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TRILINEAR FOURIER MULTIPLIERS ON HARDY SPACES

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Abstract In this paper, we obtain the $H^{p_1} \times H^{p_2} \times H^{p_3} \to H^p$ boundedness for trilinear Fourier multiplier operators, which is a trilinear analogue of the multiplier theorem of Calderón and Torchinsky [4]. Our result improves the trilinear estimate in [22] by additionally assuming an appropriate vanishing moment condition, which is natural in the boundedness into the Hardy space H^p for $0 < p \leq 1$.

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1. Introduction

For a function σ on $\mathbb{R}^n$, let T_σ be the corresponding Fourier multiplier operator given by

$$T_\sigma f(x) := \int_{\mathbb{R}^n} \sigma(\xi)\widehat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\, d\xi$$

for a Schwartz function f on $\mathbb{R}^n$, where $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{2\pi i\langle x,\xi\rangle}dx$ is the Fourier transform of f. The function σ is called an L^p multiplier if T_σ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. For several decades, figuring out a sharp condition for σ to be an L^p multiplier has been one of the most interesting problems in harmonic analysis. Although there is no complete answer to this question, we have some satisfactory results. In 1956, Mihlin [23] proved that σ is an L^p multiplier provided that

$$|\partial^\alpha \sigma(\xi)| \lesssim |\xi|^{-|\alpha|}, \quad \xi \neq 0 \qquad \text{for any multi-indices } \ |\alpha| \le [n/2]+1. \tag{1.1}$$

This result was refined by Hörmander [21] who replaced (1.1) by the weaker condition

$$\sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L^2_s(\mathbb{R}^n)} < \infty \qquad \text{for} \quad s > n/2,$$

where $L^2_s(\mathbb{R}^n)$ denotes the fractional Sobolev space on $\mathbb{R}^n$ and ψ is a Schwartz function on $\mathbb{R}^n$ generating Littlewood–Paley functions, which will be officially defined in Section 2.1. We also remark that $s > n/2$ is the best possible regularity condition for the L^p boundedness of T_σ.

Now, we define the (real) Hardy space. Let ϕ be a smooth function on $\mathbb{R}^n$ that is supported in $\{x \in \mathbb{R}^n : |x| \le 1\}$, and we define $\phi_l := 2^{ln}\phi(2^l\cdot)$. Then the Hardy space $H^p(\mathbb{R}^n)$, $0<p\le\infty$, consists of tempered distributions f on $\mathbb{R}^n$ such that

$$\|f\|_{H^p(\mathbb{R}^n)} := \Big\|\sup_{l\in\mathbb{Z}} |\phi_l * f|\Big\|_{L^p(\mathbb{R}^n)} \tag{1.2}$$

is finite. The space provides an extension to $0<p\le 1$ in the scale of classical L^p spaces for $1<p\le\infty$, which is more natural and useful in many respects than the corresponding L^p extension. Indeed, $L^p(\mathbb{R}^n) = H^p(\mathbb{R}^n)$ for $1<p\le\infty$ and several essential operators, such as singular integrals of Calderón–Zygmund type, that are well-behaved on $L^p(\mathbb{R}^n)$

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only for $1<p\le\infty$ are also well-behaved on $H^p(\mathbb{R}^n)$ for $0<p\le 1$. Now, let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space on $\mathbb{R}^n$ and $\mathcal{S}_0(\mathbb{R}^n)$ be its subspace consisting of f satisfying

$$\int_{\mathbb{R}^n} x^\alpha f(x)\,dx=0 \qquad \text{for all multi-indices } \alpha.$$

Then it turns out that

$$\mathcal{S}_0(\mathbb{R}^n) \;\text{ is dense in }\; H^p(\mathbb{R}^n) \;\text{ for all }\; 0<p<\infty. \tag{1.3}$$

We remark that $\mathcal{S}(\mathbb{R}^n)$ is also dense in $H^p(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ for $1<p<\infty$, but not for $0<p\le 1$. See [31, Chapter III, §5.2] for more details. Moreover, as mentioned in [31, Chapter III, §5.4], if $f\in L^1(\mathbb{R}^n)\cap H^p(\mathbb{R}^n)$ for $0<p\le 1$, then

$$\int_{\mathbb{R}^n} x^\alpha f(x)\,dx=0 \qquad \text{for all multi-indices }\; |\alpha|\le \frac{n}{p}-n. \tag{1.4}$$

We refer to [2, 3, 7, 31, 33] for more details.

In 1977, Calderón and Torchinsky [4] provided a natural extension of the result of Hörmander to the Hardy space $H^p(\mathbb{R}^n)$ for $0<p\le 1$. For the purpose of investigating H^p estimates for $0<p\le 1$, the operator T_σ is assumed to initially act on $\mathcal{S}_0(\mathbb{R}^n)$ and then to admit an H^p-bounded extension for $0<p<\infty$ via density, in view of (1.3). Then Calderón and Torchinsky proved

Theorem A [4]. *Let $0<p\le 1$. Suppose that $s>n/p-n/2$. Then we have*

$$\big\|T_\sigma f\big\|_{H^p(\mathbb{R}^n)} \lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L^2_s(\mathbb{R}^n)} \|f\|_{H^p(\mathbb{R}^n)}$$

for all $f\in\mathcal{S}_0(\mathbb{R}^n)$.

For more information about the theory of Fourier multipliers, we also refer the reader to [1, 13, 19, 20, 25, 28, 29, 30] and the references therein.

We now turn our attention to multilinear extensions of the above multiplier results. Let m be a positive integer greater or equal to 2. For a bounded function σ on $(\mathbb{R}^n)^m$, let T_σ now denote an m-linear Fourier multiplier operator given by

$$T_\sigma\big(f_1,\cdots,f_m\big)(x):=\int_{(\mathbb{R}^n)^m}\sigma(\vec{\boldsymbol{\xi}})\Big(\prod_{j=1}^m \widehat{f_j}(\xi_j)\Big)\,e^{2\pi i\langle x,\xi_1+\cdots+\xi_m\rangle}\,d\vec{\boldsymbol{\xi}}, \quad \vec{\boldsymbol{\xi}}:=(\xi_1,\cdots,\xi_m)$$

for $f_1,\dots,f_m\in\mathcal{S}_0(\mathbb{R}^n)$. The first important result concerning multilinear multipliers was obtained by Coifman and Meyer [5] who proved that if N is sufficiently large and

$$\big|\partial_{\xi_1}^{\alpha_m}\cdots\partial_{\xi_m}^{\alpha_m}\sigma(\xi_1,\dots,\xi_m)\big|\lesssim_{\alpha_1,\dots,\alpha_m}\big|(\xi_1,\dots,\xi_m)\big|^{-(|\alpha_1|+\cdots+|\alpha_m|)}, \quad (\xi_1,\dots,\xi_m)\neq\vec{\mathbf{0}} \tag{1.5}$$

for all $|\alpha_1|+\cdots+|\alpha_m|\le N$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1<p_1,\dots,p_m<\infty$ and $1\le p<\infty$. This result is a multilinear analogue of Mihlin's result in which Equation (1.1) is required, but the optimal regularity condition, such as

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$|\alpha| \le [n/2]+1$ in Equation (1.1), is not considered in the result of Coifman and Meyer. Afterwards, Tomita [32] provided a sharp estimate for multilinear multiplier T_σ, as a multilinear counterpart of Hörmander's result. Let $\Psi^{(m)}$ be a Schwartz function on $(\mathbb{R}^n)^m$ having the properties that

$$\mathrm{supp}(\widehat{\Psi^{(m)}}) \subset \{\vec{\boldsymbol{\xi}} := (\xi_1,\dots,\xi_m) \in (\mathbb{R}^n)^m : 1/2 \le |\vec{\boldsymbol{\xi}}| \le 2\}, \quad \sum_{j\in\mathbb{Z}} \widehat{\Psi^{(m)}}(2^{-j}\vec{\boldsymbol{\xi}}) = 1,\ \vec{\boldsymbol{\xi}} \ne \vec{\mathbf{0}}.$$

For $s \ge 0$, we define the Sobolev norm

$$\|F\|_{L^2_s((\mathbb{R}^n)^m)} := \Big(\int_{(\mathbb{R}^n)^m} \big(1+4\pi^2|\vec{\boldsymbol{\xi}}|^2\big)^s \big|\widehat{F}(\vec{\boldsymbol{\xi}})\big|^2\, d\vec{\boldsymbol{\xi}}\Big)^{1/2}. \tag{1.6}$$

Theorem B [32]. *Let $1 < p,p_1,\dots,p_m < \infty$ with $1/p = 1/p_1+\cdots+1/p_m$. Suppose that*

$$\sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\vec{\cdot}\,)\widehat{\Psi^{(m)}}\,\big\|_{L^2_s((\mathbb{R}^n)^m)} < \infty$$

for $s > mn/2$. Then we have

$$\big\|T_\sigma\big(f_1,\dots,f_m\big)\big\|_{L^p} \lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\vec{\cdot}\,)\widehat{\Psi^{(m)}}\,\big\|_{L^2_s((\mathbb{R}^n)^m)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \tag{1.7}$$

for $f_1,\dots,f_m \in \mathcal{S}_0(\mathbb{R}^n)$.

The standard Sobolev space $L^2_s((\mathbb{R}^n)^m)$ in Equation (1.7) is replaced by a product-type Sobolev space in many recent papers.

Theorem C [14, 15, 18, 24]. *Let $0 < p_1,\dots,p_m \le \infty$ and $0 < p < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$. Suppose that*

$$s_1,\dots,s_m > \frac{n}{2}, \qquad \sum_{j\in J}\Big(\frac{s_j}{n} - \frac{1}{p_j}\Big) > -\frac{1}{2} \tag{1.8}$$

for any nonempty subsets J of $\{1,\dots,m\}$, and

$$\sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L^2_{(s_1,\dots,s_m)}((\mathbb{R}^n)^m)} < \infty. \tag{1.9}$$

Then we have

$$\big\|T_\sigma\big(f_1,\dots,f_n\big)\big\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L^2_{(s_1,\dots,s_m)}((\mathbb{R}^n)^m)} \prod_{j=1}^m \|f_j\|_{H^{p_j}(\mathbb{R}^n)} \tag{1.10}$$

for $f_1,\dots,f_m \in \mathcal{S}_0(\mathbb{R}^n)$.

