

SOME RELATIONS BETWEEN DIFFERENTIAL GEOMETRIC INVARIANTS AND TOPOLOGICAL INVARIANTS OF SUBMANIFOLDS¹⁾

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§ 1. Introduction.

Let M be an n -dimensional manifold immersed in an m -dimensional euclidean space E^m and let ∇ and $\tilde{\nabla}$ be the covariant differentiations of M and E^m , respectively. Let X and Y be two tangent vector fields on M . Then the second fundamental form h is given by

$$(1.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y).$$

It is well-known that $h(X, Y)$ is a normal vector field on M and it is symmetric on X and Y . Let ξ be a normal vector field on M , we write

$$(1.2) \quad \tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi,$$

where $-A_\xi(X)$ and $D_X \xi$ denote the tangential and normal components of $\tilde{\nabla}_X \xi$. Then we have

$$(1.3) \quad \langle A_\xi(X), Y \rangle = \langle h(X, Y), \xi \rangle,$$

where \langle , \rangle denotes the scalar product in E^m . The mean curvature vector H is defined by $H = (1/n) \text{trace } h$. Let S denote the length of h and α the length of H .

In this paper we shall obtain some relations between differential geometric invariants and a topological invariants of M . In particular, we shall prove that, for any closed n -dimensional submanifold M in E^m , the geometric invariant given by the integral of S^n depends on a topol-

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ological structure of M . Moreover, if the submanifold is δ -pinching in E^m (for the definition, see §4), then the total mean curvature, i.e., the geometric invariant given by the integral of α^n , also depends on the same topological structure of M . In particular, we see that among all δ -pinching submanifolds in E^m with a fixed $\delta > -1$, the submanifolds with large homology groups must have large total mean curvature.

§2. Basic formulas.

Let ξ be a unit normal vector field on M . We define the i -th mean curvature $K_i(\xi)$ at ξ by

$$(2.1) \quad \det(I + tA_\xi) = 1 + \sum_{i=1}^n \binom{n}{i} K_i(\xi) t^i,$$

where I is the identity transformation of the tangent spaces of M , t a parameter and $\binom{n}{i} = n!/i!(n-i)!$. Let R be the curvature tensor of M , i.e.,

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

Then the Gauss equation is given by

$$(2.2) \quad \langle R(X, Y)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle.$$

Let E_1, \dots, E_n be local orthonormal tangent vector fields of M . Then the scalar curvature ρ is defined by

$$(2.3) \quad \rho = \sum_{j=1}^n \left(\sum_{i=1}^n \langle R(E_i, E_j)E_j, E_i \rangle \right).$$

From (2.2) and (2.3) we have

$$(2.4) \quad \rho = n^2 \alpha^2 - S^2.$$

§3. Integral inequality for S^n .

Let \mathcal{F} be a field and let $H_i(M; \mathcal{F})$ be the i -th homology group of M over the field \mathcal{F} . Let $\beta_i(M; \mathcal{F})$ be the dimension of the i -th homology group $H_i(M; \mathcal{F})$. We define a topological invariant $\beta(M)$ by

$$\beta(M) = \max \left\{ \sum_{i=0}^n \beta_i(M; \mathcal{F}) : \mathcal{F} \text{ fields} \right\}.$$

The main aim of this section is to prove the following.

THEOREM 1. *Let M be an n -dimensional closed manifold immersed in a euclidean m -space E^m . Then we have*

$$(3.1) \quad \int_M S^n dV \geq \left\{ \left(\frac{n}{2} \right)^{n/2} c_n \right\} \beta(M),$$

where c_n is the area of a unit n -sphere. The equality sign of (3.1) holds when and only when M is diffeomorphic to an n -sphere and M is imbedded as a hypersphere of an $(n + 1)$ -dimensional linear subspace of E^m .

Proof. Let M be an n -dimensional closed manifold immersed in E^m and ξ be any unit normal vector field on M . We denote by $S(\xi)$ the length of the second fundamental tensor A_ξ at ξ . Let ξ_1, \dots, ξ_{m-n} be local orthonormal normal vector fields of M in E^m and $\xi = \sum_{r=1}^{m-n} \cos \gamma_r \xi_r$. Then we have

$$(3.2) \quad A_\xi = \sum \cos \gamma_r A_r, \quad A_r = A_{\xi_r}.$$

Hence we have

$$(3.3) \quad S(\xi)^2 = \text{trace}(A_\xi^2) = \sum_{r,s} \cos \gamma_r \cos \gamma_s \text{trace}(A_r A_s).$$

