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A THEOREM OF PHRAGMÉN-LINDELÖF TYPE FOR SUBFUNCTIONS IN A CONE*

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Abstract. For a subfunction u, associated with the stationary Schrödinger operator, which is dominated on the boundary by a certain function on a cone, we generalise the classical Phragmén-Lindelöf theorem by making an a-harmonic majorant of u.

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1. Introduction and main results. Let $\mathbf{S}$ be an open set in $\mathbf{R}^n (n \geq 2)$, where $\mathbf{R}^n$ is the n-dimensional Euclidean space. The boundary and the closure of $\mathbf{S}$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively. In cartesian coordinate, a point P is denoted by (X, x_n), where $X = (x_1, x_2, \ldots, x_{n-1})$. Let $|P|$ be the Euclidean norm of P. Also denote $|P - Q|$ be the Euclidean distance of two points P and Q in $\mathbf{R}^n$.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with centre at P and radius r in $\mathbf{R}^n$. $S_r = \partial B(O, r)$.

A system of spherical coordinates for $P = (X, x_n)$ is given by

$$|P| = r, \qquad x_1 = r\prod_{i=1}^{n-1}\sin\theta_i \ \ (n \geq 2), \qquad x_n = r\cos\theta_1,$$

and if $n > 2$, then

$$x_{n-j+1} = r\cos\theta_j \prod_{i=1}^{j-1}\sin\theta_i \ \ (2 \leq j \leq n-1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n > 2$, then $0 \leq \theta_i \leq \pi$ $(1 \leq i \leq n-2)$.

Relative to this system, the Laplace operator Δ may be written

$$\Delta = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Delta^*}{r^2},$$

where the explicit form of the Beltrami operator Δ^* is given by V. Azarin (see [**2**]).

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Let D be an arbitrary domain in $\mathbf{R}^n$ and $\mathscr{A}_a$ denote the class of non-negative radial potentials $a(P)$, i.e. $0 \leq a(P) = a(r)$, $P = (r, \Theta) \in D$, such that $a \in L^b_{loc}(D)$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

If $a \in \mathscr{A}_a$, then the stationary Schrödinger operator

$$Sch_a = -\Delta + a(P)I = 0,$$

where Δ is the Laplace operator and I is the identical operator, can be extended in the usual way from the space $C_0^\infty(D)$ to an essentially self-adjoint operator on $L^2(D)$ (see [**7**, Ch. 13]). We will denote it Sch_a as well. This last one has a Green a-function $G^a_D(P, Q)$. Here, $G^a_D(P, Q)$ is positive on D and its inner normal derivative $\partial G^a_D(P, Q)/\partial n_Q \geq 0$, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into D. We denote this derivative $PI^a_D(P, Q)$, which is called the Poisson a-kernel with respect to D.

In the proof, we need inequalities between Green a-function $G^a_D(P, Q)$ and that of the Laplacian, hereafter denoted by $G^0_D(P, Q)$. It is well known that, for any potential $a(P) \geq 0$,

$$G^a_D(P, Q) \leq G^0_D(P, Q). \tag{1.1}$$

The inverse inequality is much more elaborate if D is a bounded domain in $\mathbf{R}^n$. Cranston, Fabes and Zhao (see [**4**], the case $n = 2$ is implicitly contained in [**3**]) have proved

$$G^a_D(P, Q) \geq M(D)G^0_D(P, Q), \tag{1.2}$$

where D is a bounded domain, a constant $M(D) = M(D, a(P))$ is positive and does not depend on points P and Q in D. If $a = 0$, then obviously, $M(D) \equiv 1$.

We call a function $u \not\equiv -\infty$ that is upper semi-continuous in D a subfunction of the Schrödinger operator Sch_a if its values belong to the interval $(-\infty, +\infty)$ and at each point $P \in D$ with $0 < r < r(P)$ the generalised mean-value inequality

$$u(P) \leq \int_{S(P,r)} u(Q)\frac{\partial G^a_{B(P,r)}(P, Q)}{\partial n_Q} d\sigma(Q)$$

is satisfied, where $S(P, r) = \partial B(P, r)$, $G^a_{B(P,r)}(P, Q)$ is the Green a-function of Sch_a in $B(P, r)$ and $d\sigma(Q)$ is the surface area element on $S(P, r)$.

The class of subfunctions in D is denoted by $SbH(a, D)$. If $-u \in SbH(a, D)$, then we call u a superfunction and denote the class of superfunctions by $SpH(a, D)$. If a function u is both subfunction and superfunction, it is, clearly, continuous and is called an a-harmonic function associated with the operator Sch_a. The class of a-harmonic functions is denoted by $H(a, D) = SbH(a, D) \cap SpH(a, D)$. In terminology, we follow B. Ya. Levin and A. Kheyfits (see [**6**]).

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$.

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$. We call it a cone. Then T_n is a special cone obtained by putting $\Omega = \mathbf{S}^{n-1}_+$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$, we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$, we denote $S_n(\Omega; (0, +\infty))$, which is $\partial C_n(\Omega) - \{O\}$. Furthermore, we denote by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r.

