introduction. Let R be a commutative ring with an identity. An ideal A of R is called a *multiplication ideal* if for every ideal $B \subseteq A$ there exists an ideal C such that $B = AC$. A ring R is called a *multiplication ring* if all its ideals are multiplication ideals. A ring R is called an *almost multiplication ring* if R_M is a multiplication ring for every maximal ideal M of R. Multiplication rings and almost multiplication rings have been extensively studied—for example, see [**4**; **8**; **9**; **11**; **12**; **15**; and **16**].

In Section 2 we investigate multiplication ideals. The key result is that a multiplication ideal in a quasi-local ring is principal. Multiplication ideals are then studied outside the quasi-local case.

In Section 3 multiplication rings and almost multiplication rings are studied. We characterize those almost multiplication rings having few zero-divisors. Finally we show that the polynomial ring $R[X]$ is an almost multiplication ring if and only if R is von Neumann regular.

In Section 4 we consider the ring $R(X)$. We show that R is an (almost) multiplication ring if and only if $R(X)$ is an (almost) multiplication ring. We also show that if R is an arithmetical ring, then $R(X)$ is a Bézout ring and that R and $R(X)$ have isomorphic lattices of ideals. Conversely, if R and $R(X)$ have isomorphic lattices of ideals, then R is arithmetical.

2. Multiplication ideals. W. W. Smith [**17**] has shown that a finitely generated multiplication ideal in a quasi-local ring is principal. Our first theorem states that every multiplication ideal in a quasi-local ring is principal.

THEOREM 1. *In a quasi-local ring every multiplication ideal is principal.*

Proof. Let (R, M) be a quasi-local ring and A a multiplication ideal in R. Suppose that $A = \sum (x_\alpha)$. Then $(x_\alpha) = AL_\alpha$ for some ideal L_α since A is a multiplication ideal. Hence $A = \sum (x_\alpha) = \sum AL_\alpha = A(\sum L_\alpha)$. If $\sum L_\alpha = R$, then $L_{\alpha_0} = R$ for some index α_0 because R is quasi-local. In this case $A = AL_{\alpha_0} = (x_{\alpha_0})$. If $\sum L_\alpha \neq R$, then $A = MA$. Suppose that $0 \neq x \in A$. Then there exists an ideal C such that $(x) = AC$. But then $(x) = AC = (MA)C = M(AC) = M(x)$, so $x = 0$ by Nakayama's Lemma.

Suppose that R is a commutative ring, S is a multiplicatively closed set in R, and that A is a multiplication ideal in R. Then A_S is a multiplication ideal in

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R_S. Indeed, suppose that $N \subseteq A_S$ is an ideal in R_S. Then $(N \cap R) \cap A \subseteq A$ so that $(N \cap R) \cap A = BA$ for some ideal B in R. Hence $N = N \cap A_S = ((N \cap R) \cap A)_S = B_S A_S$. In particular, if A is a multiplication ideal and P is a prime ideal in R, then by Theorem 1, A_P is a principal ideal in R_P.

From these remarks we draw several well-known conclusions. First, any localization of a multiplication ring is a multiplication ring. Theorem 1 yields that a quasi-local multiplication ring is a principal ideal ring and hence either a *DVR* or a special principal ideal ring. Thus an almost multiplication ring is simply a ring R such that for every maximal ideal M of R, R_M is either a DVR or a special principal ideal ring. Hence a multiplication ring is an almost multiplication ring and any localization of an almost multiplication ring is still an almost multiplication ring.

While a multiplication ideal A is locally principal (that is, A_M is a principal ideal in R_M for every maximal ideal M of R), a locally principal ideal need not be a multiplication ideal. However, it is easily verified that a finitely generated locally principal ideal is a multiplication ideal (see Theorem 3). For a discussion of such ideals, the reader is referred to [**3**] and [**13**]. As every ideal generated by idempotents is a multiplication ideal, one sees that a multiplication ideal need not be finitely generated. However, under circumstances somewhat more general than Theorem 1 we may still conclude that a multiplication ideal is principal.

