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A Class of Cellulated Spheres with Non-Polytopal Symmetries

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

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Abstract. We construct, for all $d \geq 4$, a cellulation of $\mathbb{S}^{d-1}$. We prove that these cellulations cannot be polytopal with maximal combinatorial symmetry. Such non-realizability phenomenon was first described in dimension 4 by Bokowski, Ewald and Kleinschmidt, and, to the knowledge of the author, until now there have not been any known examples in higher dimensions. As a starting point for the construction, we introduce a new class of (Wythoffian) uniform polytopes, which we call duplexes. In proving our main result, we use some tools that we developed earlier while studying perfect polytopes. In particular, we prove perfectness of the duplexes; furthermore, we prove and make use of the perfectness of another new class of polytopes which we obtain by a variant of the so-called E-construction introduced by Eppstein, Kuperberg and Ziegler.

1 Motivation and Preliminaries

A version of the famous Steinitz problem can be formulated as follows.

Problem 1.1 Given a lattice L, is it isomorphic to the face lattice $L(P)$ of some convex polytope P?

Its name stems from the fact that the first relevant characterization result is due to Steinitz (1922). By his well-known theorem, a graph is isomorphic to the 1-skeleton of a 3-polytope (*i.e.,* 3-polytopal [21]) if and only if it is simple, planar, and 3-connected (see [18, Chapter 13.1] and [32, Lecture 4]). But the problem goes back even earlier. In fact, in 1909 Brückner gave, inadvertently, the dual of a simplicial 3-sphere, the polytopality of which was decided more than fifty years later. Grünbaum and Sreedharan (1967) showed, and Bokowski (2006) confirmed, that there is no 4-polytope such that its boundary complex is isomorphic to Brückner's sphere (see [4, 18, 24, 28, 32]). Since a characterization result which would generalize Steinitz' theorem to arbitrary dimension is still lacking, the investigation of such non-polytopal examples is of continuing interest. Their existence is highly supported by some recent general results. In particular, Pfeifle and Ziegler [28] proved that for n large enough, there are far more simplicial 3-spheres than 4-polytopes on n vertices. In other words, "most" triangulations of $\mathbb{S}^3$ are not isomorphic to the boundary complex of a convex

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4-polytope. On the other hand, deciding the polytopality of questionable examples provides a great challenge to the researchers working with and on algorithmic methods; see the works of Jürgen Bokowski on the algorithmic Steinitz problem ([4, 8], and the references therein.)

Thus, we think it may be of interest for further research that in this paper we give an infinite series of examples of $(d-1)$-spheres, with $d \geq 4$, the polytopality of which is to be decided (Problem 5.1 in Section 5). These are constructed in two main steps: in Section 2 we introduce the polytopes called uniform duplexes, and in Section 4 we apply the so-called E-construction to these polytopes.

Although we cannot answer the question of polytopality of our examples, we were able to solve a restricted version of the problem. Let Σ be a combinatorial structure generalizing the boundary complex of a polytope such that one can assign to it a lattice $L(\Sigma)$ of its suitably defined substructures. In seeking the possible polytopal realizations of Σ, one may ask whether all of its combinatorial automorphisms can be realized by geometric symmetries of a polytope. Let $G(P)$ denote the group of geometric symmetries of a polytope P, *i.e.,* the group of all Euclidean isometries of the ambient space (affine hull of P) which map P to itself, and let $A(P)$ denote the group of automorphisms of its face lattice $L(P)$. Since geometric symmetries preserve incidences, they induce combinatorial automorphisms. Thus, in general, $A(P)$ is larger than $G(P)$, in the sense that $A(P)$ contains a proper subgroup isomorphic to $G(P)$. Now our question can be formulated in the following form.

Problem 1.2 Given a combinatorial structure Σ, does there exist a polytope P such that $L(\Sigma) \cong L(P)$ and $A(\Sigma) \cong G(P)$?

This goes back to the following question of Grünbaum and Shephard [19]: "Given any d-polytope P, does there always exist a polytope P' combinatorially equivalent to P such that $G(P') \cong A(P)$?"

The question is settled in three dimensions by a theorem of Mani which states that for each 3-polytopal graph G there exists a convex 3-polytope P such that every automorphism of G is induced by a symmetry of P.

The phenomenon that the full combinatorial symmetry group cannot be realized by a combinatorially prescribed polytope, appears first in four dimensions. The first examples are known as McMullen's sphere (1968) and Kleinschmidt's sphere (1984). Both were carefully investigated by Bokowski et al. (see [6], resp. [3] and [7]; they are also described in [4]). Most recently, an infinite series of examples was found in four dimensions [4, 5, 26].