For positive functions h_1 and h_2, we say that $h_1 \lesssim h_2$ if $h_1 \leq Mh_2$ for some constant $M > 0$. If $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$, we say that $h_1 \approx h_2$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary and λ be the least positive eigenvlaue for Δ^* on Ω (see [**8**, p. 41])

$$(\Delta^* + \lambda)\varphi(\Theta) = 0 \quad \text{on} \quad \Omega,$$

$$\varphi(\Theta) = 0 \quad \text{on} \quad \partial\Omega.$$

Corresponding eigenfunction is denoted by $\varphi(\Theta)$, $\int_\Omega \varphi^2(\Theta) dS_1 = 1$. In order to ensure the existence of λ and a smooth $\varphi(\Theta)$. We put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [**5**, pp. 88–89] for the definition of $C^{2,\alpha}$-domain).

Solutions of an ordinary differential equation

$$-Q''(r) - \frac{n-1}{r} Q'(r) + \left(\frac{\lambda}{r^2} + a(r) \right) Q(r) = 0, \quad 0 < r < \infty. \tag{1.3}$$

It is known (see, for example, [**11**]) that if the potential $a \in \mathscr{A}_a$, then the equation (1.3) has a fundamental system of positive solutions $\{V, W\}$ such that V is non-decreasing with

$$0 \leq V(0+) \leq V(r) \quad \text{as} \quad r \to +\infty,$$

and W is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \quad \text{as} \quad r \to +\infty.$$

We will also consider the class $\mathscr{B}_a$, consisting of the potentials $a \in \mathscr{A}_a$ such that there exists the finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0, \infty)$, moreover, $r^{-1}|r^2 a(r) - k| \in L(1, \infty)$. If $a \in \mathscr{B}_a$, then the (super)subfunctions are continuous (see [**10**]).

In the rest of paper, we assume that $a \in \mathscr{B}_a$ and we shall suppress this assumption for simplicity.

From now on, we always assume $D = C_n(\Omega)$. For the sake of brevity, we shall write $G^a_\Omega(P, Q)$ instead of $G^a_{C_n(\Omega)}(P, Q)$, $PI^a_\Omega(P, Q)$ instead of $PI^a_{C_n(\Omega)}(P, Q)$, $SpH(a)$ (resp. $SbH(a)$) instead of $SpH(a, C_n(\Omega))$ (resp. $SbH(a, C_n(\Omega))$) and $H(a)$ instead of $H(a, C_n(\Omega))$.

Denote

$$\iota_k^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k+\lambda)}}{2},$$

then the solutions to the equation (1.3) have the asymptotic (see [**5**])

$$V(r) \approx r^{\iota_k^+}, \quad W(r) \approx r^{\iota_k^-}, \quad \text{as} \quad r \to \infty. \tag{1.4}$$

REMARK 1. If $a = 0$ and $\Omega = \mathbf{S}_+^{n-1}$, then $\iota_0^+ = 1$, $\iota_0^- = 1 - n$ and $\varphi(\Theta) = (2ns_n^{-1})^{1/2} cos\theta_1$, where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

Let $u(r, \Theta)$ be a function on $C_n(\Omega)$. We introduce $M_u(r) = M(r, u) = \sup_{\Theta\in\Omega} u(r, \Theta)$, $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

We shall say that $u(P)$ $(P = (r, \Theta))$ satisfies the Phragmén-Lindelöf boundary condition on $S_n(\Omega)$, namely,

$$\limsup_{P=(r,\Theta)\in C_n(\Omega), P\to Q\in S_n(\Omega)} u(P) \leq 0. \tag{1.5}$$

For any given positive real number r, the integral

$$\int_\Omega u(r, \Theta)\varphi(\Theta) dS_1,$$

is denoted by $N_u(r)$, when it exists. The finite or infinite limit

$$\lim_{r\to\infty} \frac{N_u(r)}{V(r)} \qquad \left(\text{resp. } \lim_{r\to 0} \frac{N_u(r)}{W(r)}\right)$$

is denoted by $\mathcal{V}_u$ (resp. $\mathcal{W}_u$), when it exists.

If $f(l)$ is a real finite-valued function defined on an interval $(0, +\infty)$, then for any given l_1, l_2 $(0 < l_1 < l_2 < \infty)$ and $l \in (0, +\infty)$, we have

$$\mathscr{E}(l; f, V, W, l_1, l_2) = \begin{vmatrix} f(l) & V(l) & W(l) \\ f(l_1) & V(l_1) & W(l_1) \\ f(l_2) & V(l_2) & W(l_2) \end{vmatrix} \geq 0$$

if and only if

$$f(l) \leq \mathscr{F}(l; f, V, W, l_1, l_2),$$

where $\mathscr{F}(l; f, V, W, l_1, l_2)$ has the following expression:

$$\left\{\frac{W(l)}{W(l_1)} f(l_1)\left(\frac{V(l_2)}{W(l_2)} - \frac{V(l)}{W(l)}\right) + \frac{W(l)}{W(l_2)} f(l_2)\left(\frac{V(l)}{W(l)} - \frac{V(l_1)}{W(l_1)}\right)\right\}\left\{\frac{V(l_2)}{W(l_2)} - \frac{V(l_1)}{W(l_1)}\right\}^{-1}.$$

We shall say that $f(l)$ is (V, W)-convex on $(0, +\infty)$ if $\mathscr{E}(l; f, V, W, l_1, l_2) \geq 0$ $(l_1 \leq l \leq l_2)$ for any l_1, l_2 $(0 < l_1 < l_2 < +\infty)$.

