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AN APPLICATION OF LOGIC TO ANALYSIS

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Let V be a complex analytic subvariety of an open subset of $\mathbf{C}^n$ and $p \in V$; let $\mathcal{O}_p(V)$, $\tilde{\mathcal{O}}_p(V)$, $C_p^\infty(V)$, $C_p^k(V)$ be the germs at p of holomorphic, weakly holomorphic, infinitely differentiable, and k times continuously differentiable functions respectively. Spallek [**15**] has shown that for any $p \in V$ there exists an integer $k > 0$ such that $C_p^k(V) \cap \tilde{\mathcal{O}}_p(V) = \mathcal{O}_p(V)$, generalizing the result of Malgrange [**12**] that $C_p^\infty(V) \cap \tilde{\mathcal{O}}_p(V) = \mathcal{O}_p(V)$. In [**14**], Siu proved Spallek's result from a more sheaf theoretic point of view and showed the minimal integer function $k(p)$ is bounded on compact sets. Bloom [**7**] reproved Malgrange's result by using differential operators on varieties. None of these methods is at all enlightening as to what the minimal integer k or even an upper bound for k is. In [**3**] the first author has shown that for curves and hypersurfaces $k \leqq N$, where N is the exponent of the conductor of the variety, that is, the minimal power such that $I(\mathrm{Sg}\ V)^N \tilde{\mathcal{O}} \subset \mathcal{O}$. In this paper we give a new proof of Spallek's result that shows that for isolated singularities we can given an upper bound for k in terms of the embedding dimension n, the maximal sheeting multiplicity m [**2**], and l, the number of generators of the ideal defining the variety. Since these are all locally bounded, k is also locally bounded. This result is not the best possible: a careful look at the proof in [**11**] for the case $Z^p = W^q$ in $\mathbf{C}^2$, $p > q$, p and q relatively prime, shows $k \leqq N = [p/q(q-1)]$; however [**4**] the correct value in this case is $k = [p/q(q-2)] + 1$, where $[x]$ is the greatest integer less than or equal to x.

The basic plan is as follows. In § 1 we write down a first order theory T (see below for definitions of logical terms) which says that K is an algebraically closed field of characteristic zero normed into an ordered field; V is an algebraic variety in K^n given by l polynomials of degree $\leqq d$; $p \in V$ is an isolated singularity; $f \in \tilde{\mathcal{O}}_p(V)/\mathcal{O}_p(V)$ and $f \in C_p^k(V)$ for $k = 1, 2, \ldots$

In § 2 we show that in every model of T it is true that $f = 0$. The Gödel Completeness Theorem then tells us that there is a first order proof of this statement from the axioms of T. This proof can use only finitely many of the axioms of T and hence only finitely many of the axioms $f \in C_p^k(V)$ $k = 1, 2, \ldots,$ say for $k = 1, \ldots, k(l, d, n)$. Since every first order theorem is true it is true that if K and V are as above and $f \in \tilde{\mathcal{O}}_p(V)/\mathcal{O}_p(V)$ and $f \in C_p^{k(l,d,n)}(V)$ then $f = 0$, that is, $C_p^{k(l,d,n)}(V) \cap \tilde{\mathcal{O}}_p(V) = \mathcal{O}_p(V)$.

In § 3 we use the fact that this bound depends only on l, d and n (see above) to transfer this bound from isolated to general singularities by taking sections transversal to the singular locus.

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1. Logic. In this section we define a first order language suitable for our needs, that is, in which we can make all the above statements. First we need the observation that if V is an algebraic variety defined by l polynomials of degree $\leqq d$ in K^n then there is a function $N_1(l, n, d)$ such that the normalization W of V can be realized as a variety in $K^{N_1(l,n,d)}$ defined by $\leqq N_1$ polynomials of degree $\leqq N_1$ and such that the mapping π: $W \to V$ is a projection. We also need that there is a bound $N_2(l, n, d)$ such that the vector space $\tilde{\mathcal{O}}(V)/\mathcal{O}_p(V)$ is generated over K by monomials of degree $\leqq N_2$. That these bounds exist (constructively, in fact) is shown in [**18**]. (The same bounds that work for the algebraic closure of $\mathbf{Q}$ will work for any algebraically closed field of characteristic 0.)

