§ 8. Area.

The area of a triangle is a mean proportional between the rectangle contained by the semiperimeter and its excess above any one side and the rectangle contained by its excesses above the other two sides.

First Demonstration.

Figure 72.

Let ABC be the given triangle, and let each of its sides be given: to find the area.

Inscribe in the triangle the circle DEF whose centre is I, and join I with the points A, B, C, D, E, F.

Then the rectangle $BC \cdot ID$ = twice triangle BCI,

the rectangle $CA \cdot IE$ = twice triangle CAI,

and the rectangle $AB \cdot IF$ = twice triangle ABI;

hence the rectangle under the perimeter of triangle ABC and ID, the radius of the circle DFE = twice triangle ABC.

Produce BC, and make CD_2 equal to AE; then BD_2 is the semiperimeter, and the rectangle $BD_2 \cdot ID$ = triangle ABC. But the rectangle $BD_2 \cdot ID$ is a side of the solid contained by BD_2 and the square of ID; therefore the area of the triangle will be a side of the solid contained by BD_2 and the square of ID.

Draw IL perpendicular to IB, CL perpendicular to CB, and join BL.

Since each of the angles BIL, BCL is right,

the points B, I, C, L are concyclic;

therefore the angles BIC, BLC are equal to two right angles.

But the angles BIC, AIE are equal to two right angles,

because AI, BI, CI bisect the angles at the point I;

therefore angle AIE = angle BLC,

and triangle AIE is similar to triangle BLC.

Hence $BC : LC = AE : IE,$

$= CD_2 : ID;$

therefore $BC : CD_2 = LC : ID,$ by alternation,

$= CK : DK;$

and $BD_2 : CD_2 = CD : DK,$ by composition.

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Consequently $BD_2^2 : BD_2 \cdot CD_2 = CD \cdot BD : BD \cdot DK,$

$$= CD \cdot BD : ID^2 ;$$

therefore $BD_2^2 \cdot ID^2 = BD_2 \cdot CD_2 \cdot BD \cdot CD.$

Now each of the lines BD_2, CD_2, BD, CD is given ;
for BD_2 is the semiperimeter, CD_2 the excess of the semiperimeter above BC, BD the excess of the semiperimeter above AC, and CD the excess of the semiperimeter above AB.

The area of the triangle therefore is given.

[NUMERICAL ILLUSTRATION.]

Let AB consist of 13 parts, BC of 14, CA of 15.

Add the three together ; the result is 42, of which the half is 21. Subtract 13 ; there remain 8 : 14, there remain 7 : 15, there remain 6. 21, 8, 7, 6 into one another produce 7056, the square root of which is 84.

The area of the triangle is 84.

This useful theorem occurs in a treatise "On the Dioptra (περὶ διόπτρας) which many mathematical historians attribute to Heron of Alexandria (about 120 B.C.). See Cantor's *Vorlesungen über Geschichte der Mathematik*, I. 322-6 (1880). Mr Maximilien Marie, however (*Histoire des Sciences Mathématiques et Physiques*, I. 177-190), thinks the theorem cannot belong to so early a period, and ascribes it to Heron of Constantinople. The theorem was known to the Hindu mathematician Brahmegupta (born 598 A.D.) and to the Arabs. A good deal of historical information regarding it will be found in Chasles' *Aperçu Historique*, Note XII.

I have translated the demonstration in the text from Hultsch's *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, pp. 235-7 (1864), but I have not transliterated the notation.

SECOND DEMONSTRATION.

FIGURE 36.

Let ABC be a triangle, AX the perpendicular* from A to BC.

Then $AB^2 = BC^2 + CA^2 - 2BC \cdot CX$

that is $c^2 = a^2 + b^2 - 2a \cdot CX ;$

therefore $$CX = \frac{a^2 + b^2 - c^2}{2a}.$$

* Whatever be the shape of the triangle one of the perpendiculars will always fall inside the triangle. Let that perpendicular be AX.