The right hand side of (3.3) is a quadratic form on $\cos \gamma_1, \dots, \cos \gamma_{m-n}$. Hence, we may choose local orthonormal normal vector fields $\bar{\xi}_1, \dots, \bar{\xi}_{m-n}$ such that with respect to this frame field, we have

$$(3.4) \quad S(\bar{\xi})^2 = \sum \rho_r \cos^2 \gamma_r, \quad \rho_1 \geq \rho_2 \geq \dots \geq \rho_{m-n} \geq 0,$$

$$(3.5) \quad \rho_r = \text{trace}(A_r^2) = S(\bar{\xi}_r)^2.$$

By the definition of S and ρ_r we have

$$(3.6) \quad S^2 = \rho_1 + \dots + \rho_{m-n}.$$

In the following, let B_ν be the bundle of unit normal vectors of M in E^m so that a point of B_ν is a pair (x, ξ) where ξ is a unit normal vector at the point x in M . Then B_ν is a bundle of $(m - n - 1)$ -dimensional spheres over M and is a manifold of dimension $m - 1$. Let Σ_x be the fibre of B_ν over x . Then there is a differential form $d\sigma$ of degree $m - n - 1$ on B_ν such that its restriction to a fibre Σ_x is the volume element $d\Sigma_x$ of Σ_x . Hence $d\sigma \wedge dV$ is the volume element of the bundle B_ν . On the bundle B_ν we define a function f by

$$(3.7) \quad f(x, \xi) = S(\xi)^2.$$

For $\xi = \sum \cos \gamma_r \bar{\xi}_r$ we have

$$(3.8) \quad f(x, \xi) = \sum \rho_r \cos^2 \gamma_r .$$

Since ρ_r , $r = 1, \dots, m - n$, are nonnegative and $\sum_r \cos^2 \gamma_r = 1$, an inequality of Minkowski [1, p. 21] implies that

$$(3.9) \quad \left\{ \int_{\Sigma_x} f^{n/2} d\Sigma_x \right\}^{2/n} = \left\{ \int_{\Sigma_x} (\sum \rho_r \cos^2 \gamma_r)^{n/2} d\Sigma_x \right\}^{2/n} \\ \leq \sum \left\{ \rho_r \left(\int_{\Sigma_x} |\cos^n \gamma_r| d\Sigma_x \right)^{2/n} \right\} .$$

Moreover, we have the following identity:

$$(3.10) \quad \int_{\Sigma_x} |\cos^n \gamma_r| d\Sigma_x = 2c_{n+p-1}/c_n .$$

Thus, by combining (3.6), (3.9) and (3.10), we find

$$(3.11) \quad S^n \geq \frac{c_n}{2c_{m-1}} \int_{\Sigma_x} f^{n/2} d\Sigma_x .$$

On the other hand, from the definition of $K_n(\xi)$ and an elementary relation between elementary symmetric functions, we have $S(\xi)^n \geq \sqrt{n^n} |K_n(\xi)|$. Hence, by using (3.11), we see that

$$(3.12) \quad \int_M S^n dV \geq \sqrt{n^n} \frac{c_n}{2c_{m-1}} \int_{B_p} |K_n(\xi)| d\sigma \wedge dV .$$

By a well-known inequality of Chern-Lashof [4, II], we have

$$(3.13) \quad \int_{B_p} |K_n(\xi)| d\sigma \wedge dV \geq c_{m-1} \beta(M) .$$

Thus, by combining (3.12) and (3.13), we obtain (3.1).

The remaining part of this theorem can be proved in a similar way as the corresponding results of Theorem 4.2 in [2, p. 229]. So we omit it.

Remark 1. Theorem 1 generalizes Theorem 4.1 of [3, II]. First, Theorem 1 drops the assumption of nonnegativeness of the scalar curvature of M . Second, if n is odd, the estimation is better than the one given in Theorem 4.1 of [3, II].

§ 4. Total mean curvature.

From Proposition 2.2 of [3, II] we see that the scalar curvature ρ is always bounded from above by $(n - 1)S^2$ and bounded below by $-S^2$, i.e.,

$$(4.1) \quad -S^2 \leq \rho \leq (n - 1)S^2 .$$

In the following, a submanifold M in E^m is said to satisfy a δ -pinching in E^m if we have

$$\delta S^2 \leq \rho \leq (n - 1)S^2$$

for some $\delta \geq -1$.

THEOREM 2. *Let M be an n -dimensional closed manifold immersed in a euclidean m -space E^m . If M satisfies a δ -pinching in E^m , then we have*

$$(4.2) \quad \int_M \alpha^n dV \geq \left\{ \frac{1}{2} \left(\frac{1 + \delta}{n} \right)^{n/2} c_n \right\} \beta(M) .$$

The equality sign of (4.2) holds when and only when M is $(n - 1)$ -pinching in E^m .

Proof. If M is δ -pinching in E^m , then (2.4) implies

$$(4.3) \quad \alpha^2 \geq \frac{1 + \delta}{2} S^2 .$$

Hence, by combining Theorem 1 and (4.3) we obtain (4.2).

Now, if the equality sign of (4.2) holds, then the equality sign of (3.1) holds. Hence, Theorem 1 implies that M is imbedded as a hypersphere of an $(n + 1)$ -dimensional linear subspace of E^m . In this case we have $n^2\alpha^2 = nS^2$. Hence, by (2.4), we see that M is $(n - 1)$ -pinching in E^m . The remaining part of this Theorem is trivial.

Remark 2. If $\delta > -1$ and M is a minimal submanifold of a unit hypersphere of E^m , then M is δ -pinching in E^m when and only when the scalar curvature ρ of M satisfies the following inequality:

$$\rho \geq \frac{\delta}{1 + \delta} n^2 .$$

In this case, $\int_M \alpha^n dV$ is equal to the volume of M .

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