Our formal language will contain the following symbols:

(i) logical symbols: $\Rightarrow$ (implies), $\neg$ (not), $\vee$ (or, $\wedge$ (and), $\forall$ (for all), $\exists$ (there exists), and an infinite set of variables $x_1, x_2, \ldots$

(ii) nonlogical symbols: (a) constants $0, 1, \ldots a_1, \ldots, a_{\bar{N}}$ (to be used as coefficients for the polynomials specifying the variety, V), $b_1, \ldots, b_{\bar{N}_1}$ (to be used as coefficients for the polynomials specifying W), and $C_1, \ldots, C_{N_2}$ (to be used for the coefficients of an arbitrary element, α, of $\tilde{\mathcal{O}}/\mathcal{O}$); and constants $d_1, \ldots, d_{2n}$ to be used to specify the point p. (N_2 depends on $n, \bar{N}_1, \bar{N}$) (b) the relation symbol $<$ (c) Two binary function symbols $+$ (addition) and $\times$ (multiplication) and a $2n$-ary function symbol g (to be used to say that $\alpha \circ \pi^{-1} \in C_p^k(V)$ for $k = 1, 2, \ldots$).

For readers not familiar with formal logic we next give a precise definition of the (first order) formulas of our language and what we mean by a first order theorem and a model.

(i) Terms are built up by finitely many applications of the following rules:

(a) A variable is a term.

(b) A constant is a term.

(c) If $t_1, \ldots, t_k$ are terms and f is a k-ary function symbol (of the language) then $f(t_1, \ldots, t_2)$ is a term. (Terms are names in the language for objects.)

(ii) Formulas are built up by finitely many applications of the following rules.

(a) If t_1 and t_2 are terms then $t_1 = t_2$ and $t_1 < t_2$ are formulas.

(b) If A and B are formulas then so are $\neg A$, $A \to B$, $A \vee B$, $A \wedge B$.

(c) If A is a formula and x is a variable then $(\forall \times)A$ and $(\exists \times)A$ are formulas.

Later we shall specify some (nonlogical) axioms. We shall also assume but not specify some logical axioms and some rules of deduction. (The reader can find examples of these in any standard text on mathematical logic such as [12]. Since the exact form of these axioms and rules is irrelevant we suppress them.) We then define a (first order) proof to be a finite string of formulas $A_1, \ldots, A_l$ such that each is either an axiom as follows from some of the previous formulas by one of the rules of inference. A (first order) theorem is just the last formula of a proof.

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If we have a set of axioms (a theory) T then by a model $\mathfrak{A}$ of T we mean a set $|\mathfrak{A}|$ such that for each k-ary function symbol f (of the language) there is a k-ary function defined $|\mathfrak{A}|^k \to |\mathfrak{A}|$ and for each k-ary relation symbol (of the language) (e.g. $<$) there is a k-ary relation defined on $|\mathfrak{A}|^k$ such that interpreting the function symbols of the language as these functions and the relation symbols of the language as these relations and interpreting the logical symbols in the usual way, every axiom of T is true in $\mathfrak{A}$.

The Gödel completeness theorem then tells us that a first order formula A is provable (in the above sense) from a set of axioms T if and only if A is true in every model of T. The reader is referred to any standard text on mathematical logic (such as [**12**]) for a more detailed discussion of the above ideas.

Next we specify the axioms of the theory T_{ndl} and show that these axioms capture the desired concepts. The axioms are:

(i) Under $+$, $\times$, 0, 1, $<$ F is a real closed field, that is, the usual axioms for ordered fields (these are clearly first order) and the axiom $(\forall x)\ (x \geqq 0 \Leftrightarrow \exists\, y(x = y^2))$ and the axioms which state that every odd degree polynomial has a root, that is,

$$(\forall x_0) \dots (\forall x_{2n})(\exists\, y)(y^{2n+1} + x_{2n}y^{2n} + \dots + x_0 = 0)\ n = 1, 2, \dots .$$

(c.f. [**10**] for properties of real closed fields). We define $|x| = y$ to mean $x^2 = y^2 \wedge y \geqq 0$. We can define $K = F[i]$ (in a first order way) over F as ordered pairs with addition and multiplication defined in the usual way. It is clear that any first order statement about K can be translated into a first order statement about F. Hence we shall make statements about K. Also we can define $|x|$ for $x \in K$ in a first order way.