Now $$AX^2 = AC^2 - CX^2$$

$$= b^2 - \left(\frac{a^2+b^2-c^2}{2a}\right)^2$$

$$= \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2}$$

$$= \frac{2s \cdot 2s_1 \cdot 2s_2 \cdot 2s_3}{4a^2};$$

therefore $$AX = \frac{2}{a}\sqrt{ss_1s_2s_3}.$$

Hence $$\Delta = \tfrac{1}{2}BC \cdot AX$$

$$= \frac{a}{2} \cdot \frac{2}{a}\sqrt{ss_1s_2s_3}$$

$$= \sqrt{ss_1s_2s_3}.$$

THIRD DEMONSTRATION.

FIGURE 28.

Because triangles AFI, AF_1I_1 are similar,

therefore $AF : IF = AF_1 : I_1F_1$;

therefore $AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : I_1F_1 \cdot IF$.

Because triangles IBF, BI_1F_1 are similar,

therefore $BF : IF = I_1F_1 : BF_1$;

therefore $IF \cdot I_1F_1 = BF \cdot BF_1$.

Hence $AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : BF \cdot BF_1$:

therefore $ss_1 : \Delta = \Delta : s_2s_3$.

(1) $$\Delta = \tfrac{1}{4}\sqrt{2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4)}.$$

This expression is convenient when a, b, c are irrational quantities.

(2) *The following method will enable us to discover the expression for the area of a triangle, if it is known that the square of its area is an integral function of its sides.**

* Terquem in *Nouvelles Annales*, III. 219-220 (1844). The method is also applied by Terquem to find the expression for the area of a cyclic quadrilateral, and it had previously been applied by P. L. Cirodde in *Nouvelles Annales*, I. 117 (1842), to find the volume of a spherical segment when it is known that the volume is a function of the third degree of its height.

Let Δ denote the area of the triangle, a, b, c its sides.

Then Δ^2 is a symmetrical function of the sides of the fourth degree. If one side becomes equal to the sum of the two others, the area vanishes;

therefore Δ^2 contains the three factors $-a+b+c$, $a-b+c$, $a+b-c$.

The fourth factor must therefore be of the form $m(a+b+c)$, where m is a constant number;

therefore $\Delta^2 = m(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.

To determine the value of m, suppose the three sides of the triangle to be equal;

then $$\Delta^2 = 3ma^4.$$

But the square of the area of an equilateral triangle,

whose side is a, $$= \frac{3}{16}a^4;$$

therefore $$m = \frac{1}{16}.$$

FORMULAE FOR THE AREAS OF CERTAIN TRIANGLES.

See the notation, pp. 7–11.

Triangles connected with the Centroid.

$$A'B'C' = AC'B' = C'BA' = B'A'C = \tfrac{1}{4}\Delta \qquad (1)$$

If R, S, T be the projections of G on the sides

$$GST = \frac{4}{9}\,\frac{\Delta^3}{b^2c^2} \qquad GTR = \frac{4}{9}\,\frac{\Delta^3}{c^2a^2} \qquad GRS = \frac{4}{9}\,\frac{\Delta^3}{a^2b^2} \qquad (2)$$

$$RST = \frac{4}{9}\cdot\frac{a^2+b^2+c^2}{a^2b^2c^2}\Delta^3 \qquad (3)$$

Triangles connected with the Circumcentre.

$$\left.\begin{aligned} OBC &= \tfrac{1}{2}ak_1 = \frac{a^2(-a^2+b^2+c^2)}{16\Delta} \\ OCA &= \tfrac{1}{2}bk_2 = \frac{b^2(a^2-b^2+c^2)}{16\Delta} \\ OAB &= \tfrac{1}{2}ck_3 = \frac{c^2(a^2+b^2-c^2)}{16\Delta} \end{aligned}\right\} \qquad (4)$$

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Since O is the orthocentre of A′B′C′,

therefore $OC'B'$, $C'OA'$, $B'A'O$

stand in the same relation to A′B′C′ as

HCB, CHA, BAH

stand to ABC.