REMARK 2. A function $f(l)$ is (V, W)-convex on $(0, +\infty)$ if and only if $W^{-1}(l)f(l)$ is a convex function of $W^{-1}(l)V(l)$ on $(0, +\infty)$, or, equivalently, if and only if $V^{-1}(l)f(l)$ is a convex function of $V^{-1}(l)W(l)$ on $(0, +\infty)$.

REMARK 3. If $f(l)$ is a (V, W)-convex function on $(0, +\infty)$, then for any l_1, l_2 $(0 < l_1 < l_2 < +\infty)$, we have $\mathscr{E}(l; f, V, W, l_1, l_2) \leq 0$, where $0 < l \leq l_1$ and $l_2 \leq l < +\infty$.

Let $g(Q)$ be a locally integrable function on $S_n(\Omega)$ such that

$$\int^\infty t^{-1} V^{-1}(t) \left(\int_{\partial\Omega} |g(t, \Phi)| d_{\sigma_\Phi}\right) dt < +\infty \tag{1.6}$$

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and

$$\int_0 t^{-1} W^{-1}(t) \left(\int_{\partial\Omega} |g(t, \Phi)| d_{\sigma_\Phi} \right) dt < +\infty, \tag{1.7}$$

where d_{σ_Φ} is the surface area element of $\partial\Omega$ at $\Phi \in \partial\Omega$.

The Poisson a-integral $PI^a_\Omega[g](P)$ of g relative to $C_n(\Omega)$ is defined by

$$PI^a_\Omega[g](P) = \frac{1}{c_n} \int_{S_n(\Omega)} PI^a_\Omega(P, Q) g(Q) d\sigma_Q,$$

where

$$PI^a_\Omega(P, Q) = \frac{\partial G^a_\Omega(P, Q)}{\partial n_Q}, \qquad c_n = \begin{cases} 2\pi & n = 2, \\ (n-2)s_n & n \geq 3, \end{cases}$$

$\frac{\partial}{\partial n_Q}$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$.

Our first aim is to be concerned with the solutions of the Dirichlet problem for the Schrödinger operator Sch_a on $C_n(\Omega)$ and the growth property of them.

THEOREM 1. *Let $g(Q)$ be a continuous function on $S_n(\Omega)$ satisfying (1.6)–(1.7). Then the function $PI^a_\Omega[g](P)$ $(P = (r, \Theta))$ satisfies*

$$PI^a_\Omega[g] \in C^2(C_n(\Omega)) \cap C^0(\overline{C_n(\Omega)}),$$

$$Sch_a PI^a_\Omega[g] = 0 \quad \text{in } C_n(\Omega),$$

$$PI^a_\Omega[g] = g \quad \text{on } \partial C_n(\Omega),$$

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} V^{-1}(r)\varphi^{n-1}(\Theta) PI^a_\Omega[g](P) = 0 \tag{1.8}$$

and

$$\lim_{r\to 0, P=(r,\Theta)\in C_n(\Omega)} W^{-1}(r)\varphi^{n-1}(\Theta) PI^a_\Omega[g](P) = 0. \tag{1.9}$$

REMARK 4. If $a = 0$, $\Omega = \mathbf{S}^{n-1}_+$ and g is a continuous function on ∂T_n satisfying $\int_{\partial T_n} |g(Y)|(1 + |Y|)^{-n} dY < +\infty$, we obtain from (1.4), Remark 1 and Theorem 1 that $PI^0_{\mathbf{S}^{n-1}_+}[g](x) = o(|x| \sec^{n-1}\theta_1)$ as $|x| \to \infty$ in T_n, which is just the result of Siegel-Talvila (see [**9**, Corollary 2.1]).

It is natural to ask if 0 in (1.5) can be replaced with a general function $g(Q)$ on $S_n(\Omega)$? The following Theorem 2 gives an affirmative answer to this question. For related results, we refer the readers to the paper by B. Ya. Levin and A. Kheyfits (see [**6**, Sec. 3]).