Here, the space $L^2_{(s_1,\dots,s_m)}((\mathbb{R}^n)^m)$ indicates the product type Sobolev space on $(\mathbb{R}^n)^m$, in which the norm is defined by replacing the term $(1+4\pi^2|\vec{\boldsymbol{\xi}}|^2)^s$ in Equation (1.6) by

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$\prod_{j=1}^{m}\left(1+4\pi^2|\xi_j|^2\right)^{s_j}$. It is known in [27] that the condition (1.8) is sharp in the sense that if the condition does not hold, then there exists σ such that the corresponding operator T_σ does not satisfy Equation (1.10). We also refer the reader to [6, 11] for weighted estimates for multilinear Fourier multipliers.

As an extension of Theorem A to the whole range $0<p_1,\dots,p_m\le\infty$, in the recent paper of the authors, Lee, Heo, Hong, Park and Yang [22], we provide a multilinear multiplier theorem with standard Sobolev space conditions.

Theorem D [22]. *Let $0<p_1,\cdots,p_m\le\infty$ and $0<p<\infty$ with $1/p=1/p_1+\cdots+1/p_m$. Suppose that*

$$s>\frac{mn}{2}\quad \text{and}\quad \frac{1}{p}-\frac{1}{2}<\frac{s}{n}+\sum_{j\in J}\left(\frac{1}{p_j}-\frac{1}{2}\right) \tag{1.11}$$

for any subsets J of $\{1,\dots,m\}$, and

$$\sup_{k\in\mathbb{Z}}\left\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\right\|_{L^2_s((\mathbb{R}^n)^m)}<\infty. \tag{1.12}$$

Then we have

$$\left\|T_\sigma(f_1,\dots,f_m)\right\|_{L^p(\mathbb{R}^n)}\lesssim\sup_{k\in\mathbb{Z}}\left\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\right\|_{L^2_s((\mathbb{R}^n)^m)}\prod_{j=1}^{m}\|f_j\|_{H^{p_j}(\mathbb{R}^n)} \tag{1.13}$$

for $f_1,\dots,f_m\in\mathcal{S}_0(\mathbb{R}^n)$.

The optimality of the condition (1.11) was achieved by Grafakos, He and Hónzik [12] who proved that if Equation (1.13) holds, then we must necessarily have $s\ge mn/2$ and $1/p-1/2\le s/n+\sum_{j\in J}\left(1/p-1/2\right)$ for all subsets J of $\{1,\dots,m\}$.

We remark that in the bilinear case $m=2$, Theorem D follows from Theorem C as Equation (1.11) implies the existence of s_1 and s_2, with $s_1+s_2=s$, satisfying Equation (1.8). This is well described in the first proof of Theorem D in [22]. However, when $m\ge 3$, this inclusion is not evident even if similar types of regularity conditions are required in both theorems.

Unlike the estimate in Theorem A, the multilinear extensions in Theorems C and D consider the Lebesgue space L^p as a target space when $p\le 1$ (recall that $L^p=H^p$ for $1<p<\infty$).

If a function σ on $(\mathbb{R}^n)^m$ satisfies Equation (1.9) for $s_1,\dots,s_m>n/2$ or (1.12) for $s>mn/2$, then Theorems C and D imply that $T_\sigma(f_1,\dots,f_m)\in L^1$ for all $f_1,\dots,f_m\in\mathcal{S}_0(\mathbb{R}^n)$. Therefore, in order for $T_\sigma(f_1,\dots,f_m)$ to belong to $H^p(\mathbb{R}^n)$ for $0<p\le 1$, it should be necessary that

$$\int_{\mathbb{R}^n}x^\alpha T_\sigma\big(f_1,\dots,f_m\big)\,dx=0\qquad\text{for}\quad |\alpha|\le\frac{n}{p}-n, \tag{1.14}$$

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in view of Equation (1.4). However, this property is generally not guaranteed, even if all the functions $f_1,\dots,f_m$ satisfy the moment conditions, in the multilinear setting, while, in the linear case,

$$\int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0,\ |\alpha|\le N \quad \text{implies} \quad \int_{\mathbb{R}^n} x^\alpha T_\sigma f(x)\,dx = 0,\ |\alpha|\le N$$

for $N\ge 0$. Recently, by imposing additional cancellation conditions corresponding to (1.14), Grafakos, Nakamura, Nguyen and Sawano [16, 17] obtain a mapping property into Hardy spaces for T_σ.

Theorem E [16, 17]. *Let $0<p_1,\cdots,p_m\le\infty$ and $0<p\le 1$ with $1/p=1/p_1+\cdots+1/p_m$. Let N be sufficiently large and σ satisfy Equation (1.5) for all multi-indices $|\alpha_1|+\cdots+|\alpha_m|\le N$. Suppose that*

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma\big(a_1,\dots,a_m\big)(x)\,dx = 0$$

for all multi-indices $|\alpha|\le \frac{n}{p}-n$, where a_j's are (p_j,∞)-atoms. Then we have

$$\big\|T_\sigma(f_1,\dots,f_m)\big\|_{H^p(\mathbb{R}^n)} \lesssim_{\sigma,N} \prod_{j=1}^m \|f_j\|_{H^{p_j}(\mathbb{R}^n)} \tag{1.15}$$

for $f_1,\dots,f_m\in\mathcal{S}_0(\mathbb{R}^n)$.

Here, the (p,∞)-atom is similar, but more generalized concept than H^p-atoms defined in Section 2, and we adopt the convention that (∞,∞)-atom a simply means $a\in L^\infty(\mathbb{R}^n)$ with no cancellation condition. See [16, 17] for the definition and properties of the (p,∞)-atom.

We remark that Theorem E successfully shows the boundedness into $H^p(\mathbb{R}^n)$, but the optimal regularity conditions considered in Theorems C and D are not pursued at all as it requires sufficiently large N.

The aim of this paper is to establish the boundedness into H^p for trilinear multiplier operators, analogous to Equation (1.15), with the same regularity conditions as in Theorem D, which is significantly more difficult in general. Unfortunately, we do not obtain the desired results for general m-linear operators for $m\ge 4$ and we will discuss some obstacles for this generalization in the appendix.

To state our main result, let us write $\Psi := \Psi^{(3)}$ and in what follows, we will use the notation

$$\mathcal{L}_s^2[\sigma] := \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\vec{\cdot})\widehat{\Psi}\big\|_{L_s^2((\mathbb{R}^n)^3)}$$

for a function σ on $(\mathbb{R}^n)^3$. Let $0<p\le 1$, and we will consider trilinear multipliers σ satisfying

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma\big(f_1,f_2,f_3\big)(x)\,dx = 0 \quad \text{for all multi-indices } |\alpha|\le \frac{n}{p}-n \tag{1.16}$$

for all $f_1,f_2,f_3\in\mathcal{S}_0(\mathbb{R}^n)$. Then the main result is as follows:

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Theorem 1. *Let $0<p_1,p_2,p_3<\infty$ and $0<p\le 1$ with $1/p=1/p_1+1/p_2+1/p_3$. Suppose that*

$$s>\frac{3n}{2} \qquad \text{and} \qquad \frac{1}{p}-\frac{1}{2}<\frac{s}{n}+\sum_{j\in J}\Big(\frac{1}{p_j}-\frac{1}{2}\Big), \tag{1.17}$$

where J is an arbitrary subset of $\{1,2,3\}$. Let σ be a function on $(\mathbb{R}^n)^3$ satisfying $\mathcal{L}_s^2[\sigma]<\infty$ and the vanishing moment condition (1.16). Then we have

$$\big\|T_\sigma(f_1,f_2,f_3)\big\|_{H^p(\mathbb{R}^n)} \lesssim \mathcal{L}_s^2[\sigma]\|f_1\|_{H^{p_1}(\mathbb{R}^n)}\|f_2\|_{H^{p_2}(\mathbb{R}^n)}\|f_3\|_{H^{p_3}(\mathbb{R}^n)} \tag{1.18}$$

for $f_1,f_2,f_3\in \mathcal{S}_0(\mathbb{R}^n)$.

We remark that $(1/p_1,1/p_2,1/p_3)$ in Theorem 1 is contained in one of the following sets:

$$\begin{aligned}
\mathcal{R}_0 &:= \big\{(t_1,t_2,t_3): 0<t_1,t_2,t_3<1,\ 0<t_1+t_2,t_2+t_3,t_3+t_1<3/2, 1\le t_1+t_2+t_3<2\big\},\\
\mathcal{R}_{\mathrm{i}} &:= \big\{(t_1,t_2,t_3): 0<t_j<1/2,\ 1\le t_{\mathrm{i}}<\infty,\ j\ne \mathrm{i}\big\}, \quad \mathrm{i}=1,2,3,\\
\mathcal{R}_4 &:= \big\{(t_1,t_2,t_3): 0<t_3<1/2,\ 1/2\le t_1,t_2<\infty,\ 3/2\le t_1+t_2\big\},\\
\mathcal{R}_5 &:= \big\{(t_1,t_2,t_3): 0<t_1<1/2,\ 1/2\le t_2,t_3<\infty,\ 3/2\le t_2+t_3\big\},\\
\mathcal{R}_6 &:= \big\{(t_1,t_2,t_3): 0<t_2<1/2,\ 1/2\le t_1,t_3<\infty,\ 3/2\le t_1+t_3\big\},\\
\mathcal{R}_7 &:= \big\{(t_1,t_2,t_3): 1/2\le t_1,t_2,t_3<\infty,\ 2\le t_1+t_2+t_3<\infty\big\}.
\end{aligned}$$