Let P_j, $(j = 1, \dots, l)$, be polynomials of degree d (using the constants a_k as coefficients) and define V as the zero set (over K) of these polynomials (i.e. $\mathbf{x} \in V \Leftrightarrow p_1(\mathbf{x}) = 0 \wedge \dots \wedge p_l(\mathbf{x}) = 0$ – this is first order). Define W in the same way using the b_k's.

(ii) The axiom which says that W is the normalization of V; that is, for some projection $\pi : W \to V$, π is $1-1$ at regular points of V and $\leqq \mu$ to 1 at singular points (see below) of V (a bound for μ can be obtained from n, d, l) and that W is maximal with respect to these properties among all varieties in K^{N_1} defined by N_1 polynomials of degree N_1. (This is known to algebraic geometers as the Zariski Main Theory; for example, if R is a normal spot over a field $R \subset S$, S local domain birational with S of same Krull dimension, and $m(R)S$ primary for $m(S)$, then $R = S$.) We can quantify over all such varieties by quantifying over the (bounded) number of defining polynomials. Since there are only finitely many projections $K^{N_1} \Rightarrow K^n$ and each is first order definable, π is definable. To say that $x \in V$ is a regular point just means that $J(x) \neq 0$ where J is the Jacobian of the polynomials defining V. This is certainly a first order statement among the a_k.

If t is a term in our language we define $\lim_{x\to a} t = b$ to mean

$$(\forall \epsilon > 0)(\exists\, \delta > 0)(\forall x)(|x - a| < \delta \Rightarrow |t - b| < \epsilon).$$

This is certainly first order. Similarly it is a first order statement to say that t is uniformly continuous on bounded sets or that g is (uniformly) C^k on bounded sets.

(iii) The axioms which say that g is uniformly C^k on bounded sets (for $k = 1, 2, \ldots$).

(iv) The axiom saying that $p = (d_1 + id_2, \ldots, d_{2n-1} + id_{2n})$ is an isolated singular point of V.

(v) The axiom saying that $g|_V$ agrees with $\alpha \circ \pi^{-1}$ in some neighborhood of p (α as defined above).

In § 2 it will be shown (Proposition 1) that in every model of $T_{n,d,l}$ it is true that $\alpha = 0$. This will give the required bound for $k(n, d, l)$ for isolated singularities p.

2.

PROPOSITION 1. *If V is an algebraic variety in K^n, K an algebraically closed field of characteristic* 0, *then* $C^\infty \cap \tilde{\mathcal{O}} = \mathcal{O}$.

Proof. Let ${}_n\mathcal{O}_p$ be the localization of the affine coordinate ring of K^n at the maximal ideal at p, ${}_V\mathcal{O}_p = {}_n\mathcal{O}_p/I(V, p)$ be the local ring of V at p and ${}_v\hat{\mathcal{O}}_p$ be its completion in its m-adic topology. The simple topology on ${}_n\mathcal{O}_p$ as a Frechet space given by the semi norms $\rho_\beta(\sum a_\alpha z^\alpha) = a_\beta$ is not as fine as the Krull topology determined by the metric $||f - g|| = e^{-\mathrm{ord}_p(f-g)}$ and so has less continuous functions. These topologies and all further constructions extend in the obvious manner to the disjoint union of finitely many analytic sets.