Such examples are also interesting in the context of the universality theorems for polytopes [29]. In fact, Kleinschmidt's sphere stimulated Mnëv when he established his famous universality theorem which implies that for $d \geq 4$ the realization spaces of d-polytopes can be arbitrarily complicated [3, 4]. The realization space of the infinite series given in [5, 26] has also been investigated, see the works of Paffenholz [25, 26].

To the knowledge of the author, no such examples are known in higher dimensions. Thus we believe that our main result, formulated in Theorem 5.3 considerably extends the set of known spheres having non-polytopal symmetries.

The lattice L in Problem 1.1, or the Σ in Problem 1.2, may refer to various combinatorial structures generalizing the (boundary complex of) a polytope. Most generally, such a structure is a *strongly regular cell complex*. We recall some definitions.

A *regular cell complex* Γ is defined [1, 2] as a finite collection of closed topological balls σ in a Hausdorff space $\|\Gamma\| = \bigcup_{\sigma\in\Gamma}$ such that

(i) the interiors $\mathring{\sigma}$ partition $\|\Gamma\|$, *i.e.,* every $x \in \|\Gamma\|$ lies in exactly one $\mathring{\sigma}$,
(ii) the boundary $\partial\sigma$ is a union of some members of Γ, for all $\sigma \in \Gamma$.

The balls $\sigma \in \Gamma$ are called the *closed cells* of Γ, their interiors $\mathring{\sigma}$ are the *open cells*. The space $\|\Gamma\|$ is called the *underlying space*. Conversely, we also speak of the (regular) *cell decomposition* (or *cellulation*) of the space $\|\Gamma\|$.

A regular cell complex is also called a *regular CW complex*. If its underlying space is (topologically) a sphere, then it is called a *regular CW sphere*.

The *face poset* $F(\Gamma) = (\Gamma, \geq)$ is the set of closed cells ordered by containment. The *augmented face poset* $\widehat{F}(\Gamma) = F(\Gamma) \cup \{\hat{0}, \hat{1}\}$ is the face poset enlarged by new elements such that $\hat{0} < \sigma < \hat{1}$ for all $\sigma \in \Gamma$.

A regular cell complex has the *intersection property* if the intersection of any two non-empty closed cells is also a cell in the complex.

We adopt the following term used in [13, 27, 28]. A regular cell complex is called *strongly regular* if it has the intersection property. We note that the intersection property is necessary and sufficient for the augmented face poset of a regular cell complex to be a lattice (see [2, Problem 4.47, p. 223] and [27, p. 607].)

The concept of a strongly regular cell complex is a natural generalization of the boundary complex of a polytope. However, it is too general for our present purposes. We should like to emphasize that our construction produces objects that are conceptually very close to a convex polytope, the latter being regarded primarily as a *geometric* object. Accordingly, we use the following definition for our spheres.

Definition 1.3 A strongly regular cell complex is called a *cellulated geometric sphere*, in short, a *CG sphere*, if its underlying space is a sphere S in geometric sense, *i.e., S* is the locus of points equidistant from a fixed point in a Euclidean space $\mathbb{E}^d$.

We note that we already used this concept, implicitly, in our earlier work where the 3-spheres with non-polytopal symmetries, discovered by the author of the present paper, were also treated as CG spheres, in the guise of spherical tessellations [5].

2 Duplexes

In this section we introduce a new class of polytopes which will serve as the starting point for constructing our cellulated spheres.

Definition 2.1 Let T and T' be two concentric regular simplices of equal size such that one is the mirror image of the other with respect to their common centre. Then the intersection $T \cap T'$ is called a *uniform duplex*. A polytope combinatorially equivalent to a uniform duplex is called a *duplex*.

Here we adopt a term which was coined by Coxeter [9, 12] (see also [31]).

Definition 2.2 A convex polygon is *uniform* if it is regular. Recursively, if $d \geq 3$, a convex d-polytope is *uniform* if its facets are uniform and its (geometric) symmetry group is transitive on its vertices.

We briefly summarize some basic properties of duplexes. Since duplexes of odd and even dimension behave slightly differently, we treat the two cases separately.