Hence expressions for the areas of these triangles may be derived from (27).

Triangles connected with the Incentre and Excentres.

$$\frac{DEF}{r} = \frac{D_1E_1F_1}{r_1} = \frac{D_2E_2F_2}{r_2} = \frac{D_3E_3F_3}{r_3} = \frac{\Delta}{2R} \tag{5}$$

$$D_1E_1F_1 + D_2E_2F_2 + D_3E_3F_3 - DEF = 2\Delta \tag{6}$$

$$DEF \cdot D_1E_1F_1 \cdot D_2E_2F_2 \cdot D_3E_3F_3 = \frac{\Delta^6}{16R^4} \tag{7}$$

$$\frac{1}{D_1E_1F_1} + \frac{1}{D_2E_2F_2} + \frac{1}{D_3E_3F_3} - \frac{1}{DEF} = 0 \tag{8}$$

$$\left.\begin{aligned} \frac{AEF \cdot BFD \cdot CDE}{IEF \cdot IFD \cdot IDE} &= \frac{\Delta^2}{r^4} \\ \frac{AE_1F_1 \cdot BF_1D_1 \cdot CD_1E_1}{I_1E_1F_1 \cdot I_1F_1D_1 \cdot I_1D_1E_1} &= \frac{\Delta^2}{r_1^4} \end{aligned}\right\} \tag{9}$$

and so on.

$$\frac{AEF \cdot BFD \cdot CDE}{(DEF)^2} = \frac{AE_1F_1 \cdot BF_1D_1 \cdot CD_1E_1}{(D_1E_1F_1)^2} = \cdots\cdots = \frac{\Delta}{4} \tag{10}$$

$$\Delta_0 = 2Rs\,,\quad \Delta_1 = 2Rs_1\,,\quad \Delta_2 = 2Rs_2\,,\quad \Delta_3 = 2Rs_3 \tag{11}$$

$$\Delta_0\Delta_1\Delta_2\Delta_3 = 16R^4\Delta^2 \tag{12}$$

$$\left.\begin{aligned} \Delta_0 : \Delta &= 2R : r \\ \Delta_1 : \Delta &= 2R : r \end{aligned}\right\} \tag{13}$$

and so on.

$$\frac{\Delta}{\Delta_1}+\frac{\Delta}{\Delta_2}+\frac{\Delta}{\Delta_3}-\frac{\Delta}{\Delta_0}=2 \tag{14}$$

$$\left.\begin{aligned} 2\Delta\Delta_0 &= abcs = 4\mathrm{R}r_1r_2r_3 \\ 2\Delta\Delta_1 &= abcs_1 = 4\mathrm{R}r\,r_2r_3 \end{aligned}\right\} \tag{15}$$

and so on.

$$2r\Delta_0 = 2r_1\Delta_1 = 2r_2\Delta_2 = 2r_3\Delta_3 = abc \tag{16}$$

$$\left.\begin{aligned} \Delta_0 : \Delta &= \Delta : \mathrm{DEF} \\ \Delta_1 : \Delta &= \Delta : \mathrm{D_1E_1F_1} \end{aligned}\right\} \tag{17}$$

and so on.

$$\left.\begin{aligned} \Delta_0 : \mathrm{DEF} &= 4\mathrm{R}^2 : r^2 \\ \Delta_1 : \mathrm{D_1E_1F_1} &= 4\mathrm{R}^2 : r_1^2 \end{aligned}\right\} \tag{18}$$

and so on.