Let $\pi : W \to V$ be the normalization of V, q be the finite set $\pi^{-1}(p)$ and let $\mathcal{O}(W, q) = \bigoplus_{p' \in q} \mathcal{O}(W, p')$; we will identify $\mathcal{O}(V)$ with its image $\pi^*\mathcal{O}(V)$ in $\mathcal{O}(W)$. Since $\mathcal{O}(W)$ is a finite integral extension of $\mathcal{O}(V)$, the natural and induced Krull topologies agree on the closed subspace $\mathcal{O}(V)$ so π^* extends to an injection $\hat{\mathcal{O}}(V, p) \to \hat{\mathcal{O}}(W, q)$. Also it is not hard to see that the natural and induced Frechet topologies on $\hat{\mathcal{O}}(V, p)$ agree because this is a finite extension. Any constant coefficient differential operator

$$ {}_n\mathcal{O}_p \to \mathbf{C}, \quad f \to \sum_{|\alpha| < k} a_\alpha \frac{\partial^\alpha f}{\partial x^\alpha}(p) $$

is obviously continuous in either topology. Any continuous $\mathbf{C}$ linear function $L : {}_n\hat{\mathcal{O}}_p \to C$ is given by a constant coefficient differential operator

$$ L = \sum \frac{L(z^\alpha)}{\alpha!} \frac{\partial^\alpha}{\partial z}. $$

One notes the following which will not be used here: if the field F is archmedian, then the sum must be finite. (Or else $L(\sum z^\alpha / L(z^\alpha)) = 1 + 1 + \ldots = \infty$ where L commutes with infinite sums because it is continuous.)

Let $C(W)_q$ be the set of constant coefficient operators on W at $q = \{D : Df(q) = 0 \text{ for all } f \in \hat{I}(W, q)\} = \mathrm{Hom}_{\mathbf{C}}(\hat{\mathcal{O}}(W, q), \mathbf{C})$ and let

$\mathrm{Ann}\,({}_V\mathscr{O}_p) = \{D \in C(W)_q : Df(0) = 0 \text{ for all } f \in \hat{\mathscr{O}}(V, p)\} = \mathrm{Hom}_{\mathbf{C}}(\hat{\mathscr{O}}(W, q)/\hat{\mathscr{O}}(V, p), \mathbf{C})$. If $f \in {}_W\mathscr{O}_q$ and $\delta(f) = 0$ for each $\delta \in \mathrm{Ann}$, then $f \in {}_V\mathscr{O}_p$ (by the Hahn-Banach theorem for Frechet spaces the common zeroes of the δ's is just the zero element of $\hat{\mathscr{O}}(W, q)/\hat{\mathscr{O}}(V, p)$ so $f \in \hat{\mathscr{O}}(V, p) \cap \mathscr{O}(W, q) = \mathscr{O}(V, p)$).

Remark 1. Although the Hahn-Banach theorem is not true for normed vector spaces over an algebraically closed field of characteristic zero, we can prove a modified version which is good enough for the application in the proposition: Let B be a normed vector space over K, $A \subset B$ a closed subspace, $L : A \to K$ a bounded linear functional, and $b \in B - A$, then there is an extension of L, $L' : B \to K'$, where K' is a field extension of K, and $|L'(b)|/|b| = |L'| = |L|$. One shows this by just adjoining to the field K, the elements needed to make the usual proof work (over $\mathbf{R}$ these elements always exist by completeness).

LEMMA A. *There is countable set of irreducible curves C_i in W such that any constant coefficient operator D on W at q can be written as finite sum of constant coefficient operators on the C_i.*

Proof. [**7**, 4.5]. Choose a countable set of curves each intersecting $\mathrm{Reg}\,(W)$, so that $I(W) = \bigcap I(C_i)$ so $\hat{I}(W) = \bigcap \hat{I}(C_i)$. The natural injection $\hat{\mathscr{O}}(W) = {}_n\hat{\mathscr{O}}/\bigcap I(C_i) \to \sum_n \hat{\mathscr{O}}/I(C_i)$ induces a surjection $\mathrm{Hom}_{\mathbf{C}}(\hat{\mathscr{O}}(W), \mathbf{C}) \leftarrow \bigoplus \mathrm{Hom}_{\mathbf{C}}({}_n\hat{\mathscr{O}}/I(\mathbf{C}_i), \mathbf{C})$ since by the Hahn-Banach theorem any linear map $Z_1 \to \mathbf{C}$, $Z_1 \subset Z_2$, extends $Z_2 \to \mathbf{C}$ to a linear map. Hence $C(W)_q = \bigoplus C(C_i)_q$.