Case I. Odd dimension. We start from a regular d-simplex T^d, with $d = 2k + 1$ $(k = 1, 2, \dots)$. We represent it in the hyperplane $\sum x_i = 0$ of a Euclidean $(d + 1)$-space with vertices

$$(2k + 1, \underbrace{-1, \dots, -1}_{2k+1})^P,$$

where the superscript denotes that all permutations of the coordinates are to be taken. Then the centroids of its k-faces can be given as

$$(\underbrace{1, \dots, 1}_{k+1} \underbrace{-1, \dots, -1}_{k+1})^P. \tag{2.1}$$

Clearly, the same vectors provide the centroids of the k-faces of $-T^d$, the mirror image of T^d with respect to the origin. Alternatively, T^d and $-T^d$ are reciprocal to each other with respect to the sphere circumscribed to these points. This means, on the other hand, that the intersection of T^d and $-T^d$ is a polytope which can be considered as the k-th *simple truncation* of either of them [11, Chapter 8.1]. Thus the uniform duplex $T^d \cap -T^d$ is a (Wythoffian) uniform polytope in the sense of Definition 2.2, which justifies using this term in our definition for duplexes. Its Coxeter symbol is

$$\underbrace{\bullet\!-\cdots-\!\bullet}_{k}\!-\!\odot\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k}.$$

We shall denote it by D^d. Its vertices are exactly the points (2.1). As it follows from Coxeter's theory of Wythoffian polytopes [9], its facets are of type

$$\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}\!-\!\odot\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k} \quad \text{and} \quad \underbrace{\bullet\!-\cdots-\!\bullet}_{k}\!-\!\odot\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}.$$

Clearly, these are congruent with each other, and D^d has $d + 1$ of both of them. Furthermore, the ridges, *i.e.*, $(d - 2)$-faces, are of the following types:

$$\underbrace{\bullet\!-\cdots-\!\bullet}_{k-2}\!-\!\odot\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k}, \quad \underbrace{\bullet\!-\cdots-\!\bullet}_{k}\!-\!\odot\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-2} \tag{2.2}$$

and

$$\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}\!-\!\odot\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}. \tag{2.3}$$

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Notice that the latter is again a duplex (of odd dimension). (Here we assume that $k \geq 2$, since for $k = 1$ there is only one type of ridge, as a 3-duplex is just an octahedron.)

By adding the vector $(1, \ldots, 1)$ to (2.1) and applying the scaling factor $\frac{1}{2}$, one observes that the vertex set of a particular kind of polytope is obtained; namely, this is the *hypersimplex* $\Delta_{2k+1}(k + 1)$ [32]. Thus, in fact, our family of duplexes of odd dimension does not form a new class of polytopes (hypersimplices were introduced by Gelfand and his co-workers in 1975 [14]).

Case II. Even dimension. T^d is now a regular d-simplex with $d = 2k$ $(k = 1, 2, \ldots)$. We represent it with vertices

$$(2k, \underbrace{-1, \ldots, -1}_{2k})^P.$$

The centroids of its $(k-1)$-faces and k-faces are

$$\frac{1}{k}(\underbrace{k+1, \ldots, k+1}_{k}, \underbrace{-k, \ldots, -k}_{k+1})^P \tag{2.4}$$

and

$$\frac{1}{k+1}(\underbrace{k, \ldots, k}_{k+1}, \underbrace{-(k+1), \ldots, -(k+1)}_{k})^P, \tag{2.5}$$

respectively. Recall now that the fundamental domain of the symmetry group of T^d, given by the Coxeter graph

$$\underbrace{\bullet \cdots \bullet}_{k-1} - \bullet - \bullet - \underbrace{\bullet \cdots \bullet}_{k-1},$$

is a spherical simplex, provided that the action of the group is restricted to the unit sphere $\mathbb{S}^{d-1}$; this simplex is the radial projection on $\mathbb{S}^{d-1}$ of an orthoscheme whose vertices are just the centroids of suitably chosen j-faces of T^d, for all $j = 0, 1, \ldots, d-1$ [11, Theorem 11.23]. Take the midpoint of the edge of this spherical simplex joining the two vertices that correspond to the centroid of a $(k-1)$-face and a k-face. One sees at once that it is in the point set

$$(\underbrace{1, \ldots, 1}_{k}, 0, \underbrace{-1, \ldots, -1}_{k})^P, \tag{2.6}$$

apart from a certain scaling factor.