$$\left.\begin{aligned} \Delta_0^2 &= \mathrm{R}\,(r_2+r_3)\,(r_3+r_1)\,(r_1+r_2) \\ \Delta_1^2 &= \mathrm{R}\,(r_2+r_3)\,(r_2-r\,)\,(r_3-r\,) \\ \Delta_2^2 &= \mathrm{R}\,(r_3+r_1)\,(r_3-r\,)\,(r_1-r\,) \\ \Delta_3^2 &= \mathrm{R}\,(r_1+r_2)\,(r_1-r\,)\,(r_2-r\,) \end{aligned}\right\} \tag{19}$$

Corresponding expressions may be obtained for

ABC, HCB, CHA, BAH

by substituting instead of

$$\mathrm{R}\,,\quad r\,,\quad r_1\,,\quad r_2\,,\quad r_3$$

$$\tfrac{1}{2}\mathrm{R}\,,\quad \rho\,,\quad \rho_1\,,\quad \rho_2\,,\quad \rho_3$$

$$\left.\begin{aligned} 4\Delta_0 &= 2(\alpha_2+\alpha_3)\,\alpha_1 = 2(\beta_3+\beta_1)\,\beta_2 = 2(\gamma_1+\gamma_2)\,\gamma_3 \\ &= (\alpha_1-\alpha\,)(\alpha_2+\alpha_3) + (\beta_2-\beta\,)(\beta_3+\beta_1) + (\gamma_3-\gamma\,)(\gamma_1+\gamma_2) \\ 4\Delta_1 &= 2(\alpha_2+\alpha_3)\,\alpha = 2(\beta_2-\beta\,)\,\beta_3 = 2(\gamma_3-\gamma\,)\,\gamma_2 \\ &= -(\alpha_1-\alpha\,)(\alpha_2+\alpha_3) + (\beta_2-\beta\,)(\beta_3+\beta_1) + (\gamma_3-\gamma\,)(\gamma_1+\gamma_2) \\ 4\Delta_2 &= 2(\alpha_1-\alpha\,)\,\alpha_2 = 2(\beta_3+\beta_1)\,\beta = 2(\gamma_3-\gamma\,)\,\gamma_1 \\ &= (\alpha_1-\alpha\,)(\alpha_2+\alpha_3) - (\beta_2-\beta\,)(\beta_3+\beta_1) + (\gamma_3-\gamma\,)(\gamma_1+\gamma_2) \\ 4\Delta_3 &= 2(\alpha_1-\alpha\,)\,\alpha_2 = 2(\beta_2-\beta\,)\,\beta_1 = 2(\gamma_1+\gamma_2)\,\gamma \\ &= (\alpha_1-\alpha\,)(\alpha_2+\alpha_3) + (\beta_2-\beta\,)(\beta_3+\beta_1) - (\gamma_3-\gamma\,)(\gamma_1+\gamma_2) \end{aligned}\right\} \tag{20}$$

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$$\left.\begin{array}{lll} I_1BC = \dfrac{ar_1}{2} & = & \dfrac{r_1^2(r_2+r_3)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \\ AI_2C = \dfrac{br_2}{2} & = & \dfrac{r_2^2(r_3+r_1)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \\ ABI_3 = \dfrac{cr_3}{2} & = & \dfrac{r_3^2(r_1+r_2)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \end{array}\right\} \qquad (21)$$

The three preceding triangles are similar to $I_1I_2I_3$ and expressions for them may be derived from the expressions for $\triangle_0$ by the comparison of homologous lines in the triangles.

Thus in I_1BC, $I_1I_2I_3$ the perpendiculars I_1D_1, I_1A, or r_1, a_1, are homologous lines ;

therefore $$I_1BC : \triangle_0 = r_1^2 : a_1^2.$$

Similarly expressions may be found for

triangles	which are similar to
IBC, AI_3C, ABI_2	II_3I_2 or $\triangle_1$
I_3BC, AIC, ABI_1	I_3II_1 or $\triangle_2$
I_2BC, AI_1C, ABI	I_2I_1I or $\triangle_3$.