LEMMA B. *If C is an irreducible curve, then every constant coefficient operator can be represented as a variable coefficient operator, that is given $D \in C(W)_q$, there exists $\tilde{D} = \sum a_\alpha(z)\, \partial^\alpha/\partial z$, $a_\alpha \in \mathscr{O}(W, q)$, such that $\tilde{D}I(W, q) \subset I(W, q)$ and the specialization of $\tilde{D}$ at $q = D$.*

Proof. [**7**, 3.3]. The proof is given there for the complex numbers and convergent power series. It generalizes immediately to formal power series over an algebraically closed field of characteristic zero.

Let $f \in C_p^\infty \cap \tilde{\mathscr{O}}_p$, $\delta = \sum \delta_i$ the semirepresentation of δ as a finite sum of variable coefficient operators on curves C_i. We need to show $\delta(f) = 0$. Let T be the holomorphic part of the Taylor series of f about the point p and $g = f - T$; by construction the holomorphic Taylor series of g is zero. So by the chain rule and Leibnitz' rule we see that holomorphic derivatives of $g(\pi)$ vanish at q. Hence $\delta(g\pi) = 0$. Since T is a polynomial, $T(\pi) \in {}_V\mathscr{O}_p$ and $\delta(T\pi) = 0$. Hence $\delta(f\pi) = \delta(g\pi) + \delta(T\pi) = 0 + 0 + 0$.

In the above we have computed $\delta(g\pi)$ as follows: the operator δ is defined on formal power series on the ambient space K^m of W. but $g(\pi)$ is only C^∞ on K^m so we must replace $g(\pi)$ by some polynomial h on K^m so that $h|W = g\pi|W$. The Taylor series of h and $g(\pi)$ do not necessarily agree but $\varphi = h - g(\pi) \in I(V, C^\infty)$; we show each $\delta_i(\varphi)_p = 0$. Now δ_i is a variable operator so $\delta_i(\varphi)_p = \lim_{q \in C_i} \delta_i(\varphi)_q$. When restricting q to regular points of C_i we see that δ_i is just differentiation in the tangential directions at q and φ vanishes on V near q so $\delta_i(\varphi)_q = 0$. Hence $\delta_i(\varphi)_p = 0$.

Remark 2. In order to make the above analysis go over for a formally real closed ordered field F (instead of the reals), one must use calculus in a non-Archimedean field. Here one defines differentiability of a function in the usual manner and observes that differentiable $\Rightarrow$ uniformly continuous. It is clear that the usual concept of continuity is useless so is replaced by uniform continuity. (Since we are dealing with germs of functions and classically a continuous function on a compact set is uniformly continuous, there is no loss of generality.) It is elementary to verify the standard results of calculus: those results which claim the existence of points with certain properties are true after extending the field F to larger ordered field F'.

Since $K = F[i]$, we can define conjugation of an element of K exactly as one does in $\mathbf{C}$. Functions have Taylor series which are formal power series in z and $\bar{z}$. A holomorphic power series is a power series with only pure z terms. This completes the proof of Proposition 1.

Remark 3. The locus in V of the ideal J of universal denominators (this means $U \cdot \tilde{\mathcal{O}} \subset \mathcal{O}$) is Sg V so if p is an isolated singularity, locus (J) in W is just q so by the Hilbert Nullstellensatz (which is true over any algebraically closed field) there exists N (the conductor numbers) so that $m_q{}^N \tilde{\mathcal{O}} \subset \mathcal{O}$. So in this case $\tilde{\mathcal{O}}/\mathcal{O}$ is a finite dimensional vector space (all monomials of degree $\geqq N$ are the zero element.)

PROPOSITION 2. *Let V be an algebraic variety in K^n over an algebraically closed field of characteristic zero defined by l polynomials of degree $\leqq d$, with an isolated singular point. Then there is an integer $k(n, d, l)$ depending on n, d, l but not on V such that for $k \geqq k(n, d, l)$, $C^k(V) \cap \mathcal{O}(V) = \tilde{\mathcal{O}}(V)$.*

Proof. This follows from the Completeness Theorem [**8**] by restating the above in a first order language.

First order restatement of Proposition 1.

Hypothesis.