See Figure 1 for the regions $\mathcal{R}_{\mathrm{i}}$. Then the condition (1.17) becomes

$$s>\begin{cases} 3n/2, & (1/p_1,1/p_2,1/p_3)\in\mathcal{R}_0,\\ n/p_{\mathrm{i}}+n/2, & (1/p_1,1/p_2,1/p_3)\in\mathcal{R}_{\mathrm{i}},\ \mathrm{i}=1,2,3\\ n/p_1+n/p_2, & (1/p_1,1/p_2,1/p_3)\in\mathcal{R}_4,\\ n/p_2+n/p_3, & (1/p_1,1/p_2,1/p_3)\in\mathcal{R}_5,\\ n/p_3+n/p_1, & (1/p_1,1/p_2,1/p_3)\in\mathcal{R}_6,\\ n/p_1+n/p_2+n/p_3-n/2, & (1/p_1,1/p_2,1/p_3)\in\mathcal{R}_7. \end{cases} \tag{1.19}$$

In the proof of Theorem 1, we will mainly focus on the case $(1/p_1,1/p_2,1/p_3)\in\mathcal{R}_{\mathrm{i}}$, $\mathrm{i}=1,2,3$, in which $s>n/p_{\mathrm{i}}+n/2$ is required. Then the remaining cases follow from interpolation methods. More precisely, via interpolation,

$$\begin{aligned}
&\text{the estimates (1.18) in } \mathcal{R}_1 \text{ and } \mathcal{R}_2 \ \Rightarrow\ \text{the estimate (1.18) in } \mathcal{R}_4,\\
&\text{the estimates (1.18) in } \mathcal{R}_2 \text{ and } \mathcal{R}_3 \ \Rightarrow\ \text{the estimate (1.18) in } \mathcal{R}_5,\\
&\text{the estimates (1.18) in } \mathcal{R}_3 \text{ and } \mathcal{R}_1 \ \Rightarrow\ \text{the estimate (1.18) in } \mathcal{R}_6,\\
&\text{the estimates (1.18) in } \mathcal{R}_1,\mathcal{R}_2, \text{ and } \mathcal{R}_3 \ \Rightarrow\ \text{the estimate (1.18) in } \mathcal{R}_0,\\
&\text{the estimates (1.18) in } \mathcal{R}_1,\mathcal{R}_2, \text{ and } \mathcal{R}_3 \ \Rightarrow\ \text{the estimate (1.18) in } \mathcal{R}_7,
\end{aligned}$$

where the case $1/p_1+1/p_2+1/p_3=1$ for $(1/p_1,1/p_2,1/p_3)\in\mathcal{R}_0$ will be treated separately. Here, a complex interpolation method will be applied, but the regularity condition on s will be fixed. Moreover, the index p will be also fixed so that the vanishing moment condition (1.16) will not be damaged in the process of the interpolation. For example,

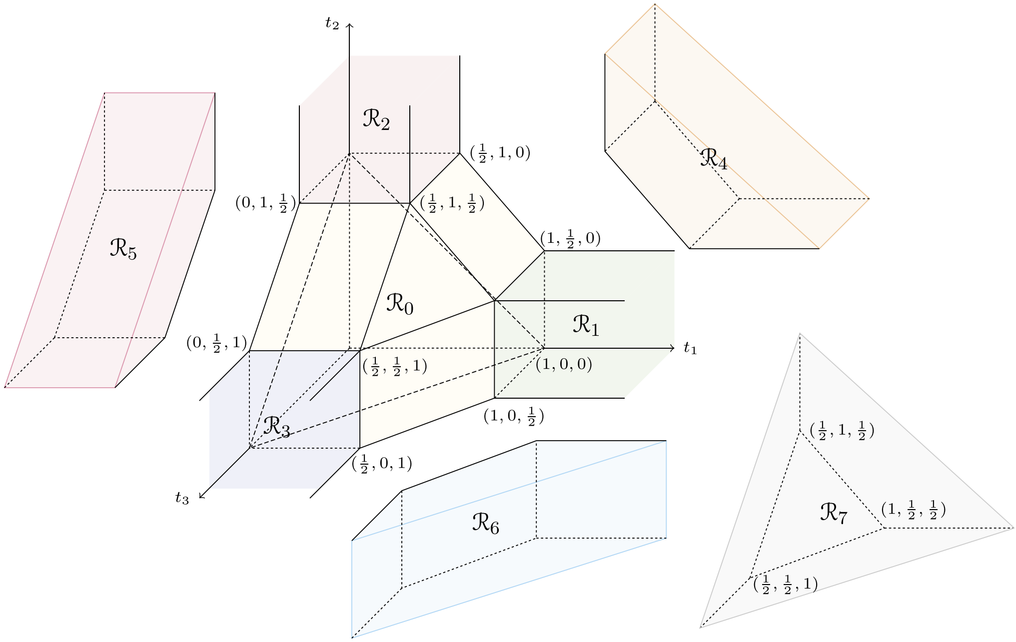

Figure 1. The regions $\mathcal{R}_{\mathrm{i}}$, $0 \le \mathrm{i} \le 7$.

when $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_4$, we set $s > n/p_1 + n/p_2$ and fix the index p with $1/p = 1/p_1 + 1/p_2 + 1/p_3$. We also fix σ satisfying the vanishing moment condition (1.16). Now, we choose $(1/p_1^0, 1/p_2^0, 1/p_3) \in R_1$ and $(1/p_1^1, 1/p_2^1, 1/p_3) \in \mathcal{R}_2$ so that

$$s > n/p_1^0 + n/2, \quad s > n/p_2^1 + n/2,$$

$$1/p = 1/p_1^0 + 1/p_2^0 + 1/p_3 = 1/p_1^1 + 1/p_2^1 + 1/p_3.$$

Then the two estimates

$$\|T_\sigma\|_{H^{p_1^0} \times H^{p_2^0} \times H^{p_3} \to H^p}, \|T_\sigma\|_{H^{p_1^1} \times H^{p_2^1} \times H^{p_3} \to H^p} \lesssim \mathcal{L}_s^2[\sigma]$$

imply

$$\|T_\sigma\|_{H^{p_1} \times H^{p_2} \times H^{p_3} \to H^p} \lesssim \mathcal{L}_s^2[\sigma].$$

The detailed arguments concerning the interpolation (for all the cases) will be provided in Section 3.

The estimates for $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_{\mathrm{i}}$, $\mathrm{i} = 1,2,3$, will be restated in Proposition 3.1 below, and they will be proved throughout three sections (Sections 5–7). Since one of p_j's is less or equal to 1, we benefit from the atomic decomposition for the Hardy space. Moreover, for other indices greater than 2, we employ the techniques of (variant) φ-transform, introduced by Frazier and Jawerth [8, 9, 10] and Park [26], which will be

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presented in Section 2. Then $T_\sigma(f_1,f_2,f_3)$ can be decomposed in the form

$$T_\sigma(f_1,f_2,f_3)=\sum_{\kappa\in \mathrm{K}} T^\kappa(f_1,f_2,f_3)$$

where K is a finite set, and then we will actually prove that each $T^\kappa(f_1,f_2,f_3)$ satisfies the estimate

$$\sup_{l\in\mathbb{Z}}\big|\phi_l*\big(T^\kappa(f_1,f_2,f_3)\big)(x)\big|\lesssim \mathcal{L}_s^2[\sigma]u_1(x)u_2(x)u_3(x), \tag{1.20}$$

where $\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)}\lesssim \|f_{\mathrm{i}}\|_{H^{p_{\mathrm{i}}}(\mathbb{R}^n)}$ for $\mathrm{i}=1,2,3$. Since the above estimate separates the left-hand side into three functions of x, we may apply Hölder's inequality with exponents $1/p=1/p_1+1/p_2+1/p_3$ to obtain, in view of Equation (1.2),

$$\begin{aligned}\big\|T^\kappa(f_1,f_2,f_3)\big\|_{H^p(\mathbb{R}^n)}&=\Big\|\sup_{l\in\mathbb{Z}}\big|\phi_l*\big(T^\kappa(f_1,f_2,f_3)\big)\big|\Big\|_{L^p(\mathbb{R}^n)}\\&\lesssim \mathcal{L}_s^2[\sigma]\|f_1\|_{H^{p_1}(\mathbb{R}^n)}\|f_2\|_{H^{p_2}(\mathbb{R}^n)}\|f_3\|_{H^{p_3}(\mathbb{R}^n)}.\end{aligned}$$

Such pointwise estimates (1.20) will be described in several lemmas in Sections 6 and 7, and the proofs will be given in Section 9 separately, which is one of the keys in this paper.

Notation

For a cube Q in $\mathbb{R}^n$ let $\mathbf{x}_Q$ be the lower left corner of Q and $\ell(Q)$ be the side-length of Q. We denote by Q^*, Q^{**} and Q^{***} the concentric dilates of Q with $\ell(Q^*)=10\sqrt{n}\ell(Q)$, $\ell(Q^{**})=\big(10\sqrt{n}\big)^2\ell(Q)$ and $\ell(Q^{***})=\big(10\sqrt{n}\big)^3\ell(Q)$. Let $\mathcal{D}$ stand for the family of all dyadic cubes in $\mathbb{R}^n$ and $\mathcal{D}_j$ be the subset of $\mathcal{D}$ consisting of dyadic cubes of side-length 2^{-j}. For each $\mathbf{x}\in\mathbb{R}^n$ and $l\in\mathbb{Z}$, let $B_{\mathbf{x}}^l:=B(\mathbf{x},100n2^{-l})$ be the ball of radius $100n2^{-l}$ and center $\mathbf{x}$. We use the notation $\langle\cdot\rangle$ to denote both the inner product of functions and $\langle y\rangle:=(1+4\pi^2|y|^2)^{1/2}$ for $y\in\mathbb{R}^M$, $M\in\mathbb{N}$. That is, $\langle f,g\rangle=\int_{\mathbb{R}^n}f(x)\overline{g(x)}\,dx$ for two functions f and g, and $\langle x_1\rangle:=(1+4\pi^2|x_1|^2)^{1/2}$, $\langle(x_1,x_2)\rangle:=\big(1+4\pi^2(|x_1|^2+|x_2|^2)\big)^{1/2}$ for $x_1,x_2\in\mathbb{R}^n$.