LEMMA 1. *Let A be an ideal in a ring R such that $(O:A)$ is contained in only finitely many maximal ideals $M_1, \ldots, M_n$ of R. If A_{M_i} is a principal ideal in R_{M_i} for $i = 1, \ldots, n$, then A is a principal ideal in R.*

Proof. Let $A_{M_i} = (x_i)_{M_i}$, where $x_i \in A$. If M is a maximal ideal of R distinct from $M_1, \ldots, M_n$, then $A_M = O_M$. Hence $A_M = (x_1, \ldots, x_n)_M$ for all maximal ideals M of R so that $A = (x_1, \ldots, x_n)$. Choose $v_i \in \bigcap_{j \neq i} M_j - M_i$ and set $v = v_1 x_1 + \ldots + v_n x_n$. Then $A = (v)$ locally, and hence globally.

The next theorem is a generalization of both our Theorem 1 and Theorem 1 of [**3**].

THEOREM 2. *Let A be a multiplication ideal in a ring R such that $(O:A)$ is contained in only finitely many maximal ideals of R. Then A is a principal ideal.*

Proof. Theorem 2 follows from Lemma 1 and the fact that multiplication ideals are locally principal.

COROLLARY 2.1. *Let R be a semi-quasi-local ring and A an ideal in R. The following statements are equivalent:*

(1) *A is a multiplication ideal,*
(2) *A is a locally principal ideal, and*
(3) *A is a principal ideal.*

COROLLARY 2.2. *A semi-quasi-local (almost) multiplication ring is a principal ideal ring.*

Let R be a ring. An ideal A of R is called a *weak-cancellation ideal* if $AB \subseteq AC$ implies that $B \subseteq C + (O : A)$. Thus every principal ideal is a weak-cancellation ideal. A *cancellation ideal* is an ideal A of R such that the condition $AB = AC$ implies that $B = C$. Hence A is a cancellation ideal if and only if A is a weak-cancellation ideal with $(O : A) = 0$.

The following theorem characterizes finitely generated multiplication ideals. Janowitz in [**11**, page 655] remarked without proof that the implication (1) $\Rightarrow$ (2) is valid.

THEOREM 3. *For an ideal A in a commutative ring R, the following statements are equivalent:*

(1) *A is both a multiplication ideal and a weak-cancellation ideal,*

(2) *A is a finitely generated multiplication ideal, and*

(3) *A is finitely generated and locally principal.*

Proof. (1) $\Rightarrow$ (2): Suppose that $A = \sum (x_\alpha)$. Then $(x_\alpha) \subseteq A$ implies that $(x_\alpha) = AL_\alpha$ so that $AR = A = \sum (x_\alpha) = \sum AL_\alpha = A(\sum L_\alpha)$. Since A is a weak-cancellation ideal, $R = (\sum L_\alpha) + (O : A)$. Because R has an identity, $R = L_1 + \ldots + L_n + (O : A)$ for some finite subset $\{L_1, \ldots, L_n\}$ of $\{L_\alpha\}$. Hence $A = A(L_1 + \ldots + L_n + (O : A)) = (x_1) + \ldots + (x_n)$, so that A is finitely generated. The implication (2) $\Rightarrow$ (3) follows from Theorem 1 and the remark following it. The implication (3) $\Rightarrow$ (1) is given by McCarthy [**13**]. Briefly, if (3) holds, then (1) holds locally and hence globally since A is finitely generated.

3. Almost multiplication rings and multiplication rings.

Mott [**15**] has shown that a ring in which every prime ideal is a multiplication ideal is actually a multiplication ring. We offer a slight extension of this result.

THEOREM 4. *For a commutative ring R with identity, the following statements are equivalent:*

(1) *R is a multiplication ring,*

(2) *R is an almost multiplication ring all of whose maximal ideals are multiplication ideals,*

(3) *every prime ideal of R is a multiplication ideal, and*

(4) *every prime ideal that is either maximal or lies directly below a maximal ideal is a multiplication ideal.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. (2) $\Rightarrow$ (3): Because R is an almost multiplication ring, $\dim R \leqq 1$. Since, by hypothesis, the maximal ideals of R are multiplication ideals, we only need show that every nonmaximal minimal prime ideal is a multiplication ideal. So let $P \subsetneqq M$ be such a prime where M is a maximal ideal containing P and let $A \subseteq P$. Now $P_M = O_M$ and $P_N = R_N$ for all maximal ideals N of R not containing P. Thus $AP = A$ because the equality is true locally. The implication (3) $\Rightarrow$ (4) is clear. We show that (4) $\Rightarrow$ (3). It is sufficient to show that $\dim R \leqq 1$. Suppose that M