Now we choose that factor 1 and apply the Wythoff construction to this particular point. In our case this means that we take in fact all the permutations of the

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coordinates and form the convex hull of the point set (2.6). What is obtained is the (uniform) polytope whose Coxeter symbol is

(2.7) $$\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}.$$

Observe the bilateral symmetry of this Coxeter symbol; it corresponds to the fact that the point set (2.6), and hence the polytope (2.7), is symmetric under reflection in the origin. Equivalently, this polytope arises from $-T^d$ as well through a procedure analogous to the above. In fact, (2.4) and (2.5) are at the same time the centroids of the k-faces and $(k-1)$-faces of $-T^d$, respectively. We conclude that the polytope (2.7) is just the (uniform) duplex $T^d \cap -T^d$. (We recall that in a continuous sequence of truncations in which the truncating hyperplanes proceed from the centroids of the $(k-1)$-faces to the centroids of the k-faces, (2.7) is an intermediate stage where these hyperplanes go through the points (2.6); in every stage the initial point of the Wythoff construction is a point on the edge connecting two such centroids of the orthoscheme mentioned above).

Analogously to the odd case, the types of its facets are

$$\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-2} \quad\text{and}\quad \underbrace{\bullet\!-\cdots-\!\bullet}_{k-2}\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1},$$

for $k \geq 2$; but we are especially interested in the ridges here as well. These are of the following types:

(2.8) $$\odot\!-\!\!-\!\bullet,\quad \bullet\!-\!\!-\!\odot,\quad \text{and}$$

(2.9) $$\odot\!-\!\!-\!\odot$$

(*i.e.*, regular triangle and hexagon) for $k = 2$, as well as

(2.10)
$$\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1}\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-3},\quad \underbrace{\bullet\!-\cdots-\!\bullet}_{k-3}\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-1},$$

and

(2.11) $$\underbrace{\bullet\!-\cdots-\!\bullet}_{k-2}\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\underbrace{\bullet\!-\cdots-\!\bullet}_{k-2},$$

for $k > 2$. Here one can observe as well that the latter is again a duplex.

The facet structure of the uniform duplex of dimension 4 is depicted in Figure 1. This polytope occurs as a Wythoffian uniform polytope in [10, 12], where its vertex coordinates in the general form (2.6) are given. It is described as a perfect polytope in [15], and also in [16]; in the latter an assignment of coordinates to all 30 vertices is also given. Its symmetry properties are investigated in more detail in [17]. It is also interesting in the context of abstract regular polytopes [20]. It has 10 Archimedean truncated tetrahedra as facets; its 2-faces are: 20 regular triangles (ridges of non-duplex type, see (2.8)) and 20 regular hexagons (ridges of duplex type, see (2.9)).

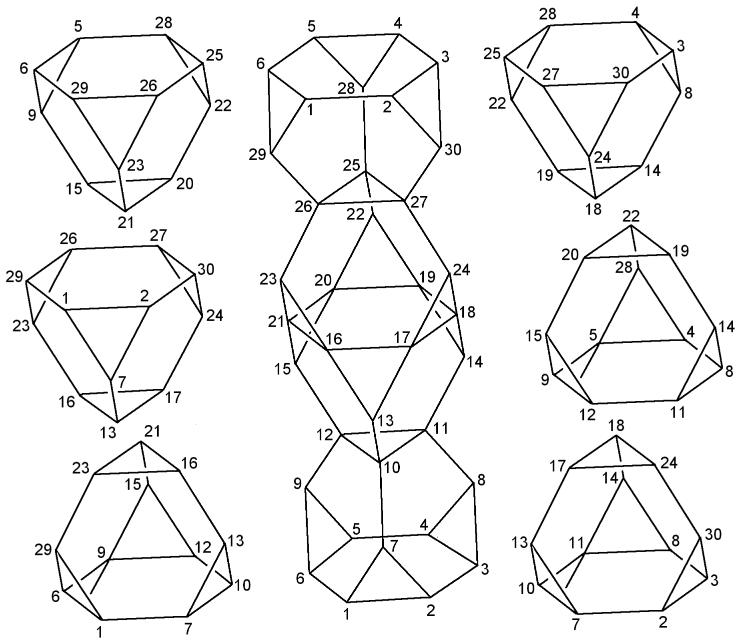

Figure 1: The uniform 4-duplex.

Actually, it was the starting point for our construction described in full generality in the next section; thus we use it here to convey the intuitive idea of that.

The following property of the ridges of uniform duplexes will be crucial for the proof of our main result.