2. Preliminaries

2.1. Hardy spaces

Let θ be a Schwartz function on $\mathbb{R}^n$ such that $\mathrm{supp}(\widehat{\theta})\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2\}$ and $\widehat{\theta}(\xi)=1$ for $|\xi|\le 1$. Let $\psi:=\theta-2^{-n}\theta(2^{-1}\cdot)$, and for each $j\in\mathbb{Z}$ we define $\theta_j:=2^{jn}\theta(2^j\cdot)$ and $\psi_j:=2^{jn}\psi(2^j\cdot)$. Then $\{\psi_j\}_{j\in\mathbb{Z}}$ forms a Littlewood–Paley partition of unity, satisfying

$$\mathrm{supp}(\widehat{\psi_j})\subset\big\{\xi\in\mathbb{R}^n:2^{j-1}\le|\xi|\le 2^{j+1}\big\}\quad\text{ and }\quad\sum_{j\in\mathbb{Z}}\widehat{\psi_j}(\xi)=1,\ \xi\ne 0.$$

We define the convolution operators Γ_j and Λ_j by

$$\Gamma_jf:=\theta_j*f,\qquad \Lambda_jf:=\psi_j*f.$$

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The Hardy space $H^p(\mathbb{R}^n)$ can be characterized with the (quasi-)norm equivalences

$$\|f\|_{H^p(\mathbb{R}^n)} \sim \big\|\{\Gamma_j f\}_{j\in\mathbb{Z}}\big\|_{L^p(\ell^\infty)}, \qquad 0<p\le\infty \tag{2.1}$$

and

$$\|f\|_{H^p(\mathbb{R}^n)} \sim \big\|\{\Lambda_j f\}_{j\in\mathbb{Z}}\big\|_{L^p(\ell^2)}, \qquad 0<p<\infty, \tag{2.2}$$

which is the Littlewood–Paley theory for Hardy spaces. In addition, when $p\le 1$, every $f\in H^p(\mathbb{R}^n)$ can be decomposed as

$$f=\sum_{k=1}^{\infty}\lambda_k a_k \quad \text{ in the sense of tempered distributions,} \tag{2.3}$$

where a_k's are H^p-atoms having the properties that $\mathrm{supp}(a_k)\subset Q_k$, $\|a_k\|_{L^\infty(\mathbb{R}^n)}\le |Q_k|^{-1/p}$ for some cube Q_k, $\int x^\gamma a_k(x)dx=0$ for all multi-indices $|\gamma|\le M$, and $\big(\sum_{k=1}^\infty|\lambda_k|^p\big)^{1/p}\lesssim\|f\|_{H^p(\mathbb{R}^n)}$, where M is a fixed integer satisfying $M\ge[n/p-n]_+$, which may be actually arbitrarily large. Furthermore, each H^p-atom a_k satisfies

$$\|a_k\|_{H^1(\mathbb{R}^n)}\lesssim|Q_k|^{-1/p+1}.$$

2.2. Maximal inequalities

Let $\mathcal{M}$ denote the Hardy–Littlewood maximal operator, defined by

$$\mathcal{M}f(x):=\sup_{Q:x\in Q}\frac{1}{|Q|}\int_Q|f(y)|dy$$

for a locally integrable function f on $\mathbb{R}^n$, where the supremum ranges over all cubes Q containing x. For given $0<r<\infty$, we define $\mathcal{M}_r f:=\big(\mathcal{M}\big(|f|^r\big)\big)^{1/r}$. Then it is well-known that

$$\big\|\{\mathcal{M}_r f_k\}_{k\in\mathbb{Z}}\big\|_{L^p(\ell^q)}\lesssim\|\{f_k\}_{k\in\mathbb{Z}}\|_{L^p(\ell^q)} \tag{2.4}$$

whenever $r<p<\infty$ and $r<q\le\infty$. We note that for $1\le r<\infty$

$$\left\|\frac{f(x-\cdot)}{\langle 2^j\cdot\rangle^t}\right\|_{L^r(\mathbb{R}^n)}\lesssim 2^{-jn/r}\mathcal{M}_r f(x) \qquad \text{if } \ t>n/r. \tag{2.5}$$

For $\boldsymbol{m}\in\mathbb{Z}^n$ and any dyadic cubes $Q\in\mathcal{D}$, we use the notation

$$Q(\boldsymbol{m}):=Q+\ell(Q)\boldsymbol{m}.$$

Then we define the dyadic shifted maximal operator $\mathcal{M}_{dyad}^{\boldsymbol{m}}$ by

$$\mathcal{M}_{dyad}^{\boldsymbol{m}}f(x):=\sup_{Q\in\mathcal{D}:x\in Q}\frac{1}{|Q|}\int_{Q(\boldsymbol{m})}|f(y)|\,dy,$$

where the supremum is taken over all dyadic cubes Q containing x. It is clear that $\mathcal{M}_{dyad}^{\boldsymbol{0}}f(x)\le\mathcal{M}f(x)$ and accordingly, $\mathcal{M}_{dyad}^{\boldsymbol{0}}$ is bounded on L^p for $p>1$. In general, the

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following maximal inequality holds: For $1 < p < \infty$ and $\boldsymbol{m} \in \mathbb{Z}^n$ we have

$$\left\| \mathcal{M}^{\boldsymbol{m}}_{dyad} f \right\|_{L^p(\mathbb{R}^n)} \lesssim \left(\log (10 + |\boldsymbol{m}|) \right)^{n/p} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.6}$$

The inequality (2.6) follows from the repeated use of the inequality in one-dimensional setting that appears in [31, Chapter II, §5.10], and we omit the detailed proof here. Refer to [22, Appendix] for the argument.

2.3. Variants of φ-transform

For a sequence of complex numbers $\boldsymbol{b} := \{b_Q\}_{Q \in \mathcal{D}}$, we define

$$\|\boldsymbol{b}\|_{\dot{f}^{p,q}} := \left\| g^q(\boldsymbol{b}) \right\|_{L^p(\mathbb{R}^n)}$$

for $0 < p < \infty$, where

$$g^q(\boldsymbol{b})(x) := \left\| \left\{ |b_Q| |Q|^{-1/2} \chi_Q(x) \right\}_{Q \in \mathcal{D}} \right\|_{\ell^q}, \qquad 0 < q \le \infty.$$

Let $\widetilde{\psi_j} := \psi_{j-1} + \psi_j + \psi_{j+1}$ for $j \in \mathbb{Z}$. Observe that $\widetilde{\psi_j}$ enjoys the properties that $\mathrm{supp}(\widehat{\widetilde{\psi}}) \subset \{\xi \in \mathbb{R}^n : 2^{j-2} \le |\xi| \le 2^{j+2}\}$ and $\psi_j = \psi_j * \widetilde{\psi_j}$. Then we have the representation

$$\Lambda_j f(x) = \sum_{Q \in \mathcal{D}_j} b_Q \psi^Q(x), \tag{2.7}$$

where $\psi^Q(x) := |Q|^{1/2} \psi_j(x - \mathbf{x}_Q)$, $\widetilde{\psi}^Q(x) := |Q|^{1/2} \widetilde{\psi_j}(x - \mathbf{x}_Q)$ for each $Q \in \mathcal{D}_j$, and $b_Q = \langle f, \widetilde{\psi}^Q \rangle$. This implies that

$$f = \sum_{j \in \mathbb{Z}} \Lambda_j f = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j} b_Q \psi^Q \quad \text{ in } \mathcal{S}'/\mathcal{P},$$

where $\mathcal{S}'/\mathcal{P}$ stands for a tempered distribution modulo polynomials. Moreover, in this case, we have

$$\|\boldsymbol{b}\|_{\dot{f}^{p,q}} \sim \left\| \{\Lambda_j f\}_{j \in \mathbb{Z}} \right\|_{L^p(\ell^q)}. \tag{2.8}$$

Therefore, the Hardy space $H^p(\mathbb{R}^n)$ can be characterized by the discrete function space $\dot{f}^{p,2}$, in view of the equivalence in Equation (2.2). We refer to [8, 9, 10] for more details.

It is also known in [26] that $\Gamma_j f$ has a representation analogous to (2.7) with an equivalence similar to (2.8), while $f \neq \sum_{j \in \mathbb{Z}} \Gamma_j f$ generally. Let $\widetilde{\theta} := 2^n \theta(2\cdot)$ and $\widetilde{\theta_j} := 2^{jn} \widetilde{\theta}(2^j \cdot) = \theta_{j+1}$ so that $\theta_j = \theta_j * \widetilde{\theta_j}$. Let $\theta^Q(x) := |Q|^{1/2} \theta_j(x - \mathbf{x}_Q)$, $\widetilde{\theta}^Q(x) := |Q|^{1/2} \widetilde{\theta}_j(x - \mathbf{x}_Q)$, and $b_Q = \langle f, \widetilde{\theta}^Q \rangle$ for each $Q \in \mathcal{D}_j$. Then we have

$$\Gamma_j f(x) = \sum_{Q \in \mathcal{D}_j} b_Q \theta^Q(x) \tag{2.9}$$

and for $0 < p < \infty$ and $0 < q \le \infty$

$$\left\| \{\Gamma_j f\}_{j \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \sim \|\boldsymbol{b}\|_{\dot{f}^{p,q}}. \tag{2.10}$$

We refer to [26, Lemma 3.1] for more details.

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3. Proof of Theorem 1: reduction and interpolation

The proof of Theorem 1 can be obtained by interpolating the estimates in the following propositions.