THEOREM 2. *Let $g(Q)$ be a continuous function on $S_n(\Omega)$ satisfying (1.6)–(1.7) and let $u(P)$ be a subfunction on $C_n(\Omega)$ such that*

$$\limsup_{P\in C_n(\Omega), P\to Q\in S_n(\Omega)} u(P) \leq g(Q). \tag{1.10}$$

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Then all of the limits $\mathcal{V}_u$, $\mathcal{W}_u$, $\mathcal{V}_{u^+}$ *and* $\mathcal{W}_{u^+}$ $(-\infty < \mathcal{V}_u, \mathcal{W}_u \leq +\infty, 0 \leq \mathcal{V}_{u^+}, \mathcal{W}_{u^+} \leq +\infty)$ *exist, and if*

$$\mathcal{V}_{u^+} < +\infty \quad and \quad \mathcal{W}_{u^+} < +\infty, \tag{1.11}$$

then

$$u(P) \leq PI^a_\Omega[g](P) + (\mathcal{V}_u V(r) + \mathcal{W}_u W(r))\varphi(\Theta) \tag{1.12}$$

for any $P = (r, \Theta) \in C_n(\Omega)$.

As an application of Theorems 1 and 2, we obtain the following result.

THEOREM 3. *Let* $g(Q)$ *be defined as in Theorem 2 and* $h(P)$ *be any solution of the Dirichlet problem for the Schrödinger operator* Sch_a *on* $C_n(\Omega)$ *with g. Then all of the limits* $\mathcal{V}_h$, $\mathcal{W}_h$, $\mathcal{V}_{|h|}$ *and* $\mathcal{W}_{|h|}$ $(-\infty < \mathcal{V}_h, \mathcal{W}_h \leq +\infty, 0 \leq \mathcal{V}_{|h|}, \mathcal{W}_{|h|} \leq +\infty)$ *exist, and if*

$$\mathcal{V}_{|h|} < +\infty \quad and \quad \mathcal{W}_{|h|} < +\infty, \tag{1.13}$$

then

$$h(P) = PI^a_\Omega[g](P) + (\mathcal{V}_h V(r) + \mathcal{W}_h W(r))\varphi(\Theta) \tag{1.14}$$

for any $P = (r, \Theta) \in C_n(\Omega)$.

REMARK 5. Theorems 2 and 3 for $a = 0$ are due to H. Yoshida (see [**13**, Theorems 2 and 3 (II)]).

2. Some Lemmas. In our discussions, the following estimates for the kernel functions $PI^a_\Omega(P, Q)$, $G^a_\Omega(P, Q)$ and $\partial G^a_{\Omega,R}(P, Q)/\partial R$ are fundamental, which follow from [**6**] and [**2**, Lemma 4 and Remark].

LEMMA 1.

$$PI^a_\Omega(P, Q) \approx t^{-1} V(t) W(r) \varphi(\Theta) \frac{\partial \varphi(\Phi)}{\partial n_\Phi}, \tag{2.1}$$

$$\left(resp.\ PI^a_\Omega(P, Q) \approx V(r) t^{-1} W(t) \varphi(\Theta) \frac{\partial \varphi(\Phi)}{\partial n_\Phi} \right) \tag{2.2}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ *and any* $Q = (t, \Phi) \in S_n(\Omega)$ *satisfying* $0 < \frac{t}{r} \leq \frac{4}{5}$ (*resp.* $0 < \frac{r}{t} \leq \frac{4}{5}$);

$$PI^0_\Omega(P, Q) \lesssim \frac{\varphi(\Theta)}{t^{n-1}} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} + \frac{r\varphi(\Theta)}{|P - Q|^n} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} \tag{2.3}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ *and any* $Q = (t, \Phi) \in S_n(\Omega; (\frac{4}{5}r, \frac{5}{4}r))$.

LEMMA 2. *If* $h(r, \Theta)$ *is an a-harmonic function on* $C_n(\Omega)$ *vanishing continuously on* $S_n(\Omega)$*, then*

$$\mathscr{E}(r; N_h, V, W, r_1, r_2) = 0$$

for any r_1, r_2 $(0 < r_1 < r_2 < +\infty)$ *and every* r $(0 < r < +\infty)$.

Proof. Making use of the assumptions on h and self-adjoint of the Laplace-Beltrami operator Δ^*, one can check directly (by differentiating under the integral sign) that the functions $N_h(r)$ satisfy the equation (1.3). This equation has a general solution

$$N_h(r) = AV(r) + BW(r),$$

where $r \in (0, +\infty)$, A and B are two constants. Since $N_h(r)$ takes value $N_h(r_i)$ $(i = 1, 2)$, then

$$N_h(r) = \mathscr{F}(r; N_h, V, W, r_1, r_2),$$

which gives the conclusion of Lemma 2. □

LEMMA 3. *If $f(l)$ is (V, W)-convex on $(0, d_1)$ $(0 < d_1 \leq +\infty)$, then*

$$\beta = \lim_{l\to 0} \frac{f(l)}{W(l)} \quad (-\infty < \alpha \leq +\infty),$$

exists. Further, if $\beta \leq 0$, then $V^{-1}(l)f(l)$ is non-decreasing on $(0, d_1)$.