A) V is an algebraic variety in K^n over an algebraically closed field of characteristic zero defined by polynomials of degree $\leqq d$.

B) W is the normalization of V (min multi $(W) \leqq$ min multi $(V) \leqq$ max multi (V) so we again have an upper bound for the number of generators for W; the normalization is just the graph of the elements of the integral closure so W is defined by polynomials of degree $\leqq \max\{d, N\}$).

C) $h \in \tilde{\mathcal{O}}/\mathcal{O}$, $\dim_K \tilde{\mathcal{O}}/\mathcal{O} \leqq \binom{N+n}{n}$ = number of monomials of degree $\leqq N$ in n variables.

D) $h \in C^\infty(V)$.

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Conclusion. $h = 0$.

THEOREM. *Let V be a complex analytic variety in $\mathbf{C}^n$, $p \in V$, maximal multiplicity at p of V be m, the conductor number of V at p be N. Then there exists $k(n, m, N)$ so that for $k \geqq k(n, m, N)$, $C_p^k \cap \tilde{\mathcal{O}}_p = \mathcal{O}_p$.*

Proof. We already have the result for isolated analytic singularities since these are algebraic. By Lemma 4 and Proposition 2, it suffices to know there are upper bounds for $n(V \cap T)$, $m(V \cap T)$, $l(V \cap T)$ over generic slices $V \cap T$ transversal to the Sg V. These facts are established in Lemmas 2 and 3.

Section 3.

Explanation of Maximal Multiplicity. Let V be the germ of a r dimensional complex analytic variety in $\mathbf{C}^n$, then most (in the sense each plane $T \in G(n - r, n)$, the Grassman manifold of $n - r$ complex dimensional planes in $\mathbf{C}^n$ determines a projection $\pi_T : \mathbf{C}^n \rightarrow \mathbf{C}^r$ so that $\pi_T^{-1}(0) = T$) projections $\mathbf{C}^n \rightarrow \mathbf{C}^r$ give a branched covering of V; this is a proper continuous function with finite fibers with an analytic set $A \subset \mathbf{C}^r$, $A \neq \mathbf{C}^r$ such that $\pi : V - \pi^{-1}A \rightarrow \mathbf{C}^r - A$ is a covering map — the number of points in the fibers is called the sheeting order. The sheeting order depends, of course, on the projection. In [**2**], it was shown that at each point $p \in V$, there is a maximum sheeting order $m(p)$ and that the function $m : V \rightarrow \mathbf{Z}$ is bounded on compact subsets of V, and bounded for algebraic varieties. A brief sketch of the existence of the maximum multiplicity for hypersurfaces goes as follows: If V is a hypersurface, the ideal $I(V, 0)$ is principal and generated by some

$$f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha z^\alpha.$$

Now T is just spanned by some nonzero vector b, write $\mathbf{C}^n = \mathbf{C}b \oplus L$, $\dim L = n - 1$; $\pi_b : \mathbf{C}^n \rightarrow L$ has sheeting order $u \Leftrightarrow \mathrm{ord}_t f(tb, 0) = u \Leftrightarrow 0 = \sum_{|\alpha|=k} a_\alpha b^\alpha$, all $k \leqq u - 1$. Consider the ideal of $\mathcal{O}_p$ generated by the countable number of elements $g_k(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$. Since $\mathcal{O}_p$ is strong Noetherian (any ideal generated by some set of elements can be generated by some finite subset of these elements), there exists m such that $g_1, \ldots, g_m$ generate the rest of the g_i's. Hence if $g_i(b) = 0$, $1 \leqq i \leqq m$, then all $g_i(b) = 0$ and the line $\mathbf{C}b$ lies in the locus of f so π_b is not a branched covering of V. Hence the maximum multiplicity $< m$.

Unfortunately the maximum multiplicity is not an invariant of the local ring of the variety; it depends upon the choice of coordinates (consider $t \rightarrow (t^5, t^7, t^{71})$). But if V is embedded in minimal codimension, then there is a maximal multiplicity (over all analytic changes of co-ordinates) which is an invariant of the embedded variety [**6**]. In the special case of algebraic varieties one can compute the maximum multiplicity explicity (for hyper-surfaces, max mult = degree of polynomial defining variety).