Lemma 2.3 *Let D^d be a uniform d-duplex such that its vertex set is* (2.1) *if $d = 2k+1$, and is* (2.6) *if $d = 2k$ ($k = 2, 3\dots$). Then both types of the ridges of D^d have a circumsphere. The radius of this sphere for the ridges of duplex type, resp. non-duplex type, is*

$$r_D^{2k+1} = \sqrt{2k}, \quad \textit{resp. } r_N^{2k+1} = \sqrt{2\Big(k - \frac{1}{k}\Big)}, \quad \textit{if } d = 2k+1, \tag{2.12}$$

and

$$r_D^{2k} = \sqrt{2(k-1)}, \quad \textit{resp. } r_N^{2k} = \sqrt{2\Big(k - 1 - \frac{2}{2k-1}\Big)}, \quad \textit{if } d = 2k. \tag{2.13}$$

Proof Since D^d is a Wythoffian polytope (both in odd and in even case), it is vertex-transitive; thus it has a circumsphere. A supporting hyperplane of D^d containing any facet $F \subset D^d$ intersects that circumsphere in a sphere which is circumscribed

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around F. Likewise, any face of lower dimension of D^d has its own circumsphere. We are interested, in particular, in the radius of two such spheres.

It is clear that for all $i = 1, 2, \ldots, d + 1$, the hyperplanes

$$x_i = 1 \quad \text{and} \quad x_i = -1 \tag{2.14}$$

are supporting hyperplanes of D^d. Let H be such a hyperplane for some fixed i. H contains exactly those vertices of D^d (from (2.1) or (2.6)) the i-th coordinate of which is fixed as either 1 or -1. Among these vertices one easily finds d affinely independent points. Hence the vertices in H form actually the vertex set of a facet of D^d. Thus the supporting hyperplanes (2.14) determine the facets of D^d originating (as truncations) from the facets of T or T', depending on the sign on the right-hand side of the equations (2.14) (clearly there are no other facets).

It follows that, fixing two (non-zero) coordinates of the vertices, one obtains the vertex set of a ridge of D^d. Namely, the ridges of non-duplex type (such as (2.2) or (2.10)) have vertices with fixed coordinates of identical sign, while the vertices of the ridges of duplex type (such as (2.3) or (2.11) have fixed coordinates of distinct sign. For the ridges of the latter type an obvious consequence is that they are centrally symmetric. It follows directly that their centroids are

$$(1, \underbrace{0, \ldots, 0}_{d-1}, -1)^P.$$

This together with (2.1) resp. (2.6), taking into consideration (2.14), yields the radius vectors and then the equality (2.12) for r_D^{2k+1} resp. the equality (2.13) for r_D^{2k}.

In the vertex vector of a ridge of non-duplex type two coordinates, say the i-th and j-th, are fixed as 1 (or -1). Thus in the vector of the centroid of this ridge the i-th and j-th coordinate will be 1 (or -1) as well. On the other hand, it is clear that the rest of the coordinates of this centroid (using vertices either (2.1) or (2.6)) will be equal to each other. Hence from the condition that the coordinate sum must be zero one obtains at once the vectors

$$\pm\frac{1}{k}(-k, -k, \underbrace{1, \ldots, 1}_{2k})^P \quad \text{and} \quad \pm\frac{1}{2k-1}(-2k+1, -2k+1, \underbrace{2, \ldots, 2}_{2k-1})^P,$$

in the odd and even case, respectively. This again, together with (2.1) and (2.6) yields the equalities (2.12) and (2.13) for r_N^{2k+1} and r_N^{2k}, respectively. ∎

Note that for $k = 1$, r_N^{2k+1} is zero, which is consistent with the fact that in the case of the octahedron the "ridge" of non-duplex type shrinks in fact to a point.

Corollary 2.4 *In a uniform duplex D of dimension at least four the distance of a ridge of duplex type from the centre of D is smaller than that of a ridge of non-duplex type.*

Proof This is equivalent to saying that the size of the circumsphere of a ridge of duplex type is greater than that of a ridge of non-duplex type, see (2.12) and (2.13). ∎

We shall need the following observation, which comes from a comparison of the Coxeter symbols of the facets and edges given above (we used it implicitly in the proof of Lemma 2.3).

Observation 2.5 Let a duplex of dimension d be constructed from a pair of simplices T and T' as in Definition 2.1. Consider a pair of its adjacent facets F_1 and F_2. For $d \geq 4$, if both F_1 and F_2 originate from (the facets of) the same simplex (either T or T'), then they meet in a ridge of non-duplex type. If they originate from different simplices, then they meet in a ridge of duplex type.

3 The Uniform Duplexes as Perfect Polytopes

We shall prove our main result in the context of perfect polytopes; as a preparation, in this section we prove that the uniform duplexes are perfect polytopes.