Proposition 3.1. *Let* $0 < p_1, p_2, p_3 < \infty$ *and* $0 < p < 1$ *with* $1/p = 1/p_1 + 1/p_2 + 1/p_3$. *Suppose that* $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ *and*

$$s > \frac{n}{\min\{p_1, p_2, p_3\}} + \frac{n}{2}.$$

Let σ *be a function on* $(\mathbb{R}^n)^3$ *satisfying* $\mathcal{L}_s^2[\sigma] < \infty$ *and the vanishing moment condition (1.16). Then we have*

$$\big\|T_\sigma(f_1, f_2, f_3)\big\|_{H^p(\mathbb{R}^n)} \lesssim \mathcal{L}_s^2[\sigma] \|f_1\|_{H^{p_1}(\mathbb{R}^n)} \|f_2\|_{H^{p_2}(\mathbb{R}^n)} \|f_3\|_{H^{p_3}(\mathbb{R}^n)}$$

for $f_1, f_2, f_3 \in \mathcal{S}_0(\mathbb{R}^n)$.

Proposition 3.2. *Let* $0 < p \le 1$. *Suppose that one of* p_1, p_2, p_3 *is equal to* p *and the other two are infinity. Suppose that* $s > n/p + n/2$. *Let* σ *be a function on* $(\mathbb{R}^n)^3$ *satisfying* $\mathcal{L}_s^2[\sigma] < \infty$ *and the vanishing moment condition (1.16). Then we have*

$$\big\|T_\sigma(f_1, f_2, f_3)\big\|_{H^p(\mathbb{R}^n)} \lesssim \mathcal{L}_s^2[\sigma] \|f_1\|_{H^{p_1}(\mathbb{R}^n)} \|f_2\|_{H^{p_2}(\mathbb{R}^n)} \|f_3\|_{H^{p_3}(\mathbb{R}^n)}$$

for $f_1, f_2, f_3 \in \mathcal{S}_0(\mathbb{R}^n)$.

We present the proof of Proposition 3.1 in Sections 5, 6 and 7 and that of Proposition 3.2 in Section 8. For now, we proceed with the following interpolation argument, simply assuming the above propositions hold.

Lemma 3.1. *Let* $0 < p_1^0, p_2^0, p_3^0 \le \infty$, $0 < p_1^1, p_2^1, p_3^1 \le \infty$ *and* $0 < p^0, p^1 < \infty$. *Suppose that*

$$\big\|T_\sigma\big\|_{H^{p_1^l} \times H^{p_2^l} \times H^{p_3^l} \to H^{p^l}} \lesssim \mathcal{A}, \qquad l = 0, 1.$$

Then for any $0 < \theta < 1$, $0 < p_1, p_2, p_3 \le \infty$ *and* $0 < p < \infty$ *satisfying*

$$1/p_j = (1-\theta)/p_j^0 + \theta/p_j^1 \qquad \text{for } \; j = 1, 2, 3,$$

$$1/p = (1-\theta)/p^0 + \theta/p^1,$$

we have

$$\|T_\sigma\|_{H^{p_1} \times H^{p_2} \times H^{p_3} \to H^p} \lesssim \mathcal{A}.$$

The proof of the lemma is essentially same as that of [22, Lemma 2.4], so it is omitted here.

3.1. Proof of Equation (1.18) when $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_4 \cup \mathcal{R}_5 \cup \mathcal{R}_6$

We need to work only with $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_4$ since the other cases are just symmetric versions. In this case, $2 < p_3 < \infty$ and as mentioned in Equation (1.19), the condition (1.17) is equivalent to

$$s > n/p_1 + n/p_2.$$

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Now, choose $\widetilde{p_1}, \widetilde{p_2} < 1$ such that

$$1/p_1 + 1/p_2 = 1/\widetilde{p_1} + 1/2 = 1/2 + 1/\widetilde{p_2}$$

and thus

$$s > n/\widetilde{p_1} + n/2, \quad s > n/2 + n/\widetilde{p_2}.$$

Let $\epsilon_1, \epsilon_2 > 0$ be numbers with

$$s > n/(\widetilde{p_1} - \epsilon_1) + n/2, \quad s > n/2 + n/(\widetilde{p_2} - \epsilon_2)$$

and select $q_1, q_2 > 2$ such that

$$1/p = 1/(\widetilde{p_1} - \epsilon_1) + 1/q_1 + 1/p_3 = 1/q_2 + 1/(\widetilde{p_2} - \epsilon_2) + 1/p_3.$$

Then we observe that

$$(1-\theta)\Big(\frac{1}{\widetilde{p_1} - \epsilon_1}, \frac{1}{q_1}, \frac{1}{p_3}\Big) + \theta\Big(\frac{1}{q_2}, \frac{1}{\widetilde{p_2} - \epsilon_2}, \frac{1}{p_3}\Big) = \Big(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\Big)$$

for some $0 < \theta < 1$. Let $C_1 := (1/(\widetilde{p_1} - \epsilon_1), 1/q_1, 1/p_3)$ and $C_2 := (1/q_2, 1/(\widetilde{p_2} - \epsilon_2), 1/p_3)$. It is obvious that $C_1 \in \mathcal{R}_1$, $C_2 \in \mathcal{R}_2$, and thus it follows from Proposition 3.1 that

$$\|T_\sigma\|_{H^{\widetilde{p_1} - \epsilon_1} \times H^{q_1} \times H^{p_3} \to H^p} \lesssim \mathcal{L}_s^2[\sigma] \quad \text{at} \quad C_1 = (1/(\widetilde{p_1} - \epsilon_1), 1/q_1, 1/p_3) \in \mathcal{R}_1,$$
$$\|T_\sigma\|_{H^{q_2} \times H^{\widetilde{p_2} - \epsilon_2} \times H^{p_3} \to H^p} \lesssim \mathcal{L}_s^2[\sigma] \quad \text{at} \quad C_2 = (1/q_2, 1/(\widetilde{p_2} - \epsilon_2), 1/p_3) \in \mathcal{R}_2.$$

Finally, the assertion (1.18) for $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_4$ is derived by means of interpolation in Lemma 3.1. See Figure 2 for the interpolation.

3.2. Proof of Equation (1.18) when $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_0$

We first fix $1/2 < p < 1$ such that $1/p_1 + 1/p_2 + 1/p_3 = 1/p$ and assume that, in view of Equation (1.19),

$$s > 3n/2 = n/1 + n/2.$$

Then we choose $2 < p_0 < \infty$ such that $1 + 1/2 + 1/p_0 = 1/p$. Then it is clear that $(1/p_1, 1/p_2, 1/p_3)$ is located inside the hexagon with the vertices $(1, 1/p_0, 1/2)$, $(1, 1/2, 1/p_0)$, $(1/2, 1, 1/p_0)$, $(1/p_0, 1, 1/2)$, $(1/p_0, 1/2, 1)$ and $(1/2, 1/p_0, 1)$. Now, we choose a sufficiently small $\epsilon > 0$ and $2 < \widetilde{p_0} < \infty$ such that

$$\frac{1}{2+\epsilon} + \frac{1}{\widetilde{p_0}} = \frac{1}{2} + \frac{1}{p_0},$$

and the point $(1/p_1, 1/p_2, 1/p_3)$ is still inside the smaller hexagon with $D_1 := (1, 1/\widetilde{p_0}, 1/(2+\epsilon))$, $D_2 := (1, 1/(2+\epsilon), 1/\widetilde{p_0})$, $D_3 := (1/(2+\epsilon), 1, 1/\widetilde{p_0})$, $D_4 := (1/\widetilde{p_0}, 1, 1/(2+\epsilon))$, $D_5 := (1/\widetilde{p_0}, 1/(2+\epsilon), 1)$, and $D_6 := (1/(2+\epsilon), 1/\widetilde{p_0}, 1)$. Now, Proposition 3.1 deduces that

$$\|T_\sigma\|_{H^{q_1} \times H^{q_2} \times H^{q_3} \to H^p} \lesssim \mathcal{L}_s^2[\sigma]$$

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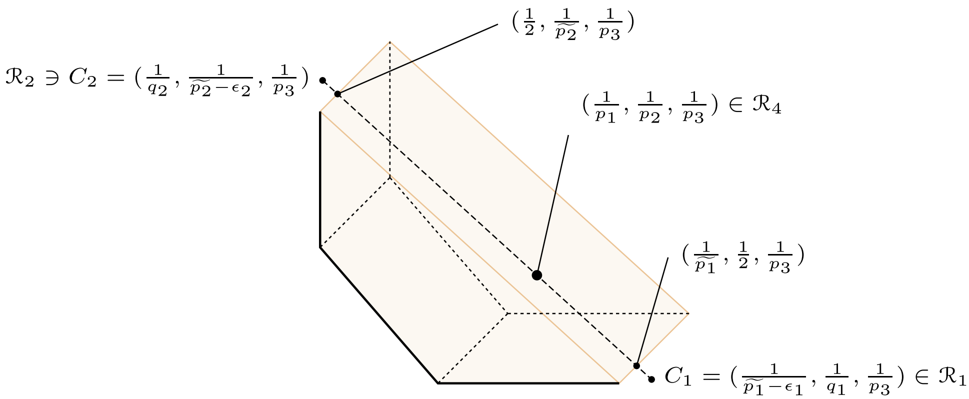

Figure 2. $(1-\theta)(\frac{1}{\widetilde{p_1}-\epsilon_1}, \frac{1}{q_1}, \frac{1}{p_3}) + \theta(\frac{1}{q_2}, \frac{1}{\widetilde{p_2}-\epsilon_2}, \frac{1}{p_3}) = (\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}) \in \mathcal{R}_4$.