Proof. Put

$$G(s) = \frac{f(l(s))}{V(l(s))} \quad \text{on} \quad (l^{-1}(d_1), +\infty),$$

where $W(l(s)) = sV(l(s))$, l^{-1} denotes the inverse $l(s)$ (see [**6**, Appendix C] for the existence of $l(s)$). Notice that $l \to 0$ as $s \to \infty$. Then $G(s)$ is a convex function on $(l^{-1}(d_1), +\infty)$ from Remark 2. Hence by Lemma 3.1 (see [**12**, p. 275])

$$\beta = \lim_{s\to\infty} \frac{G(s)}{s} = \lim_{s\to\infty} \frac{f(l(s))}{W(l(s))} = \lim_{l\to 0} \frac{f(l)}{W(l)} \quad (-\infty < \beta \leq +\infty)$$

exists. Further, if $\beta \leq 0$, then $G(s)$ is non-increasing and hence $V^{-1}(l)f(l)$ is non-decreasing on $(0, d_1)$. Thus, we complete the proof of Lemma 3. □

It is known that $C_n(\Omega)$ is regular, the Dirichlet problem for Δ and Sch_a is solvable in it (see [**6**]). Based on this fact, Lemmas 4, 5 and 6 could be derived from (1.1), (1.2), (1.4), Remarks 2 and 3, Lemmas 2 and 3 with its means of proof essentially due to H. Yoshida (see [**12**, Theorems 3.1, 5.1] and [**13**, Lemma 3]). Herein, we remove its detailed proof information.

LEMMA 4. *If $u(r, \Theta)$ is a subfunction on $C_n(\Omega)$ satisfying the Phragmén-Lindelöf boundary condition on $S_n(\Omega)$, then*

$$N_u(r) > -\infty$$

for $0 < r < +\infty$ and $N_u(r)$ is (V, W)-convex on $(0, +\infty)$. If there are three numbers r_1, r_2 and r_0 satisfying $0 < r_1 < r_0 < r_2 < +\infty$ such that

$$\mathscr{E}(r_0; N_u, V, W, r_1, r_2) = 0,$$

then we have that

(1) $\mathscr{E}(r; N_u, V, W, r_1, r_2) = 0 \quad (r_1 \leq r \leq r_2)$.

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(2) *$u(r,\Theta)$ is an a-harmonic function on $C_n(\Omega;(r_1,r_2))$ and vanishes continuously on $S_n(\Omega;(r_1,r_2))$.*

LEMMA 5. *Let $g(Q)$ be defined as in Theorem 2. Then $PI^a_\Omega[g](P)$ (resp. $PI^a_\Omega[|g|](P)$) is an a-harmonic function on $C_n(\Omega)$ such that both of the limits $\mathcal{V}_{PI^a_\Omega[g]}$ and $\mathcal{W}_{PI^a_\Omega[g]}$ (resp. $\mathcal{V}_{PI^a_\Omega[|g|]}$ and $\mathcal{W}_{PI^a_\Omega[|g|]}$) exist, and*

$$\mathcal{V}_{PI^a_\Omega[g]}=\mathcal{W}_{PI^a_\Omega[g]}=0 \quad (resp.\ \mathcal{V}_{PI^a_\Omega[|g|]}=\mathcal{W}_{PI^a_\Omega[|g|]}=0).$$

LEMMA 6. *Let $u(P)$ be a subfunction on $C_n(\Omega)$ satisfying the Phragmén-Lindelöf boundary condition on $S_n(\Omega)$. If (1.11) is satisfied, then*

$$u(P)\le(\mathcal{V}_uV(r)+\mathcal{W}_uW(r))\varphi(\Theta)$$

for any $P=(r,\Theta)\in C_n(\Omega)$.

*By the Kelvin transformation (see [**1**, p. 59]), Lemmas 3 and 4, we immediately have the following result, which is due to H. Yoshida in the case $a=0$ (see [**12**, Theorem 3.3]).*

LEMMA 7. *Let $u(P)$ be defined as in Lemma 6. Then*

(1) *Both of the limits $\mathcal{V}_u$ and $\mathcal{W}_u$ $(-\infty<\mathcal{V}_u,\mathcal{W}_u\le+\infty)$ exist.*

(2) *If $\mathcal{W}_u\le 0$, then $V^{-1}(r)N_u(r)$ is non-decreasing on $(0,+\infty)$.*

(3) *If $\mathcal{V}_u\le 0$, then $W^{-1}(r)N_u(r)$ is non-increasing on $(0,+\infty)$.*

3. Proof of the Theorem 1. For any fixed $P=(r,\Theta)\in C_n(\Omega)$, take two numbers R_1, R_2 satisfying $R_1<\frac{4}{5}r$, $R_2>\frac{5}{4}r$. By Lemma 1, we have

$$\frac{1}{c_n}\int_{S_n(\Omega;(R_2,+\infty))}PI^a_\Omega(P,Q)|g(Q)|d\sigma_Q\lesssim V(r)\varphi(\Theta)\int_{R_2}^{+\infty}t^{-1}V^{-1}(t)\left(\int_{\partial\Omega}|g(t,\Phi)|d_{\sigma_\Phi}\right)dt$$

and

$$\frac{1}{c_n}\int_{S_n(\Omega;(0,R_1))}PI^a_\Omega(P,Q)|g(Q)|d\sigma_Q\lesssim W(r)\varphi(\Theta)\int_0^{R_1}t^{-1}W^{-1}(t)\left(\int_{\partial\Omega}|g(t,\Phi)|d_{\sigma_\Phi}\right)dt.$$