is a maximal ideal with rank $M \geqq 1$. Since M is a multiplication ideal, M_M is a nonminimal principal prime ideal in R_M. Thus $Q = \bigcap_{n=1}^{\infty} M_M{}^n$ is a prime ideal in R_M and every prime ideal of R_M other than M_M is contained in Q (this observation is due to J. Matijevic). Thus $Q \cap R$ lies directly below M. By hypothesis $Q \cap R$ is a multiplication ideal in R and hence $Q = (Q \cap R)_M$ is a principal prime ideal in R_M. Since $Q \subsetneqq M_M$ are principal prime ideals, $Q = M_M Q$ and thus $Q = O_M$ by Nakayama's Lemma. Thus rank $M = 1$ and R_M is a DVR. The implication (3) $\Rightarrow$ (1) is proved in [**15**].

A ring R is said to have *few zero-divisors* if $Z(R)$, the set of zero-divisors of R, is a finite union of prime ideals. Mott [**15**] has shown that an almost multiplication ring with few zero-divisors is a finite direct product of almost Dedekind domains and special principal ideal rings. (An almost multiplication ring without zero-divisors is called an *almost Dedekind domain.*) We give another characterization of such almost multiplication rings. We need the following lemma concerning the zero-divisors in an almost multiplication ring.

LEMMA 2. *Let R be an almost multiplication ring. An ideal not contained in any minimal prime ideal of R has zero annihilator. Thus $Z(R)$ is the union of the minimal prime ideals of R.*

Proof. Suppose that A is an ideal of R that is not contained in any minimal prime ideal of R. Suppose that $As = 0$ for some $s \in R$. Since A is not contained in any minimal prime ideal of R, s is contained in every minimal prime ideal of R, and hence s is nilpotent. Let M be a maximal ideal of R. We show that $s/1 = 0/1$ in R_M. In any case $s/1$ is nilpotent in R_M. Thus if rank $M = 1$, then $s/1 = 0/1$ since R_M is a DVR. Suppose that rank $M = 0$. Then $A_M = R_M$ and $A_M(s/1) = (0/1)$ implies that $s/1 = 0/1$ in R_M. The second statement follows from the first because any minimal prime ideal consists of zero-divisors.

THEOREM 5. *For an almost multiplication ring R, the following statements are equivalent:*

(1) *R has few zero-divisors,*
(2) *R has only finitely many minimal prime ideals,*
(3) *R is a finite direct product of special principal ideal rings and almost Dedekind domains, and*
(4) *the minimal prime ideals of R are finitely generated.*

Proof. The equivalence of (1) and (2) follows from Lemma 2 and the fact that any maximal ideal in an almost multiplication ring contains a unique minimal prime ideal. (2) $\Rightarrow$ (3): Suppose that $P_1, \ldots, P_n$ are the minimal prime ideals of R. Since each P_{iP_i} is nilpotent in R_{P_i}, there exists an integer s such that $(P_1 \ldots P_n)^s$ is locally zero and hence equal to the zero ideal of R. Hence (0) is a product of powers of the P_i's and $P_1, \ldots, P_n$ are comaximal. Thus R splits into the direct product $R \cong R/P_1{}^s \times \ldots \times R/P_n{}^s$. If P_i is a maximal ideal, then $R/P_i{}^s$ is a special principal ideal ring. If P_i is not a

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maximal ideal, then $P_i = P_i^s$ (for the equality holds locally) and $R/P_i^s = R/P_i$ is an almost Dedekind domain. (The equivalence of (2) and (3) is also proved in [**15**].) As (3) $\Rightarrow$ (4) is obvious, it remains to prove (4) $\Rightarrow$ (2). Suppose that R is an almost multiplication ring in which the minimal prime ideals are finitely generated. It suffices to show that the total quotient ring T of R is zero-dimensional. For then the prime ideals of T will all be extensions of the minimal prime ideals of R and hence will be finitely generated. Thus by Cohen's Theorem T will be Noetherian and hence will have only finitely many prime ideals. Let M be a rank one prime ideal in R; we show that M contains a non-zero-divisor. Let P be the unique minimal prime contained in M. By hypothesis, P is finitely generated. Also, $P = P^2$. Hence $P = (p)$ where p is idempotent. Now $(1 - p)M \neq 0$, so there exists an $m \in M - (p)$ such that $(1 - p)m \neq 0$. Let $x = (1 - p)m + p$ so that $x \in M - (p)$. If $x \in Z(R)$, then by Lemma 2, x belongs to a minimal prime Q of R distinct from $P = (p)$. But then $p = px \in Q$, a contradiction.