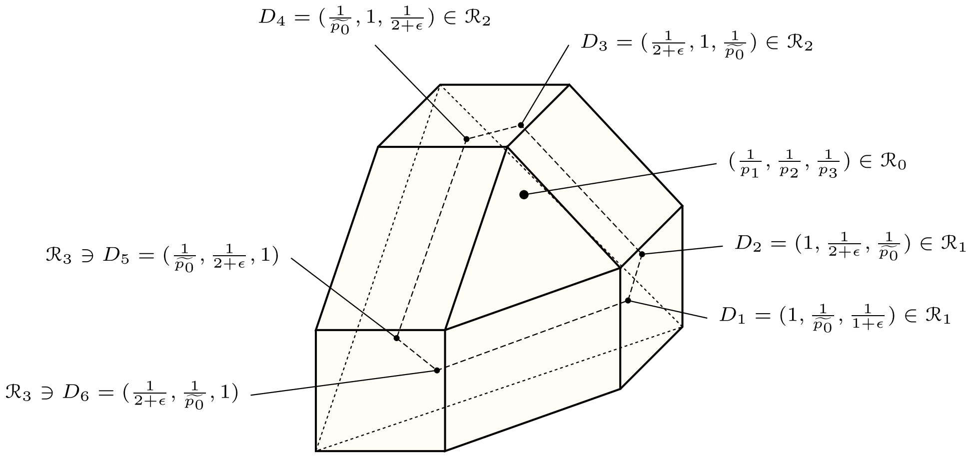

Figure 3. $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}) \in \mathcal{R}_0$.

for $(1/q_1, 1/q_2, 1/q_3) \in \{D_1, D_2, D_3, D_4, D_5, D_6\}$, as $D_1, D_2 \in \mathcal{R}_1$, $D_3, D_4 \in \mathcal{R}_2$ and D_5, $D_6 \in \mathcal{R}_3$. This implies, via interpolation in Lemma 3.1,

$$\big\|T_\sigma(f_1,f_2,f_3)\big\|_{H^p(\mathbb{R}^n)} \lesssim \mathcal{L}_s^2[\sigma]\|f_1\|_{H^{p_1}(\mathbb{R}^n)}\|f_2\|_{H^{p_2}(\mathbb{R}^n)}\|f_3\|_{H^{p_3}(\mathbb{R}^n)}.$$

See Figure 3 for the interpolation.

For the case $p=1$, we interpolate the estimates in Proposition 3.2. To be specific, for any given $0 < p_1, p_2, p_3 < \infty$ with $1/p_1 + 1/p_2 + 1/p_3 = 1$, the estimate (1.18) with $p=1$ follows from interpolating

$$\big\|T_\sigma(f_1,f_2,f_3)\big\|_{H^1(\mathbb{R}^n)} \lesssim \mathcal{L}_s^2[\sigma]\|f_1\|_{H^1(\mathbb{R}^n)}\|f_2\|_{H^\infty(\mathbb{R}^n)}\|f_3\|_{H^\infty(\mathbb{R}^n)},$$
$$\big\|T_\sigma(f_1,f_2,f_3)\big\|_{H^1(\mathbb{R}^n)} \lesssim \mathcal{L}_s^2[\sigma]\|f_1\|_{H^\infty(\mathbb{R}^n)}\|f_2\|_{H^1(\mathbb{R}^n)}\|f_3\|_{H^\infty(\mathbb{R}^n)},$$
$$\big\|T_\sigma(f_1,f_2,f_3)\big\|_{H^1(\mathbb{R}^n)} \lesssim \mathcal{L}_s^2[\sigma]\|f_1\|_{H^\infty(\mathbb{R}^n)}\|f_2\|_{H^\infty(\mathbb{R}^n)}\|f_3\|_{H^1(\mathbb{R}^n)}.$$

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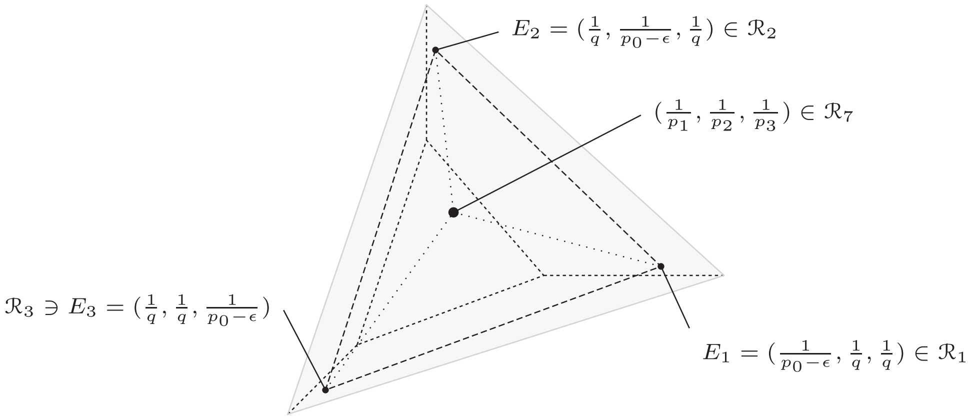

Figure 4. $\theta_1\big(\frac{1}{p_0-\epsilon},\frac{1}{q},\frac{1}{q}\big)+\theta_2\big(\frac{1}{q},\frac{1}{p_0-\epsilon},\frac{1}{q}\big)+\theta_3\big(\frac{1}{q},\frac{1}{q},\frac{1}{p_0-\epsilon}\big)=\big(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p_3}\big)\in\mathcal{R}_7$.

3.3. Proof of Equation (1.18) when $(1/p_1,1/p_2,1/p_3)\in\mathcal{R}_7$

Let $0<p\le 1/2$ be such that $1/p=1/p_1+1/p_2+1/p_3$, and assume that

$$s>n/p-n/2.$$

We choose $0<p_0\le 1$, satisfying $1/p_0+1=1/p$, so that

$$s>n/p_0+n/2.$$

Then there exist $\epsilon>0$ and $2<q<\infty$ so that $s>n/(p_0-\epsilon)+n/2$ and $1/p=1/(p_0-\epsilon)+2/q$. Let $E_1:=\big(1/(p_0-\epsilon),1/q,1/q\big)$, $E_2:=\big(1/q,1/(p_0-\epsilon),1/q\big)$, and $E_3:=\big(1/q,1/q,1/(p_0-\epsilon)\big)$. Then it is immediately verified that $E_1\in\mathcal{R}_1$, $E_2\in\mathcal{R}_2$, $E_3\in\mathcal{R}_3$, and

$$\theta_1\Big(\frac{1}{(p_0-\epsilon)},\frac{1}{q},\frac{1}{q}\Big)+\theta_2\Big(\frac{1}{q},\frac{1}{(p_0-\epsilon)},\frac{1}{q}\Big)+\theta_3\Big(\frac{1}{q},\frac{1}{q},\frac{1}{(p_0-\epsilon)}\Big)=\Big(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p_3}\Big)$$

for some $0<\theta_1,\theta_2,\theta_3<1$ with $\theta_1+\theta_2+\theta_3=1$. Therefore, Proposition 3.1 yields that

$$\begin{aligned}
&\|T_\sigma\|_{H^{p_0-\epsilon}\times H^q\times H^q\to H^p}\lesssim\mathcal{L}_s^2[\sigma] \quad \text{at } E_1=\big(1/(p_0-\epsilon),1/q,1/q\big)\in\mathcal{R}_1,\\
&\|T_\sigma\|_{H^q\times H^{p_0-\epsilon}\times H^q\to H^p}\lesssim\mathcal{L}_s^2[\sigma] \quad \text{at } E_2=\big(1/q,1/(p_0-\epsilon),1/q\big)\in\mathcal{R}_2,\\
&\|T_\sigma\|_{H^q\times H^q\times H^{p_0-\epsilon}\to H^p}\lesssim\mathcal{L}_s^2[\sigma] \quad \text{at } E_3=\big(1/q,1/q,1/(p_0-\epsilon)\big)\in\mathcal{R}_3,
\end{aligned}$$

and using the interpolation method in Lemma 3.1, we conclude the estimate (1.18) holds for $(1/p_1,1/p_2,1/p_3)\in\mathcal{R}_7$. See Figure 4 for the interpolation.

4. Auxiliary lemmas

This section is devoted to providing several technical results which will be repeatedly used in the proof of Propositions 3.1 and 3.2.

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Lemma 4.1. *Let $N \in \mathbb{N}$ and $a \in \mathbb{R}^n$. Suppose that a Schwartz function f, defined on $\mathbb{R}^n$, satisfies*

$$\int_{\mathbb{R}^n} x^\alpha f(x)dx = 0 \quad \text{for all multi-indices } \alpha \quad \text{with} \quad |\alpha| \le N. \tag{4.1}$$

Then for any $0 \le \epsilon \le 1$, there exists a constant $C_\epsilon > 0$ such that

$$\big\|\phi_l * f\big\|_{L^\infty(\mathbb{R}^n)} \le C_\epsilon 2^{l(N+n+\epsilon)} \int_{\mathbb{R}^n} |y-a|^{N+\epsilon}|f(y)|\, dy.$$

Proof. Using the Taylor theorem for ϕ_l, we write

$$\begin{aligned}\phi_l(x-y) = &\sum_{|\alpha|\le N-1} \frac{\partial^\alpha \phi_l(x-a)}{\alpha!}(a-y)^\alpha \\ &+ N \sum_{|\alpha|=N} \frac{1}{\alpha!}\Big(\int_0^1 (1-t)^{N-1}\partial^\alpha \phi_l\big(x-a+t(a-y)\big)dt\Big)(a-y)^\alpha.\end{aligned}$$

Then it follows from the condition (4.1) that

$$\begin{aligned}\big|\phi_l * f(x)\big| &\lesssim_N \sum_{|\alpha|=N} \Big|\int_{\mathbb{R}^n} \Big(\int_0^1 (1-t)^{N-1}\partial^\alpha \phi_l\big(x-a+t(a-y)\big)dt\Big)(a-y)^\alpha f(y)\, dy\Big| \\ &\lesssim_N \sum_{|\alpha|=N} \Big|\int_{\mathbb{R}^n} \Big[\int_0^1 (1-t)^{N-1}\partial^\alpha \phi_l\big(x-a+t(a-y)\big)\, dt \\ &\qquad - \int_0^1 (1-t)^{N-1}\partial^\alpha \phi_l(x-a)dt\Big](a-y)^\alpha f(y)\, dy\Big| \\ &\lesssim \sum_{|\alpha|=N} \int_{\mathbb{R}^n} \Big(\int_0^1 \big|\partial^\alpha \phi_l\big(x-a+t(a-y)\big) - \partial^\alpha \phi_l(x-a)\big|dt\Big)|y-a|^N|f(y)|\, dy.\end{aligned}$$