From these again, by making the appropriate changes in the letters, corresponding expressions may be found for

triangles	which are similar to
AYZ, XBZ, XYC	ABC
HYZ, XCZ, XYB	HCB
CYZ, XHZ, XYA	CHA
BYZ, XAZ, XYH	BAH.

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Triangles connected with the Angular Bisectors.

$$\left.\begin{aligned}
\mathrm{LMN} &= \frac{2abc\Delta}{(b+c)(c+a)(a+b)} = \frac{l_1 l_2 l_3}{4s} \\
\mathrm{LM'N'} &= \frac{2abc\Delta}{(b+c)(a-c)(a-b)} = \frac{l_1 \lambda_2 \lambda_3}{4s_1} \\
\mathrm{L'MN'} &= \frac{2abc\Delta}{(b-c)(c+a)(a-b)} = \frac{\lambda_1 l_2 \lambda_3}{4s_2} \\
\mathrm{L'M'N} &= \frac{2abc\Delta}{(b-c)(a-c)(a+b)} = \frac{\lambda_1 \lambda_2 l_3}{4s_3}
\end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned}
\mathrm{LL'M} &= \frac{2abc\Delta}{(c+a)(b^2-c^2)} & \mathrm{LL'N} &= \frac{2abc\Delta}{(a+b)(b^2-c^2)} \\
\mathrm{MM'N} &= \frac{2abc\Delta}{(a+b)(a^2-c^2)} & \mathrm{MM'L} &= \frac{2abc\Delta}{(b+c)(a^2-c^2)} \\
\mathrm{NN'L} &= \frac{2abc\Delta}{(b+c)(a^2-b^2)} & \mathrm{NN'M} &= \frac{2abc\Delta}{(c+a)(a^2-b^2)}
\end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned}
\mathrm{LL'M'} &= \frac{2abc\Delta}{(a-c)(b^2-c^2)} & \mathrm{LL'N'} &= \frac{2abc\Delta}{(a-b)(b^2-c^2)} \\
\mathrm{MM'N'} &= \frac{2abc\Delta}{(a-b)(a^2-c^2)} & \mathrm{MM'L'} &= \frac{2abc\Delta}{(b-c)(a^2-c^2)} \\
\mathrm{NN'L'} &= \frac{2abc\Delta}{(b-c)(a^2-b^2)} & \mathrm{NN'M'} &= \frac{2abc\Delta}{(a-c)(a^2-b^2)}
\end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned}
\mathrm{LMN} &: \Delta_0 = 4s_1s_2s_3 : (b+c)(c+a)(a+b) \\
\mathrm{LM'N'} &: \Delta_1 = 4s\,s_2s_3 : (b+c)(a-c)(a-b) \\
\mathrm{L'MN'} &: \Delta_2 = 4s\,s_3s_1 : (b-c)(c+a)(a-b) \\
\mathrm{L'M'N} &: \Delta_3 = 4s\,s_1s_2 : (b-c)(a-c)(a+b)
\end{aligned}\right\} \quad (25)$$

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Triangles connected with the Orthocentre.

$$\left.\begin{array}{ll} ABX = \dfrac{(a^2-b^2+c^2)\Delta}{2a^2} & ACX = \dfrac{(a^2+b^2-c^2)\Delta}{2a^2} \\ BCY = \dfrac{(a^2+b^2-c^2)\Delta}{2b^2} & BAY = \dfrac{(-a^2+b^2+c^2)\Delta}{2b^2} \\ CAZ = \dfrac{(-a^2+b^2+c^2)\Delta}{2c^2} & CBZ = \dfrac{(a^2-b^2+c^2)\Delta}{2c^2} \end{array}\right\} \quad (26)$$

