ON THE TIMES OF BIRTHS IN A LINEAR BIRTHPROCESS

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Abstract: If $S_1, S_2, \ldots$ are the times of successive births in a pure birth-process with linear birthrate and $i > 0$ individuals initially, then given that there are $k > 0$ births in $(0, t)$, the random variables $e^{\lambda S_1}-1, e^{\lambda S_2}-1, \ldots e^{\lambda S_k}-1$ are distributed (conditionally) like the order statistics of k independent random variables, all uniformly distributed on the interval $(0, e^{\lambda t}-1)$, regardless of the initial population size. A similar property holds for the pure death process.

1. Introduction

A well-known property of the homogeneous Poisson process says that, given that in the interval $(0, t)$ there are $k > 0$ occurrences, the times of these occurrences are distributed like the order statistics of k independent uniform random variables on $(0, t)$, [1, p. 183].

We asked ourselves what the corresponding result might be for a pure birth process with linear birthprocess – also known as the Yule process [1, p. 180]. The answer to our question is, though elementary, somewhat surprising. This for two reasons.

a. The conditional distribution of the birthtimes does not depend on the initial population size.

b. It is now easy to construct a point process from the Yule process, which shares the property of the Poisson process stated above, but is not a Poisson process. This process supplies an elementary example to show that the stated property does not characterize the Poisson process.

2. Main theorem

We consider a Yule process with initial population size $i > 0$ and a birthrate proportional to the population size with parameter λ. Let S_j, $j = 1, 2, \cdots$ be the time of the j-th birth. We will prove that:

THEOREM 1. *Given that* $S_k < t \leqq S_{k+1}$, *the random variables* $e^{\lambda S_1}-1, \cdots,$ $e^{\lambda S_k}-1$, *are distributed as the order statistics of* k *independent random variables, uniformly distributed on* $(0, e^{\lambda t}-1)$ $k > 0$.

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PROOF. If $X(t)$ denotes the population size at time t, then it is well-known that:

$$(1)\qquad P\{X(t) = i+k|X(0) = i\} = (-1)^k \binom{-i}{k} e^{-i\lambda t}(1-e^{-\lambda t})^k, \qquad k \geqq 0, \quad i > 0$$

[1, p. 180].

Furthermore, we recall that the random variables $S_1, S_2-S_1, \cdots, S_{j+1}-S_j, \cdots$ are independent, negative exponentials with parameters $\lambda i, \lambda(i+1), \cdots, \lambda(i+j), \cdots$.

It follows that for any k numbers s_j satisfying $0 \leqq s_1 \leqq \cdots \leqq s_k \leqq t$, we have that:

$$(2)\qquad P\{S_1 \leqq s_1, \cdots, S_k \leqq s_k, S_{k+1} \geqq t\}$$

$$= \int_0^{s_1} e^{-\lambda i t_1}\lambda i \, dt_1 \int_0^{s_2-t_1} e^{-\lambda(i+1)t_2}\lambda(i+1)dt_2 \cdots$$

$$\int_0^{s_k-(t_1+\cdots+t_{k-1})} e^{-\lambda(i+k-1)t_k}\lambda(i+k-1)\,dt_k \int_{t-(t_1+\cdots+t_k)}^{\infty} e^{-\lambda(i+k)t_{k+1}}\lambda(i+k)\,dt_{k+1}$$

$$= \frac{\lambda^k e^{-\lambda(i+k)t}(i+k-1)!}{(i-1)!} \int_0^{s_1}\int_{v_1}^{s_2}\cdots\int_{v_{k-1}}^{s_k} e^{\lambda(v_1+\cdots+v_k)}dv_k\,dv_{k-1}\cdots dv_1$$

upon evaluating the last integral and making the change of variables

$$(3)\qquad v_j = t_0+\cdots+t_j, \qquad j = 1, \cdots, k, \quad t_0 = 0.$$

From (1) and (2) we obtain:

$$(4)\qquad P\{S_1 \leqq s_1, \cdots, S_k \leqq s_k|S_k < t \leqq S_{k+1}\} = \frac{\lambda^k e^{-\lambda k t}k!}{(1-e^{-\lambda t})^k}\int_0^{s_1}\int_{v_1}^{s_2}\cdots\int_{v_{k-1}}^{s_k} e^{\lambda(v_1+\cdots+v_k)}dv_k\cdots dv_1.$$

It follows that for any nymbers v_j with $0 \leqq v_1 \leqq \cdots \leqq v_k \leqq e^{\lambda t}-1$, we have:

$$(5)\qquad P\{e^{\lambda S_1}-1 \leqq v_1, \cdots, e^{\lambda S_k}-1 \leqq v_k|S_k < t \leqq S_{k+1}\} = \frac{k!}{(e^{\lambda t}-1)^k}\int_0^{v_1}\int_{u_1}^{v_2}\cdots\int_{u_{k-1}}^{v_k} du_k\cdots du_1.$$

This completes the proof.

If $S_1, \cdots, S_j, \cdots$ are the times of death in a pure death process with $i+k$ individuals initially and death rate μ, then the following result may be proved in the same manner as above.

THEOREM 2. *Given that $S_k < t \leqq S_{k+1}$, the random variables $1-e^{-\mu S_1}, \cdots, 1-e^{-\mu S_k}$ are distributed like the order statistics of k independent random variables uniformly distributed on* $(0, 1-e^{\mu t})$.

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Returning to the Yule process, we note that clearly it is not transformed into a homogeneous Poisson process by the change-of-timescale $\tau = e^{\lambda t} - 1$. *Yet, the point process generated by the random variables*:

$$Y_j = \frac{t}{e^{\lambda t} - 1}(e^{\lambda S_j} - 1) \qquad j = 1, 2, \cdots,$$

has the property of the Poisson process which we stated in the introduction.

The point process induced by the random variables Y_j is a very simple example of a point process which shares the conditional uniformity with the Poisson process, yet is not a Poisson process.

Reference

[1] Samuel Karlin, *A First Course in Stochastic Processes* (Academic Press, New York and London, 1966).

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