For $|\alpha| = N$, we note that

$$\big|\partial^\alpha \phi_l\big(x-a+t(a-y)\big) - \partial^\alpha \phi_l(x-a)\big| \lesssim 2^{l(N+n+1)}|y-a| \tag{4.2}$$

and

$$\begin{aligned}&\big|\partial^\alpha \phi_l\big(x-a+t(a-y)\big) - \partial^\alpha \phi_l(x-a)\big| \\ &\quad \le \big|\partial^\alpha \phi_l\big(x-a+t(a-y)\big)\big| + \big|\partial^\alpha \phi_l(x-a)\big| \lesssim 2^{l(N+n)}.\end{aligned} \tag{4.3}$$

Then by averaging both Equation (4.2) and Equation (4.3), we obtain that

$$\big|\partial^\alpha \phi_l\big(x-a+t(a-y)\big) - \partial^\alpha \phi_l(x-a)\big| \lesssim_\epsilon 2^{l(N+n+\epsilon)}|y-a|^\epsilon, \qquad 0 \le \epsilon \le 1,$$

which completes the proof. □

Now, we recall that $\widetilde{\psi_j} = \psi_{j-1} + \psi_j + \psi_{j+1}$ and $\widetilde{\theta_j} = 2^n\theta_j(2\cdot)$, and then define $\widetilde{\Lambda_j} g := \widetilde{\psi_j} * g$ and $\widetilde{\Gamma_j} g := \widetilde{\theta_j} * g$.

Lemma 4.2. *Let $2 \le q < \infty$, $s > n/q$, and $L > n,s$. Let φ be a function on $\mathbb{R}^n$ satisfying*

$$|\varphi(x)| \lesssim_M \frac{1}{(1+|x|)^M} \qquad \text{for all } M > 0.$$

For $j \in \mathbb{Z}$ and for each $Q \in \mathcal{D}_j$, let

$$\varphi^Q(x) := 2^{jn/2}\varphi\big(2^j(x - \mathbf{x}_Q)\big),$$

and for a Schwartz function g on $\mathbb{R}^n$ let

$$\mathcal{B}_Q(g) := \Big\langle \big|\widetilde{\Lambda_j} g\big|, \frac{2^{jn/2}}{\langle 2^j(\cdot - \mathbf{x}_Q)\rangle^L} \Big\rangle \quad \text{or} \quad \Big\langle \big|\widetilde{\Gamma_j} g\big|, \frac{2^{jn/2}}{\langle 2^j(\cdot - \mathbf{x}_Q)\rangle^L} \Big\rangle.$$

Then we have

$$\begin{aligned} &\Big\| \sum_{Q \in \mathcal{D}_j} |\mathcal{B}_Q(g)| \frac{\chi_{Q^c}(x)}{\langle 2^j(x - \mathbf{x}_Q)\rangle^s} \big|\varphi^Q(z)\big| \Big\|_{L^q(z)} \\ &\quad \lesssim_L 2^{-jn/q} \bigg(\sum_{Q \in \mathcal{D}_j} \Big(|\mathcal{B}_Q(g)||Q|^{-1/2} \frac{\chi_{Q^c}(x)}{\langle 2^j(x - \mathbf{x}_Q)\rangle^s} \Big)^q \bigg)^{1/q}. \end{aligned}$$

Proof. For $2 \le q < \infty$, we have

$$\begin{aligned} &\Big(\sum_{Q \in \mathcal{D}_j} |\mathcal{B}_Q(g)| \frac{\chi_{Q^c}(x)}{\langle 2^j(x - \mathbf{x}_Q)\rangle^s} \big|\varphi^Q(z)\big| \Big)^q \\ &\quad \lesssim \Big(\sum_{Q \in \mathcal{D}_j} |\mathcal{B}_Q(g)|^{q/2} \frac{\chi_{Q^c}(x)}{\langle 2^j(x - \mathbf{x}_Q)\rangle^{qs/2}} \big|\varphi^Q(z)\big| \Big)^2 \Big(\sum_{Q \in \mathcal{D}_j} \big|\varphi^Q(z)\big| \Big)^{q-2}, \end{aligned}$$

where Hölder's inequality is applied if $2 < q < \infty$. Clearly,

$$\begin{aligned} \Big(\sum_{Q \in \mathcal{D}_j} \big|\varphi^Q(z)\big| \Big)^{q-2} &\lesssim_M 2^{jn(q-2)/2} \Big(\sum_{Q \in \mathcal{D}_j} \frac{1}{\langle 2^j(z - \mathbf{x}_Q)\rangle^M} \Big)^{q-2} \\ &= 2^{jn(q-2)/2} \Big(\sum_{\boldsymbol{m} \in \mathbb{Z}^n} \frac{1}{\langle 2^j z - \boldsymbol{m}\rangle^M} \Big)^{q-2} \lesssim_M 2^{jn(q-2)/2} \end{aligned}$$

for sufficiently large $M > n$. Therefore, the left-hand side of the claimed estimate is less than a constant times

$$2^{jn(1/2-1/q)} \Big\| \sum_{Q \in \mathcal{D}_j} |\mathcal{B}_P(g)|^{q/2} \frac{\chi_{Q^c}(x)}{\langle 2^j(x - \mathbf{x}_Q)\rangle^{qs/2}} \big|\varphi^Q(z)\big| \Big\|_{L^2(z)}^{2/q}. \tag{4.4}$$

The L^2 norm is dominated by

$$\Big(\sum_{Q \in \mathcal{D}_j} |\mathcal{B}_Q(g)|^{q/2} \frac{\chi_{Q^c}(x)}{\langle 2^j(x - \mathbf{x}_Q)\rangle^{qs/2}} \sum_{R \in \mathcal{D}_j} |\mathcal{B}_R(g)|^{q/2} \frac{1}{\langle 2^j(x - \mathbf{x}_R)\rangle^{qs/2}} \big\langle |\varphi^Q|, |\varphi^R| \big\rangle \Big)^{1/2}.$$

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Note that

$$\langle|\varphi^Q|,|\varphi^R|\rangle \lesssim_{q,L} \frac{1}{\langle 2^j(\mathbf{x}_Q-\mathbf{x}_R)\rangle^{3Lq/2}}$$

and thus the preceding term is controlled by a constant multiple of

$$\Big(\sum_{Q\in\mathcal{D}_j}|\mathcal{B}_Q(g)|^q\frac{\chi_{Q^c}(x)}{\langle 2^j(x-\mathbf{x}_Q)\rangle^{qs}}\sum_{R\in\mathcal{D}_j}\frac{1}{\langle 2^j(\mathbf{x}_Q-\mathbf{x}_R)\rangle^{Lq/2}}\Big)^{1/2}.$$

Here, we used the facts that

$$\frac{|\mathcal{B}_R(g)|}{(1+2^j|\mathbf{x}_Q-\mathbf{x}_R|)^L}\le|\mathcal{B}_Q(g)|$$

and

$$\frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{qs/2}\langle 2^j(\mathbf{x}_Q-\mathbf{x}_R)\rangle^{Lq/2}}\le\frac{1}{\langle 2^j(x-\mathbf{x}_Q)\rangle^{qs/2}}.$$

Since the sum over $R\in\mathcal{D}_j$ converges, we deduce

$$(4.4)\lesssim 2^{-jn/q}\Big(\sum_{Q\in\mathcal{D}_j}\Big(|\mathcal{B}_Q(g)||Q|^{-1/2}\frac{\chi_{Q^c}(x)}{\langle 2^j(x-\mathbf{x}_Q)\rangle^s}\Big)^q\Big)^{1/q}$$

and thus the desired result follows. □

Lemma 4.3. *Let $2\le p,q<\infty$, $s>n/\min\{p,q\}$ and $L>n,s$. For $j\in\mathbb{Z}$ and $Q\in\mathcal{D}_j$, let*

$$\mathcal{B}_Q(g):=\Big\langle|\widetilde{\Lambda_j}g|,\frac{2^{jn/2}}{\langle 2^j(\cdot-\mathbf{x}_Q)\rangle^L}\Big\rangle,$$

where g is a Schwartz function on $\mathbb{R}^n$. Then we have

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}\Big(|\mathcal{B}_Q(g)||Q|^{-1/2}\frac{\chi_{Q^c}(\cdot)}{\langle 2^j(\cdot-\mathbf{x}_Q)\rangle^s}\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\lesssim\|g\|_{L^p(\mathbb{R}^n)}. \tag{4.5}$$

Proof. It is easy to verify that for $Q\in\mathcal{D}_j$

$$\frac{1}{\langle 2^j|x-\mathbf{x}_Q|\rangle^s}\chi_{Q^c}(x)\lesssim\mathcal{M}_{\frac{n}{s}}\chi_Q(x)$$

and thus the left-hand side of Equation (4.5) is less than a constant multiple of

$$\Big\|\Big(\sum_{Q\in\mathcal{D}}\Big(|\mathcal{B}_Q(g)||Q|^{-1/2}\mathcal{M}_{\frac{n}{s}}\chi_Q(\cdot)\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\lesssim\Big\|\Big(\sum_{Q\in\mathcal{D}}\Big(|\mathcal{B}_Q(g)||Q|^{-1/2}\chi_Q(\cdot)\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$