We end this section by characterizing the multiplication rings and almost multiplication rings which are polynomial rings. A ring R is called an *arithmetical ring* if $L(R)$, its lattice of ideals, is distributive.

THEOREM 6. *For a commutative ring R the following statements are equivalent:*
(1) *R is von Neumann regular,*
(2) *$R[X]$ is an almost multiplication ring, and*
(3) *$R[X]$ is an arithmetical ring.*

Proof. (1) $\Rightarrow$ (2). Suppose that R is von Neumann regular and let M be a maximal ideal in $R[X]$. Then $P = M \cap R$ is a maximal ideal in R and hence R_P is a field. Thus $R[X]_M \cong (R_P[X])_M$ is a localization of a *PID* and hence a *DVR*. (2) $\Rightarrow$ (3). Clearly any almost multiplication ring has a distributive lattice of ideals since locally its lattice of ideals is totally ordered. (3) $\Rightarrow$ (1). Suppose that $R[X]$ has a distributive lattice of ideals. Then for $a \in R$, $(a) = (a) \cap \{(X - a) + (X)\} = (a) \cap (X - a) + (a) \cap (X)$, so $a = f(X)(X - a) + g(X)X$, where $f(X)$, $g(X) \in R[X]$ and $f(X)(X - a) \in (a)$, $g(X)X \in (a)$. Let $f(X) = b_0X^n + \ldots + b_n$, then $b_0X^{n+1} + (b_1 - ab_0)X^n + \ldots + (b_n - ab_{n-1})X - b_na = f(X)(X - a) \in (a)$. Thus $(b_n - ab_{n-1})X \in (a)$ so $b_n \in aR$. Letting $b_n = ra$, $a = -b_na = a(-r)a$. Thus R is von Neumann regular.

The implication (3) $\Rightarrow$ (1) is found in Camillo [**5**], but our proof is simpler. The equivalence of (1) and (3) also occurs as an exercise in [**7**, page 321]. It follows from Theorem 3.2 [**18**] that $R[X]$ is Bézout whenever R is von Neumann regular.

COROLLARY 6.1. *For a ring R, the following conditions are equivalent:*
(1) *R is a finite direct product of fields,*
(2) *$R[X]$ is a multiplication ring.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Conversely suppose that $R[X]$ is a multiplication ring. Theorem 6 implies that R is von Neumann regular. Let M be a maximal ideal in R. It suffices to show that M is finitely generated, for Cohen's Theorem then implies that R is Noetherian. The ideal $M' = MR[X] + (X)$ is a rank one prime ideal in $R[X]$. Thus M' is locally a cancellation ideal, and hence a cancellation ideal. By Theorem 3, M' is finitely generated. (In particular any rank one prime ideal in a multiplication ring is finitely generated.) Hence M is finitely generated.

Let R be a non-Noetherian von Neumann regular ring. Then $R[X]$ is an almost multiplication ring which is not a multiplication ring. We note that $R[X]$ is locally a *DVR*. In fact, $R[X]$ is semihereditary [**14**]. Also note that for every minimal prime ideal P in $R[X]$, $R[X]/P = R[X]/(P \cap R)R[X] \cong (R/P \cap R)[X]$ is a *PID*.

4. The ring R(X). Let R be a ring and let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a set of indeterminates over R. For $f \in R[\{X_\alpha\}]$ we let $C(f)$ be the ideal of R generated by the coefficients of R. Let $S = \{f \in R[\{X_\alpha\}] | C(f) = R\}$. Then $S = R[\{X_\alpha\}] - \bigcup \{MR[\{X_\alpha\}] | M$ is a maximal ideal in $R\}$ is a multiplicatively closed set consisting entirely of regular elements. The ring $R[\{X_\alpha\}]_S$ is denoted by $R(\{X_\alpha\})$. For properties of $R(\{X_\alpha\})$, the reader is referred to [**7**]. While all the results of this section are true for $R(\{X_\alpha\})$, where $\{X_\alpha\}$ is an arbitrary set of indeterminates, for simplicity of notation we only state our results for $R(X)$. The following proposition is well-known.