Thus $PI^a_\Omega[g](P)$ is finite for any $P\in C_n(\Omega)$ for (1.6) and (1.7). Since $PI^a_\Omega(P,Q)$ is an a-harmonic function of $P\in C_n(\Omega)$ for any $Q\in S_n(\Omega)$, $PI^a_\Omega[g](P)\in H(a)$.

Now we study the boundary behaviour of $PI^a_\Omega[g](P)$. Let $Q'=(t',\Phi')\in S_n(\Omega)$ be any fixed point and L be any positive number such that $L>\max\{t'+1,\frac{4}{5}R_2\}$.

Set $\chi_{S(L)}$ is the characteristic function of $S(L)=\{Q=(t,\Phi);Q\in S_n(\Omega;[R_1,\frac{5}{4}L])\}$ and write

$$PI^a_\Omega[g](P)=PI^a_{\Omega,1}[g](P)+PI^a_{\Omega,2}[g](P)+PI^a_{\Omega,3}[g](P),$$

where

$$PI^a_{\Omega,1}[g](P)=\frac{1}{c_n}\int_{S_n(\Omega;(0,R_1))}PI^a_\Omega(P,Q)g(Q)d\sigma_Q,$$

$$PI^a_{\Omega,2}[g](P)=\frac{1}{c_n}\int_{S_n(\Omega;(0,[R_1,\frac{5}{4}L]))}PI^a_\Omega(P,Q)g(Q)d\sigma_Q$$

and

$$PI^a_{\Omega,3}[g](P) = \frac{1}{c_n}\int_{S_n(\Omega;(\frac{5}{4}L,\infty))} P^a_\Omega(P,Q)g(Q)d\sigma_Q.$$

Notice that $PI^a_{\Omega,2}[g](P)$ is the Poisson a-integral of $g(Q)\chi_{S(L)}$, we have

$$\lim_{P\in C_n(\Omega), P\to Q'\in S_n(\Omega)} PI^a_{\Omega,2}[g](P) = g(Q').$$

$PI^a_{\Omega,1}[g](P) = O(W(r)\varphi(\Theta))$ and $PI^a_{\Omega,3}[g](P) = O(V(r)\varphi(\Theta))$, which tend to zero from $\lim_{\Theta\to\Phi'}\varphi(\Theta) = 0$. So the function $PI^a_\Omega[g](P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P\in C_n(\Omega), P\to Q'\in S_n(\Omega)} PI^a_\Omega[g](P) = g(Q')$$

from the arbitrariness of L.

For any $\epsilon > 0$, there exists $R_\epsilon > 1$ such that

$$\int_{R_\epsilon}^\infty t^{-1}V^{-1}(t)\left(\int_{\partial\Omega}|g(t,\Phi)|d_{\sigma_\Phi}\right)dt < \epsilon. \tag{3.1}$$

Take any point $P = (r,\Theta) \in C_n(\Omega)$ such that $r > \frac{5}{4}R_\epsilon$, and write

$$PI^a_\Omega[g](P) \lesssim PI_1(P) + PI_2(P) + PI_3(P) + PI_4(P) + PI_5(P),$$

where

$PI_1(P) = \int_{S_n(\Omega;(0,1])} |PI^a_\Omega(P,Q)||g(Q)|d\sigma_Q,$
$PI_2(P) = \int_{S_n(\Omega;(1,R_\epsilon])} |PI^a_\Omega(P,Q)||g(Q)|d\sigma_Q,$
$PI_3(P) = \int_{S_n(\Omega;(R_\epsilon,\frac{4}{5}r])} |PI^a_\Omega(P,Q)||g(Q)|d\sigma_Q,$
$PI_4(P) = \int_{S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} |PI^a_\Omega(P,Q)||g(Q)|d\sigma_Q,$
$PI_5(P) = \int_{S_n(\Omega;[\frac{5}{4}r,\infty))} |PI^a_\Omega(P,Q)||g(Q)|d\sigma_Q.$