The notion a perfect polytope was introduced by Stewart Robertson [30]. We recall some definitions (see [15]). Let $G(P)$ and $L(P)$ denote the (geometric) symmetry group and the face lattice of a d-polytope P, respectively, as in Section 1. Two d-polytopes P and Q in $\mathbb{E}^d$ are said to be *symmetry equivalent* if there exists an isometry φ of $\mathbb{E}^d$ and a lattice isomorphism $\lambda\colon L(P) \to L(Q)$ such that for each $g \in G(P)$ and each $F \in L(P)$, $\lambda(g(F)) = (\varphi g \varphi^{-1})(\lambda(F))$. Now a polytope P is said to be *perfect* if all polytopes symmetry equivalent to P are similar to P (where similarity is meant in the usual geometric sense). Intuitively speaking, a polytope is perfect if its shape cannot be changed without changing the action of its symmetry group on its face lattice.

The following result (due to M. R. Pinto, as cited in [22]) shows that the perfectness is preserved in a restricted case of polarity.

Proposition 3.1 *Let P be a perfect polytope with its centroid at the origin. Then its polar dual $P^{\triangle} = \{y \in \mathbb{E}^d \mid \langle x, y\rangle \leq 1$ for all $x \in P\}$ is perfect as well.*

Let G be a finite group of isometries in $\mathbb{E}^d$ fixing the origin and denote by e the identity in G. Then the *symmmetry scaffolding* of G is the union of the fixed point sets of all transformations in $G \setminus \{e\}$ and is denoted by $\operatorname{scaf} G$. We use the same term (and notation) for the intersection of this set with the unit sphere $\mathbb{S}^{d-1}$. (To avoid confusion, we use the attribute *spherical* in this latter case, where the distinction is important). For a point $A \in \operatorname{scaf} G$, the *fixed point set* of A is defined as the set

$$\operatorname{fix}_A = \{X \in \mathbb{R}^d \mid g(X) = X \text{ for all } g \in \operatorname{Stab} A\},$$

where $\operatorname{Stab} A$ is the stabilizer of A in G. The dimension of fix_A is called the *degree of freedom* of A. A point in the spherical symmetry scaffolding of G is called a *node* if it has zero degree of freedom. Intuitively, a node is a point that cannot be displaced from its position within a small neighbourhood without increasing the cardinality of its G-orbit.

Let P be a d-polytope. We locate it so that its centroid coincides with the centre of the unit sphere $\mathbb{S}^{d-1}$. We radially project it onto $\mathbb{S}^{d-1}$. Then the image of its boundary complex yields a cellulation of $\mathbb{S}^{d-1}$, which we call the *spherical image* of P

and denote by $\widehat{P}$. A vertex of a polytope P is called *nodal* if in the spherical image of P it coincides with a node in scaf $G(P)$. A *nodal polytope* is a polytope such that all its vertices are nodal.

The notion of a nodal polytope was introduced by the author [15], and several classes of examples were studied in [15–17]. In particular, we have [15, Theorem 2.3] the following.

Theorem 3.2 *Every vertex-transitive nodal polytope is perfect.*

Consider now a uniform duplex D. It symmetry group is easily established as follows.

Proposition 3.3 *The symmetry group of a uniform duplex of dimension d is*

$$A_d \times \langle I \rangle \cong \mathcal{S}_{d+1} \times C_2,$$

where I is a central inversion.

Proof This follows directly from Definition 2.1. In fact, if the dimension of the pair of simplices (T, T') used in the definition is d, then the union of them is obviously invariant under the action of the reflection group A_d, which is known to be isomorphic to $\mathcal{S}_{d+1}$, the symmetric group of degree $d + 1$; in addition, there is a central inversion I that swaps the two simplices. Clearly, all these symmetries are preserved by the intersection. ■

Now we are ready to prove the perfectness of uniform duplexes.

Theorem 3.4 *Every uniform duplex D is perfect.*

Proof By Proposition 3.1 it is sufficient to prove that $D^{\triangle}$ is perfect (the direct way would be a bit longer). For this, it is sufficient to prove that $D^{\triangle}$ is vertex-transitive and nodal, by Theorem 3.2. Thus consider the vertices of $D^{\triangle}$. They coincide, up to rescaling, with the centroids of the facets of D, from which the transitivity is obvious. To see that these centroids are nodal, consider the spherical fundamental tessellation of a Coxeter group of type A_d, that is, the tessellation consisting of the fundamental domain and all its transforms under the action of the group on the unit sphere $\mathbb{S}^{d-1}$. It is a basic theorem of the finite reflection groups that this fundamental domain is a (spherical) simplex [11, Theorem 11.23]. Clearly, the vertices of this fundamental domain (and hence all the vertices of the tessellation) are nodes in the (spherical) symmetry scaffolding of the group [15]. Now observe that the centroids of the two kinds of facets of D correspond just to the vertices of the fundamental domain which are represented by the two terminal points of the Coxeter graph of the group. Thus they are nodes. ■