PROPOSITION 1. *Let R be a commutative ring. Then*

(1) *There is a one-to-one correspondence between the (minimal prime) maximal ideals of R and the (minimal prime) maximal ideals of $R(X)$ given by $M \leftrightarrow MR(X)$.*

(2) *If Q is an ideal of R, then $QR(X) \cap R = Q$. If Q is P-primary, then $QR(X)$ is $PR(X)$-primary.*

The following theorem, while probably well-known, could not be found in the literature.

THEOREM 7. *Let $f \in R[X]$ be a polynomial with $C(f)$ locally principal. Then $C(f)R(X) = fR(X)$. If $g \in R[X]$ satisfies $C(g) \subseteq C(f)$, then $gR(X) \subseteq fR(X)$.*

Proof. By localization we may assume that R is quasi-local and that $C(f)$ is a principal ideal in R. Let $f = a_0 + a_1X + \ldots + a_nX^n$, so that $C(f) = (a_0, \ldots, a_n)$. Since $C(f)$ is principal, $C(f) = (a_{i_0})$ for some i_0 with $1 \leqq i_0 \leqq n$. Let $a_i = r_i a_{i_0}$ and $h = r_0 + r_1X + \ldots + r_nX^n$. Note that $h \in S$ because $r_{i_0} = 1$. Hence $C(f)R(X) = (a_{i_0})R(X) = (a_{i_0})hR(X) = fR(X)$. Suppose that $C(g) \subseteq C(f)$. Let $g = c_0 + c_1X + \ldots + c_mX^m$. Then $(c_i) \subseteq C(g) \subseteq C(f) = (a_{i_0})$, so that $c_i = e_i a_{i_0}$ for $i = 1, \ldots, m$. Hence $g \in (a_{i_0})R(X) = fR(X)$.

COROLLARY 7.1. *Let A be a finitely generated locally principal ideal in R. Then $AR(X)$ is a principal ideal in $R(X)$.*

Proof. Let $A = (a_0, \ldots, a_n)$. If $f = a_0 + a_1X + \ldots + a_nX^n$, then $C(f) = A$. By Theorem 7, $AR(X) = C(f)R(X) = fR(X)$.

Our next theorem gives another construction for multiplication rings and almost multiplication rings. A ring in which every finitely generated ideal is principal is called a *Bézout ring*. We note that (1) of Theorem 8 is proved in the domain case by Arnold [**2**]. Another generalization of Arnold's results is found in [**10**].

THEOREM 8. *Let R be a commutative ring.*

(1) *R is an arithmetical ring if and only if $R(X)$ is an arithmetical ring, and in that case $R(X)$ is actually a Bézout ring.*

(2) *If R is an arithmetical ring, then the map $\theta : L(R) \to L(R(X))$ given by $\theta(A) = AR(X)$ is a lattice isomorphism which preserves multiplication. Conversely, if θ is surjective, then R is arithmetical.*

(3) *R is an almost multiplication ring if and only if $R(X)$ is an almost multiplication ring.*

(4) *R is a multiplication ring if and only if $R(X)$ is a multiplication ring.*

Proof. (1). Suppose that R is an arithmetical ring. We show that $R(X)$ is a Bézout ring. Let $0 \neq f_1, f_2 \in R[X]$ and define $f = f_1 + X^n f_2$, where $n =$ (degree of f_1) $+ 1$. Then $C(f) = C(f_1) + C(f_2)$, and hence $f_1R(X) + f_2R(X) = C(f_1)R(X) + C(f_2)R(X) = C(f)R(X)$ is principal. Thus any finitely generated ideal in $R(X)$ is principal. Conversely, suppose that $R(X)$ is arithmetical. Let A, B and C be three ideals in R. Then $\{A \cap (B + C)\}R(X) = AR(X) \cap \{BR(X) + CR(X)\} = AR(X) \cap BR(X) + AR(X) \cap CR(X) = (A \cap B + A \cap C)R(X)$. Contracting back into R yields $A \cap (B + C) = A \cap B + A \cap C$.