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LEMMA 1. *For algebraic varieties, one can give an upper and lower bound for* max multi m *in terms of the degree of certain polynomials whose zero locus is the variety.*

Proof. Let $V \subset \mathbf{C}^n$, dim $V = r$; pick a branched covering $\pi : V \to \mathbf{C}^r$ of sheeting order $\leqq m$ and $(n - r - 1)\, u + 1$ linear functions $L_i : \mathbf{C}^{n-r} \to \mathbf{C}$ such that every set of $n - r$ of these L_i are linear independent. The minimal polynomial P_i of $L_i(z)$ in $\mathbf{C}[z_1, \ldots, z_n]/I(V)$ over $\mathbf{C}[z_1, \ldots, z_r]$ has degree $\leqq m$ and the common locus of the P_i is just V. If m is the maximal multiplicity one easily sees by considering partial projections $\mathbf{C}^n \to \mathbf{C}^{r+1}$ that degree of P_i in all variables $\leqq m$.

Conversely given polynomials P of deg $\leqq d$ defining V, consider the generic birational partial projection $\pi : V \to \mathbf{C}^{r+1}$, $V' = \pi(V)$. We have that $m(V) = m(V')$ and $m(V') <$ deg P' where P' is a polynomial which defines V' in $\mathbf{C}^{r+1}$. Furthermore one can find an upper bound for the degree of P' by computing P' explicitly by plugging the P's into the determinant formula for the resultant.

LEMMA 2. *Let V be a complex analytic variety. For a generic slice $V \cap T$ of V, and $p \in V \cap T$, $m(V \cap T, p) \leqq m(V, p)$.*

Proof. Let dim $V = r$ and T be a $(n - k)$-dimensional plane in $\mathbf{C}^n$ so that dim $V \cap T = r - k$. Choose a k-dimensional subspace S of $\mathbf{C}^n$ so $T \oplus S = \mathbf{C}^n$ and $(r - k)$ and $(n - r)$-dimensional subspaces T', T'' of T respectively so that $T' \oplus T'' = T$ and $\pi_{T'} : V \cap T \to T''$ is a branched covering of sheeting order u. Then $\pi_{T'}$ extends to a branched covering of V, $(\pi_{T'}, 1)$: $V \subset T \oplus S \to T'' \oplus S$ which has sheeting order $\geqq u$.

LEMMA 3. *There is an upper bound L for the number of generators of the ideal of linear slices of a complex analytic variety transversal to the singular locus near p.*

Proof. First it suffices to consider only parallel transversal slices by replacing V by $W = \{(q, p, T) \in C^n \times \text{Sg } V \times G\colon q \in V \cap (T + p)\}$ since $W \cap (C^n \times p \times T) = V \cap (T + p)$, where G is the Zariski open subset of the Grassman $G(n - k, n)$, $k =$ dim Sg V, of T such that dim $V \cap T = \text{codim}_V$ Sg V. The lemma then follows as in [**5**] by stratifying V into "equisingular" varieties (here equisingular mean that the parallel slices having simultaneous normalization by the same blow-ups) and letting $N =$ maximum of the numbered generators of the finite number of resulting strata.

LEMMA 4. *If V is complex analytic variety, h a weakly holomorphic function on V which is holomorphic on the generic linear slice of V transversal to* Sg V, *then h is holomorphic on V.*

Proof. It is trivial to modify [**7**, 4.5] to read as follows: let W be the normalization of V, $0 < j <$ dim $W = r$, $W \subset \mathbf{C}^n$, then there exist $(n - r + j)$-dimensional planes $\{T_i\}_{i=1}^{\infty}$, $W_i = W \cap T_i$, dim $W_i = j$, such that any constant coefficient differential operator D in Ann $(\mathscr{O}(V))$ can be written as a finite sum of constant coefficient operators on W_i which annihilate $\mathscr{O}(V)$, $D = D_1 + \ldots + D_k$. To show $h \in \mathscr{O}(V)$, we need only check $D(h) = 0$, but each $D_i(h) = 0$ by hypothesis.

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