By (1.4), (2.1), (2.2) and (3.1) we have the following growth estimates:

$$\begin{aligned} PI_2(P) &\lesssim W(r)\varphi(\Theta)\int_{S_n(\Omega;(1,R_\epsilon])} t^{-1}V(t)|g(Q)|d\sigma_Q \\ &\lesssim W(r)R_\epsilon^{2\iota_k^+ + n-2}\varphi(\Theta). \end{aligned} \tag{3.2}$$

$$PI_1(P) \lesssim W(r)\varphi(\Theta), \tag{3.3}$$

$$PI_3(P) \lesssim \epsilon V(r)\varphi(\Theta), \tag{3.4}$$

$$PI_5(P) \lesssim \epsilon V(r)\varphi(\Theta). \tag{3.5}$$

By (2.3), we consider the inequality

$$PI_4(P) \lesssim PI_{41}(P) + PI_{42}(P),$$

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where

$$PI_{41}(P) = \varphi(\Theta) \int_{S_n(\Gamma;(\frac{4}{5}r,\frac{5}{4}r))} \frac{V(t)W(t)}{t} |g(Q)| d\sigma_Q,$$

$$PI_{42}(P) = r\varphi(\Theta) \int_{S_n(\Gamma;(\frac{4}{5}r,\frac{5}{4}r))} \frac{|g(Q)|}{|P-Q|^n} d\sigma_Q.$$

We first have

$$PI_{41}(P) \lesssim \epsilon V(r)\varphi(\Theta) \tag{3.6}$$

from (3.1).

Next, we shall estimate $PI_{42}(P)$. Take a sufficiently small positive number d_2 such that $S_n(\Omega; (\frac{4}{5}r, \frac{5}{4}r)) \subset B(P, \frac{1}{2}r)$ for any $P = (r, \Theta) \in \Pi(d_2)$, where

$$\Pi(d_2) = \{P = (r, \Theta) \in C_n(\Omega);\ \inf_{z \in \partial\Omega} |(1, \Theta) - (1, z)| < d_2,\ 0 < r < \infty\}$$

and divide $C_n(\Omega)$ into two sets $\Pi(d_2)$ and $C_n(\Omega) - \Pi(d_2)$.

If $P = (r, \Theta) \in C_n(\Omega) - \Pi(d_2)$, then there exists a positive d_2' such that $|P - Q| \geq d_2' r$ for any $Q \in S_n(\Omega)$, and hence

$$PI_{42}(P) \lesssim \epsilon V(r)\varphi(\Theta). \tag{3.7}$$

We shall consider the case $P = (r, \Theta) \in \Pi(d_2)$. Now put

$$H_i(P) = \left\{Q \in S_n\left(\Omega; \left(\frac{4}{5}r, \frac{5}{4}r\right)\right);\ 2^{i-1}\delta(P) \leq |P - Q| < 2^i \delta(P)\right\},$$

where $\delta(P) = \inf_{Q \in \partial C_n(\Omega)} |P - Q|$.

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P - Q| < \delta(P)\} = \varnothing$, we have

$$PI_{42}(P) = \sum_{i=1}^{i(P)} \int_{H_i(P)} \frac{r\varphi(\Theta)}{|P-Q|^n} |g(Q)| d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \leq \frac{r}{2} < 2^{i(P)}\delta(P)$.

Since $r\varphi(\Theta) \leq \delta(P)$ $(P = (r, \Theta) \in C_n(\Omega))$, and hence by (3.1)

$$\int_{H_i(P)} \frac{r\varphi(\Theta)}{|P-Q|^n} |g(Q)| d\sigma_Q \lesssim V(r)\varphi^{1-n}(\Theta) \int_{S_n(\Omega;(\frac{4}{5}r,+\infty))} \frac{W(t)}{t} |g(Q)| d\sigma_Q$$
$$\lesssim V(r)\varphi^{1-n}(\Theta)\epsilon$$

for $i = 0, 1, 2, \ldots, i(P)$.

So

$$PI_{42}(P) \lesssim V(r)\varphi^{1-n}(\Theta)\epsilon. \tag{3.8}$$

Combining (3.2)–(3.8), (1.8) is proved.

Consider the Kelvin transformation (see [**1**, p. 59]) $K : (r, \Theta) \to (r^{-1}, \Theta)$ and apply (1.8) to the following function $u^*(r, \Theta) = r^{2-n}(u \circ K)(r, \Theta))$, we obtain (1.9) from (1.7).

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Thus we complete the proof of Theorem 1.

4. Proof of the Theorem 2. We remark that

$$\lim_{P\in C_n(\Omega),P\to Q\in S_n(\Omega)} PI^a_\Omega[g](P)=g(Q) \quad \text{and} \quad \lim_{P\in C_n(\Omega),P\to Q\in S_n(\Omega)} PI^a_\Omega[|g|](P)=|g(Q)| \tag{4.1}$$

from Theorem 1. For the two subfunctions

$$U(P)=u(P)-PI^a_\Omega[g](P) \quad \text{and} \quad U'(P)=u^+(P)-PI^a_\Omega[|g|](P)$$

on $C_n(\Omega)$, we have

$$\limsup_{P\in C_n(\Omega),P\to Q\in S_n(\Omega)} U(P)\le 0 \quad \text{and} \quad \limsup_{P\in C_n(\Omega),P\to Q\in S_n(\Omega)} U'(P)\le 0$$

from (1.10) and (4.1). Hence Lemma 7 (1) gives that the four limits $\mathcal{V}_U$, $\mathcal{W}_U$, $\mathcal{V}_{U'}$ and $\mathcal{W}_{U'}$ $(-\infty<\mathcal{V}_U,\mathcal{W}_U,\mathcal{V}_{U'},\mathcal{W}_{U'}\le+\infty)$ exist.