(2). The map θ is one-to-one by Proposition 1. It is easily seen that θ preserves order, arbitrary sums, finite intersections, and products. Moreover θ is onto because $fR(X) = C(f)R(X)$ for every $f \in R[X]$ (Theorem 7). Hence θ preserves arbitrary intersections and thus θ is a complete lattice isomorphism. Conversely suppose that the map $\theta : L(R) \to L(R(X))$ is surjective. Then for each maximal ideal M of R, the map $\theta : L(R_M(X)) \to L(R_M(X))$ is a surjection. Hence we may assume that R is quasi-local. Let $a, b \in R$; we show that (a, b) is principal and hence that R is Bézout. Now $(a + bX)R(X) = BR(X)$ for some ideal B in R. Since R is quasi-local, one sees that B must actually be principal, say $B = (c)$, so that $(a + bX)R(X) = cR(X)$. Hence $a + bX = c(f/g)$ where $f, g \in R[X]$ with $C(g) = R$. Let $g = a_0 + a_1X + \ldots + a_nX^n$ and assume that i_0 is the greatest integer such that a_{i_0} is a unit. (Since R is quasi-local and $C(g) = R$, some coefficient of g must be a unit). We may assume that $a_{i_0} = 1$. Let $f = b_0 + b_1X + \ldots + b_mX^m$. Hence

$$aa_0 + (aa_1 + ba_0)X + \ldots + (aa_n + ba_{n-1})X^n + ba_nX^{n+1} = g(a + bX)$$
$$= cf = cb_0 + cb_1 + \ldots + cb_mX^m.$$

Hence $aa_{i_0+1} + ba_{i_0} = cb_{i_0+1}$ and $aa_{i_0} + ba_{i_0-1} = cb_{i_0}$. (If $i_0 = n$, then we take $a_{i_0+1} = 0$). Substituting $a_{i_0} = 1$ yields $b = -aa_{i_0+1} + cb_{i_0+1}$ and $a = -ba_{i_0-1} + cb_{i_0}$.

Substituting the value for b in the second equation gives $(1 - a_{i_0+1}a_{i_0-1})a \in (c)$ and hence $a \in (c)$ since $1 - a_{i_0+1}a_{i_0-1}$ is a unit. Hence also $b \in (c)$. Set $a = a'c$ and $b = b'c$ so that $a + bX = c(a' + b'X)$. Hence $(cR(X))(a' + b'X)R(X) = (a + bX)R(X) = cR(X)$. By Nakayama's Lemma, $(a' + b'X)R(X) = R(X)$. Hence $(a', b') = C(a' + b'X) = R$. Thus $(a, b) = c(a', b') = (c)$.

To prove (3) we make the following observation. For M a maximal ideal in R, the rings $R(X)_{MR(X)}$, $R_M(X)$, $R[X]_{MR[X]}$, and $R_M[X]_{M_MR_M[X]}$ are all naturally isomorphic. (Thus when dealing with the ring $R(X)$ we can often reduce to the case where R is quasi-local.) Hence the rings $R(X)_{MR(X)}$ and R_M are simultaneously principal ideal rings and therefore the rings R and $R(X)$ are simultaneously almost multiplication rings. To complete the proof of Theorem 8, we observe that (4) follows immediately from (2).

COROLLARY 8.1. *Given any arithmetical ring R, there exists a Bézout ring R' such that R and R' have isomorphic lattices of ideals.*

Corollary 8.1 generalizes the result in [**1**] that given a Prüfer domain D there exists a Bézout domain D' such that D and D' have isomorphic lattices of ideals. A different method of proof was used, however. Combining implications (1) through (4) we see that given an (almost) multiplication ring R there exists an (almost) multiplication ring R' such that R' is a Bézout ring and R and R' have isomorphic lattices of ideals. Of special interest is the case where D is a Dedekind domain. Then $D(X)$ is a *PID* and infact an Euclidean domain as follows from theorem 5.3 [**6**].

The $R(X)$ construction may be extended to R-modules. Let A be an R-module. With the notation preceding Proposition 1, we define $A(X) = A[X]_S$ and note that $A(X) \approx A \otimes R(X)$ is an $R(X)$-module. Theorem 8 generalized to modules yields

THEOREM 9. *Let R be a commutative ring and A an R-module.*

(1) *A is arithmetical if and only if $A(X)$ is arithmetical and in this case $A(X)$ is actually Bézout.*

(2) *If A is arithmetical, then the map $\theta : L_R(A) \to L_{R(X)}(A(X))$ given by $\theta(N) = R(X)N$ is a lattice isomorphism which preserves the scalar product. Conversely, if θ is surjective, then A is arithmetical.*

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