$$\left.\begin{array}{l} \Delta_a = HCB = \dfrac{(a^2-b^2+c^2)(a^2+b^2-c^2)}{16\Delta} \\ \Delta_b = CHA = \dfrac{(a^2+b^2-c^2)(-a^2+b^2+c^2)}{16\Delta} \\ \Delta_c = BAH = \dfrac{(-a^2+b^2+c^2)(a^2-b^2+c^2)}{16\Delta} \end{array}\right\} \quad (27)$$

$$\left.\begin{array}{l} HBX = \dfrac{(a^2-b^2+c^2)^2(a^2+b^2-c^2)}{32a^2\Delta} \\ HCX = \dfrac{(a^2-b^2+c^2)(a^2+b^2-c^2)^2}{32a^2\Delta} \end{array}\right\} \quad (28)$$

$$\left.\begin{array}{c} AYZ = \dfrac{(-a^2+b^2+c^2)^2\Delta}{4b^2c^2}, \quad XBZ = \dfrac{(a^2-b^2+c^2)^2\Delta}{4c^2a^2}, \\ XYC = \dfrac{(a^2+b^2-c^2)^2\Delta}{4a^2b^2} \end{array}\right\} \quad (29)$$

$$XYZ = \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)\Delta}{4a^2b^2c^2} \quad (30)$$

$$\left.\begin{array}{l} DEF : XYZ = r : 2\rho \\ D_1E_1F_1 : XYZ = r_1 : 2\rho \end{array}\right\} \quad (31)$$

and so on.

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$$\left.\begin{aligned} \triangle_0 &: \triangle = 4s : x+y+z \\ \triangle_1 &: \triangle = 4s_1 : x+y+z \end{aligned}\right\} \tag{32}$$

and so on.

Miscellaneous.

$$\left.\begin{aligned} \mathrm{I\,I_1O} \cdot 8\triangle &= abc\,(c-b) \\ \mathrm{I\,I_2O} \cdot 8\triangle &= abc\,(c-a) \\ \mathrm{I\,I_3O} \cdot 8\triangle &= abc\,(a-b) \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} \mathrm{I\,I_1H} \cdot 8\triangle &= a(c-b)\,(b^2+c^2-a^2) \\ \mathrm{I\,I_2H} \cdot 8\triangle &= b(c-a)\,(c^2+a^2-b^2) \\ \mathrm{I\,I_3H} \cdot 8\triangle &= c(a-b)\,(a^2+b^2-c^2) \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} (2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3O})\;8\triangle &= abc\,(b+c) \\ (2\,\mathrm{I_3I_1I} - \mathrm{I_3I_1O})\;8\triangle &= abc\,(c+a) \\ (2\,\mathrm{I_1I_2I} - \mathrm{I_1I_2O})\;8\triangle &= abc\,(a+b) \end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned} (2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3H})\;8\triangle &= a(b+c)\,(b^2+c^2-a^2) \\ (2\,\mathrm{I_3I_1I} - \mathrm{I_3I_1H})\;8\triangle &= b(c+a)\,(c^2+a^2-b^2) \\ (2\,\mathrm{I_1I_2I} - \mathrm{I_1I_2H})\;8\triangle &= c(a+b)\,(a^2+b^2-c^2) \end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned} \mathrm{I\,H\,O} \cdot 8\triangle &= s\,(c-b)\,(c-a)\,(a-b) \\ \mathrm{I_1H\,O} \cdot 8\triangle &= s_1(c-b)\,(c+a)\,(a+b) \\ \mathrm{I_2H\,O} \cdot 8\triangle &= s_2(b+c)\,(c-a)\,(a+b) \\ \mathrm{I_3H\,O} \cdot 8\triangle &= s_3(b+c)\,(c+a)\,(a-b) \end{aligned}\right\} \tag{37}$$

$$\left.\begin{array}{l} \mathrm{I\,I_1O} \cdot 8\Delta = (r_1 - r)\,(r_3 - r_2)\,(r_2r_3 + rr_1) \\ \text{and so on.} \end{array}\right\} \quad (38)$$