4 The E-Construction for Uniform Duplexes

Here we apply a construction, called the E-construction, which was introduced by Eppstein, Kuperberg, and Ziegler in order to produce certain 4-polytopes and

3-spheres with prescribed combinatorial properties [13]. For its extension and various applications, see [25–28].

The construction is essentially as follows. It assigns to a polytope P a regular CW sphere $E(P)$ by the following two steps: (1) in the boundary of P, one stellarly subdivides each facet of P; (2) one merges facets of the subdivision sharing a ridge of P. In our particular case we describe it in more detail in order to emphasize certain metrical and symmetry properties.

We start from a uniform duplex D of dimension $d \geq 4$ such that it is centered at the origin and it is scaled so that its ridges of duplex type are tangent to the unit sphere $\mathbb{S}^{d-1}$. (Such a scaling is possible, since ridges of the same type form a single orbit under the action of the symmetry group of D.) Then in the polar dual $D^{\triangle}$ (that, in our case, is the reciprocal with respect to $\mathbb{S}^{d-1}$) there is a corresponding edge $E^{\triangle}$ for each ridge E of D. Moreover, if E is of duplex type, then $E^{\triangle}$ is tangent to $\mathbb{S}^{d-1}$ as well. In this case the affine hulls of E and $E^{\triangle}$ form orthogonal complements in the tangent space of $\mathbb{S}^{d-1}$. Thus the convex hull $\operatorname{conv}(E \cup E^{\triangle})$ forms an orthogonal bipyramid (see Eppstein et al. [13, p. 240]; here we adopt their approach in describing the E-construction). More closely, it has the property formulated in the following definition.

Definition 4.1 A pyramid P is called a *uniform pyramid* if its basis B is a uniform polytope and the altitude of P meets B orthogonally at its centroid. A bipyramid is called a *uniform bipyramid* if it is the union of two congruent uniform pyramids such that are mirror images of each other with respect to their common basis.

Thus $\operatorname{conv}(E \cup E^{\triangle})$ is a uniform bipyramid. Consider now the polytope $\operatorname{conv}(D \cup D^{\triangle})$. We shall denote it by $D^{\diamond}$. We establish its facet structure.

Theorem 4.2 *The facets of the polytope $D^{\diamond}$ are uniform bipyramids over the ridges of duplex type of D and uniform pyramids over the ridges of non-duplex type of D.*

Proof The uniform bipyramids considered above form just one type of the facets; thus it remains to show that the only other facets are uniform pyramids. Choose a facet F of D and consider the apex A which the bipyramids in $D^{\diamond}$ have in common whose bases are ridges of duplex type of F. Clearly $A = F^{\triangle}$ and, as such, for symmetry reasons, it lies on the line connecting the origin with the centroid C_F of F. For any ridge $E \subset F$ of non-duplex type the convex hull $\operatorname{conv}(E \cup A)$ is obviously a pyramid. Consider now the affine hull $\operatorname{aff} E$ of such a ridge $E \subset F$. Take the plane P passing through the centroid of C_E of E such that $\operatorname{aff} E$ and P form a pair of orthogonal complementary affine subspaces. The symmetry properties of D imply that P contains both C_F and the origin. It is also a consequence of symmetry that the pyramid $\operatorname{conv}(E \cup C_F)$ is a uniform pyramid. Now since A is on the line connecting the origin with C_F, A lies in P. It follows that $\operatorname{conv}(E \cup A)$ is also a uniform pyramid. (One could think of the pyramid $\operatorname{conv}(E \cup C_F)$ being transformed to $\operatorname{conv}(E \cup A)$ by a rotation about the hyperline $\operatorname{aff} E$ and by an elongation while preserving its symmetry.)