Since

$$N_U(r)=N_u(r)-N_{PI^a_\Omega[g]}(r) \quad \text{and} \quad N_{U'}(r)=N_{u^+}(r)-N_{PI^a_\Omega[|g|]}(r),$$

it follows that the four limits $\mathcal{V}_u$, $\mathcal{W}_u$, $\mathcal{V}_{u^+}$ and $\mathcal{W}_{u^+}$ exist and that

$$\mathcal{V}_U=\mathcal{V}_u,\quad \mathcal{W}_U=\mathcal{W}_u,\quad \mathcal{V}_{U'}=\mathcal{V}_{u^+},\quad \mathcal{W}_{U'}=\mathcal{W}_{u^+} \tag{4.2}$$

from Lemma 5.

Since

$$U^+(P)\le u^+(P)+(PI^a_\Omega[g])^-(P),$$

we have

$$\mathcal{V}_{U^+}\le\mathcal{V}_{u^+}<+\infty \quad \text{and} \quad \mathcal{W}_{U^+}\le\mathcal{W}_{u^+}<+\infty$$

from Lemma 5 and (1.11).

By applying Lemma 6 to U, we can obtain (1.12) from (4.2).

5. Proof of the Theorem 3. Put $u(P)=h(P)$ and $-h(P)$ in Theorem 2. Meanwhile, Theorem 2 gives the existence of all limits $\mathcal{V}_h$, $\mathcal{W}_h$, $\mathcal{V}_{h^+}$, $\mathcal{W}_{h^+}$,

$$\mathcal{V}_{(-h)^+}=\mathcal{V}_{h^-} \quad \text{and} \quad \mathcal{W}_{(-h)^+}=\mathcal{W}_{h^-}. \tag{5.1}$$

Since

$$\mathcal{V}_{|h|}=\mathcal{V}_{h^+}+\mathcal{V}_{h^-} \quad \text{and} \quad \mathcal{W}_{|h|}=\mathcal{W}_{h^+}+\mathcal{W}_{h^-}, \tag{5.2}$$

it follows that both limits $\mathcal{V}_{|h|}$ and $\mathcal{W}_{|h|}$ exist. Then we see that $\mathcal{V}_{h^+}$, $\mathcal{V}_{h^-}$, $\mathcal{W}_{h^+}$ and $\mathcal{W}_{h^-}<+\infty$ from (5.1), (5.2) and (1.13). Hence, by applying Theorem 2 to $u(P)=h(P)$ and $-h(P)$ again, we obtain from (1.12)

$$h(P)\le PI^a_\Omega[g](P)+(\mathcal{V}_hV(r)+\mathcal{W}_hW(r))\varphi(\Theta)$$

and

$$h(P) \geq PI_{\Omega}^{a}[g](P) + (\mathcal{V}_h V(r) + \mathcal{W}_h W(r))\varphi(\Theta)$$

respectively, which give (1.14).

REFERENCES

1. S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory* (Springer-Verlag, New York, 1992).

2. V. S. Azarin, Generalization of a theorem of Hayman on subharmonic functions in an m-dimensional cone, *Amer. Math. Soc. Trans.* **80**(2) (1969), 119–138.

3. M. Cranston, E. Fabes and Z. Zhao, Conditional gauge and potential theory for the Schrödinger operator, *Trans. Amer. Math. Soc.* 307 (1988), 415–425.

4. M. Cranston, Conditional Brownian motion, Whitney squares and the conditional gauge theorm, in *Seminar on stochastic processes, 1988* (Birkhäuser Verlag, Basel-Boston-Berlin, 1989), 109–119.

5. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* (Springer Verlag, Berlin, 1977).

6. B. Levin and A. Kheyfits, Asymptotic behavior of subfunctions of the stationary Schrödinger operator, Preprint, http://arxiv.org/abs/math/0211328v1, 2002.

7. M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. 3 (Academic Press, New York, 1970).

8. G. Rosenblum, M. Solomyak and M. Shubin, *Spectral theory of differential operators* (VINITI, Moscow, 1989).

9. D. Siegel and E. O. Talvila, Uniqueness for the n-dimensional half space Dirichlet problem, *Pacific J. Math.* **175**(2) (1996), 571–587.

10. B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982), 447–526.

11. G. M. Verzhbinskii and V. G. Maz'ya, Asymptotic behavior of solutions of elliptic equations of the second order close to a boundary, *Sibirsk. Math. J.* **12**(6) (1971), 874–899.

12. H. Yoshida, Nevanlinna norm of a subharmonic function on a cone or on a cylinder, *Proc. Lond. Math. Soc.* **54**(3) (1987), 267–299.

13. H. Yoshida, Harmonic majorization of a subharmonic function on a cone or on a cylinder, *Pacific J. Math.* **148**(2) (1991), 369–395.