$$\left.\begin{array}{l} \mathrm{I\,I_1H} \cdot 4\Delta = (r_1 - r)\,(r_3 - r_2)\,(r_2r_3 - rr_1) \\ \text{and so on.} \end{array}\right\} \quad (39)$$

$$\left.\begin{array}{l} (2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3O})\,8\Delta = (r + r_1)\,(r_2 + r_3)\,(r_2r_3 + rr_1) \\ \text{and so on.} \end{array}\right\} \quad (40)$$

$$\left.\begin{array}{l} (2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3H})\,4\Delta = (r + r_1)\,(r_2 + r_3)\,(r_2r_3 - rr_1) \\ \text{and so on.} \end{array}\right\} \quad (41)$$

$$\left.\begin{array}{l} \mathrm{I\,H\,O} \cdot 32\Delta = \dfrac{1}{\mathrm{R}r}(r_1 - r_2)(r_1 - r_3)(r_2 - r_3)(r_1 - r)(r_2 - r)(r_3 - r) \\ \mathrm{I_1H\,O} \cdot 32\Delta = \dfrac{1}{\mathrm{R}r_1}(r_1 + r_2)(r_1 + r_3)(r_3 - r_2)(r_1 - r)(r_2 + r)(r_3 + r) \\ \text{and so on.} \end{array}\right\} \quad (42)$$

It is probable that some of the preceding 42 formulae may belong to earlier dates, and to other authors than those indicated below. I shall be glad if any reader, who knows of earlier sources than those I have recorded, will take the trouble to inform me.

(2), (3) Mr E. Hain in *Nouvelle Correspondance Mathématique*, I. 75 (1874-5).

(5), (6) Feuerbach, *Eigenschaften...des...Dreiecks*, §§ 8, 9 (1822).

(7), (8) Mr B. Möllmann in Grunert's *Archiv*, XVII., 396 (1851).

(9), (10) Mr Combier gives the first of each of these expressions in the *Journal de Mathématiques Élémentaires*, III., 351 (1879).

(11), (12), (14) Mr B. Möllmann in Grunert's *Archiv*, XVII., 393-6 (1851).

(13) C. J. Matthes gives the first expression in *Commentatio de Proprietatibus Quinque Circulorum*, p. 55 (1831); T. S. Davies gives the others in the *Lady's and Gentleman's Diary* for 1842, p. 87.

(15) The first expression is given by Mr A. R. Clarke in 1847 in the *Mathematician*, III., 45 (1856); all are given by Mr C. Hellwig in Grunert's *Archiv*, XIX., 43 (1852).

(17) These proportions are implied in Feuerbach, *Eigenschaften...des...Dreiecks*, § 61 (1822). The first of them is given by C. J. Matthes in his *Commentatio*, p. 55, and also the first of (18).

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(19) The first expression is given by C. Adams in his *Eigenschaften...des... Dreiecks*, p. 61 (1846); the corresponding expression for ABC is given on p. 62.

(20) The last values of each of these sets are given by Thomas Weddle in the *Lady's and Gentleman's Diary* for 1845, p. 75.

(21) T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 88.

(22)-(25) Mr Georges Dostor in the *Journal de Mathématiques Élémentaires et Spéciales*, IV. 21-23 (1880). The first expression for LMN, however, is given by Grunert in his article "Dreieck," quoted on p. 25. C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 15 (1825), gives the expression

$$\frac{2\Delta}{(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})-1}.$$

(26)-(28) Mr C. Hellwig in Grunert's *Archiv*, XIX., 25-26 (1852).

(30) Feuerbach, *Eigenschaften...des...Dreiecks*, §23 (1822).

(31)-(32) C. Adams, *Eigenschaften...des...Dreiecks*, pp. 54, 53 (1846).

(33)-(42) Mr C. Hellwig in Grunert's *Archiv*, XIX., 43-50 (1852).