Let F' be another facet of D such that it is adjacent to F at the ridge E. Then A' is the other apex of the bipyramid over E whose one apex is A. We show that $\operatorname{conv}(E \cup A)$ and $\operatorname{conv}(E \cup A')$ together cannot form a bipyramidal facet of $D^{\diamond}$. To see this, first

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observe that the union $(\mathrm{conv}(E \cup A)) \cup (\mathrm{conv}(E \cup A'))$ must be invariant under the action of $\mathrm{Stab}(E)$, the subgroup of the symmetry group of D stabilizing the ridge E. We denote this figure by U. It follows that if U were a bipyramid, it would be a uniform bipyramid. In fact, E is a ridge of non-duplex type, thus, by Observation 2.5, F and F' originate from one and the same simplex. Hence they are mirror images of each other with respect to the hyperplane $\mathrm{aff}(E \cup O)$, where O is the origin (a well-known property of regular polytopes, in particular, simplices). If U were a uniform bipyramid, then its apices A and A' would be collinear with C_E, by definition. But these points cannot be collinear. For, if they were, C_E would be at the same distance from the origin as the centroids of the ridges of duplex type. This is impossible, by Corollary 2.4. Thus U cannot form a bipyramidal facet of $D^{\diamondsuit}$. Actually, the corollary says that C_E is farther then the apices; this implies just that the set U decomposes to two pyramidal facets (in the converse case U would decompose to a larger number of facets, which would even be of other type).

Since the set of ridges of D consists of two orbits under the action of the symmetry group of D, so does the set of facets of $D^{\diamondsuit}$. Thus we have found all the facets of $D^{\diamondsuit}$. ■

Now our next step is to radially project the polytope $D^{\diamondsuit}$ onto $\mathbb{S}^{d-1}$. The spherical image of the boundary complex of $D^{\diamondsuit}$ then forms a CG sphere, which we shall denote by $\widehat{D^{\diamondsuit}}$. Finally, in $\widehat{D^{\diamondsuit}}$, for each pair of the pyramids with common basis we take their union to form a figure that is combinatorially equivalent to a bipyramid. This yields $E(D)$, as desired.

5 Non-Realizability of the Symmetries of $E(D)$

The sphere $E(D)$ obtained in the previous section is clearly a CG sphere. We pose the following problem.

Problem 5.1 Decide the polytopality of the CG sphere $E(D)$ constructed above. That is, for each $d \geq 4$, decide the existence of a d-polytope such that its boundary complex is combinatorially equivalent to the CG sphere $E(D)$ obtained in our construction from a d-dimensional uniform duplex D.

Investigation of this problem is beyond the scope of the present paper. Instead, we focus on a particular aspect of it. Namely, we show that $E(D)$ cannot be polytopal with full symmetry. To this end, first we establish the perfectness of the polytope $D^{\diamondsuit}$ constructed in the previous section.

Theorem 5.2 *$D^{\diamondsuit}$ is perfect.*

Proof D is perfect, by Theorem 3.4. Furthermore, $D^{\triangle}$ is nodal, as we have shown just in the proof of Theorem 3.4. It follows that if we fix the vertices of $D^{\diamondsuit}$ originating from D, then there is only one possibility to displace those originating from $D^{\triangle}$ without changing the action of the symmetry group. This is nothing else than shifting them in radial direction (all to the same extent). However, this is also impossible. For, in doing that, it would take the apices opposite to each other of each bipyramidal facet

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to a position such that they would not be collinear with the centroid of the basis of the bipyramid; thus, again, the symmetry of the whole figure would change. ■

Now we are ready to prove our main result. With the notation introduced in Section 1, we have the following.

Theorem 5.3 *For each $d \geq 4$, there is no d-polytope P such that $L(E(D)) \cong L(P)$ and $A(E(D)) \cong G(P)$.*

Proof First we note that it is sufficient to work with $G(E(D))$, the (geometric) symmetry group of $E(D)$ (it can be shown, however, that the isomorphism $A(E(D)) \cong G(E(D))$ holds). The group $G(E(D))$ can just as well be defined as the symmetry group of a polytope, since $E(D)$ is the cellulation of a *geometric* sphere. Clearly

$$G(E(D)) = G\big(\widehat{D^{\diamondsuit}}\big) = G\left(D^{\diamondsuit}\right).$$

Assume now the contrary of the statement, *i.e.,* there exists a polytope P satisfying the conditions. Since in $\widehat{D^{\diamondsuit}}$ all the vertices, which are the same as those of $E(D)$, coincide with nodes, the vertices of P must lie on the radial lines connecting the centre of $\mathbb{S}^{d-1}$ with the vertices of $D^{\diamondsuit}$. Moreover, P must possess facets which are uniform bipyramids with basis of duplex type. Since $D^{\diamondsuit}$ is perfect, it is the only polytope with these properties (up to similarity). But its other type of facets are pyramids, contrary to what is desired. ■

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