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by virtue of the maximal inequality (2.4) with $s > n/\min\{p,q\}$. We see that

$$\Big(\sum_{Q\in\mathcal{D}}\Big(\big|\mathcal{B}_Q(g)\big||Q|^{-1/2}\chi_Q(x)\Big)^q\Big)^{1/q} \le \Big(\sum_{Q\in\mathcal{D}}\Big(\big|\mathcal{B}_Q(g)\big||Q|^{-1/2}\chi_Q(x)\Big)^2\Big)^{1/2}$$

$$= \Big(\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}\chi_Q(x)\Big(\int_{\mathbb{R}^n}|\widetilde{\Lambda_j}g(y)|\frac{2^{jn}}{\langle 2^j(y-\mathbf{x}_Q)\rangle^L}\,dy\Big)^2\Big)^{1/2}$$

$$\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\mathbb{R}^n}|\widetilde{\Lambda_j}g(y)|\frac{2^{jn}}{\langle 2^j(y-x)\rangle^L}\,dy\Big)^2\Big)^{1/2} \lesssim \big\|\{\mathcal{M}\Lambda_j g(x)\}_{j\in\mathbb{Z}}\big\|_{\ell^2}$$

since $\ell^2 \hookrightarrow \ell^q$, $L > n$ and $\langle 2^j(y-\mathbf{x}_Q)\rangle \gtrsim \langle 2^j(y-x)\rangle$ for $Q \in \mathcal{D}_j$ and $x \in Q$. Using Equation (2.4) again, the left-hand side of Equation (4.5) is less than a constant times

$$\big\|\{\Lambda_j g\}_{j\in\mathbb{Z}}\big\|_{L^p(\ell^2)} \sim \|g\|_{L^p(\mathbb{R}^n)}. \qquad \square$$

Lemma 4.4. *Let $1 \le q < \infty$, $s > n/q$ and $L > n,s$. For $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_j$, let*

$$\mathcal{B}_Q(g) := \Big\langle |\widetilde{\Gamma_j}g|, \frac{2^{\frac{jn}{2}}}{\langle 2^j(\cdot - c_Q)\rangle^L}\Big\rangle,$$

where g is a Schwartz function on $\mathbb{R}^n$. Then for $1 < p \le \infty$ with $q \le p$ we have

$$\Big\|\sup_{j\in\mathbb{Z}}\Big(\sum_{Q\in\mathcal{D}_j}\Big(\big|\mathcal{B}_Q(g)\big||Q|^{-1/2}\frac{1}{\langle 2^j(\cdot - c_Q)\rangle^s}\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}. \tag{4.6}$$

Proof. For any $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_j$, there exists a unique lattice $\boldsymbol{m}_Q \in \mathbb{Z}^n$ such that $\mathbf{x}_Q = 2^{-j}\boldsymbol{m}_Q$. For any $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $Q_{j,x}$ be a unique dyadic cube in $\mathcal{D}_j$ containing x. Then we have the representations $\mathbf{x}_{Q_{j,x}} = 2^{-j}\boldsymbol{m}_{Q_{j,x}}$ for $\boldsymbol{m}_{Q_{j,x}} \in \mathbb{Z}^n$ and

$$x = 2^{-j}(\boldsymbol{m}_{Q_{j,x}} + u_x) \qquad \text{for some} \quad u_x \in [0,1)^n.$$

Now, for $Q \in \mathcal{D}_j$, we write

$$|\mathcal{B}_Q(g)||Q|^{-1/2} \lesssim_L \int_{\mathbb{R}^n}\big|\mathcal{M}g(y)\big|\frac{2^{jn}}{\langle 2^j(y-\mathbf{c}_Q)\rangle^L}dy \lesssim \int_{\mathbb{R}^n}\big|\mathcal{M}g(y)\big|\frac{2^{jn}}{\langle 2^j(y-\mathbf{x}_Q)\rangle^L}\,dy$$

$$= \int_{\mathbb{R}^n}\big|\mathcal{M}g(y)\big|\frac{2^{jn}}{\big\langle 2^j(y-x) + \boldsymbol{m}_{Q_{j,x}} - \boldsymbol{m}_Q + u_x\big\rangle^L}\,dy$$

$$\lesssim_L \int_{\mathbb{R}^n}\big|\mathcal{M}g(y)\big|\frac{2^{jn}}{\big\langle 2^j(y-x) + \boldsymbol{m}_{Q_{j,x}} - \boldsymbol{m}_Q\big\rangle^L}\,dy$$

$$\lesssim \mathcal{M}_{dyad}^{\boldsymbol{m}_{Q_{j,x}} - \boldsymbol{m}_Q}\mathcal{M}g(x),$$

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where the penultimate inequality follows from the fact that $u_x \in [0,1)^n$. This deduces

$$\begin{aligned}
&\sup_{j\in\mathbb{Z}}\Big(\sum_{Q\in\mathcal{D}_j}\Big(\big|\mathcal{B}_Q(g)\big||Q|^{-1/2}\frac{1}{\langle 2^j(x-c_Q)\rangle^s}\Big)^q\Big)^{1/q}\\
&\lesssim \sup_{j\in\mathbb{Z}}\Big(\sum_{\boldsymbol{m}\in\mathbb{Z}^n}\Big(\mathcal{M}_{dyad}^{\boldsymbol{m}_{Q_{j,x}}-\boldsymbol{m}}\mathcal{M}g(x)\frac{1}{\langle \boldsymbol{m}_{Q_{j,x}}-\boldsymbol{m}\rangle^s}\Big)^q\Big)^{1/q}\\
&=\Big(\sum_{\boldsymbol{m}\in\mathbb{Z}^n}\Big(\mathcal{M}_{dyad}^{\boldsymbol{m}}\mathcal{M}g(x)\frac{1}{\langle \boldsymbol{m}\rangle^s}\Big)^q\Big)^{1/q}.
\end{aligned}$$

Therefore, the left-hand side of Equation (4.6) is less than a constant times

$$\begin{aligned}
\Big(\sum_{\boldsymbol{m}\in\mathbb{Z}^n}\langle\boldsymbol{m}\rangle^{-sq}\big\|\mathcal{M}_{dyad}^{\boldsymbol{m}}\mathcal{M}g\big\|_{L^p(\mathbb{R}^n)}^q\Big)^{1/q} &\lesssim \|\mathcal{M}g\|_{L^p(\mathbb{R}^n)}\Big(\sum_{\boldsymbol{m}\in\mathbb{Z}^n}\langle\boldsymbol{m}\rangle^{-sq}\big(\log\big(10+|\boldsymbol{m}|\big)\big)^{qn/p}\Big)^{1/q}\\
&\lesssim \|g\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

as $sq > n$, where we applied Minkowski's inequality if $p > q$ and the maximal inequality (2.6). This completes the proof. □

Lemma 4.5. *Let a be an H^p-atom associated with Q, satisfying*

$$\int_{\mathbb{R}^n} x^\gamma a(x)\,dx = 0 \qquad \textit{for all multi-indices } |\gamma|\le M, \tag{4.7}$$

and fix $L_0 > 0$. Then we have

$$\big|\Lambda_j a(x)\big| \lesssim_{L_0} l(Q)^{-n/p}\min\big\{1,\big(2^j l(Q)\big)^{M+n+1}\big\}\Big(\chi_{Q^*}(x)+\chi_{(Q^*)^c}(x)\frac{1}{\langle 2^j(x-\mathbf{x}_Q)\rangle^{L_0}}\Big), \tag{4.8}$$

and

$$\big|\Gamma_j a(x)\big| \lesssim_{L_0} l(Q)^{-n/p}\min\big\{1,\big(2^j l(Q)\big)^{M+n+1}\big\}\Big(\chi_{Q^*}(x)+\chi_{(Q^*)^c}(x)\frac{1}{\langle 2^j(x-\mathbf{x}_Q)\rangle^{L_0}}\Big). \tag{4.9}$$

Moreover, for $1\le r\le\infty$,

$$\|\Lambda_j a\|_{L^r(\mathbb{R}^n)}, \|\Gamma_j a\|_{L^r(\mathbb{R}^n)} \lesssim l(Q)^{-n/p+n/r}\min\{1,(2^j l(Q))^{M+n-n/r+1}\}. \tag{4.10}$$

Proof. We will prove only the estimates for $\Lambda_j a$, and the exactly same argument is applicable to $\Gamma_j a$ as well. Let us first assume $2^j\ell(Q)\ge 1$. Then we have

$$\begin{aligned}
\big|\Lambda_j a(x)\big| &\le \ell(Q)^{-n/p}\Big(\chi_{Q^*}(x)\|\psi_j\|_{L^1(\mathbb{R}^n)}+\chi_{(Q^*)^c}(x)\int_{y\in Q}\big|\psi_j(x-y)\big|\,dy\Big)\\
&\lesssim_{L_0} \ell(Q)^{-n/p}\Big(\chi_{Q^*}(x)+\chi_{(Q^*)^c}(x)\int_{y\in Q}\frac{2^{jn}}{\langle 2^j(x-y)\rangle^{n+1}}\frac{1}{\langle 2^j(x-y)\rangle^{L_0}}\,dy\Big)\\
&\lesssim \ell(Q)^{-n/p}\Big(\chi_{Q^*}(x)+\chi_{(Q^*)^c}(x)\frac{1}{\langle 2^j(x-\mathbf{x}_Q)\rangle^{L_0}}\Big)
\end{aligned}$$

since $|x-y|\gtrsim|x-\mathbf{x}_Q|$ for $x\in(Q^*)^c$ and $y\in Q$.

Now, suppose that $2^j\ell(Q) < 1$. By using the vanishing moment condition (4.7), we obtain

$$\left|\Lambda_j a(x)\right| \le 2^{j(M+n+1)} \int_Q \int_0^1 \frac{1}{\langle 2^j(x-ty-(1-t)\mathbf{x}_Q)\rangle^{L_0}} |y-\mathbf{x}_Q|^{M+1}|a(y)|\, dtdy.$$

If $x \in Q^*$, then it is clear that

$$\left|\Lambda_j a(x)\right| \lesssim \left(2^j\ell(Q)\right)^{M+n+1}\ell(Q)^{-n/p}.$$

If $x \in (Q^*)^c$, then we have

$$\langle 2^j(x-ty-(1-t)\mathbf{x}_Q)\rangle^{-1} \lesssim \langle 2^j(x-\mathbf{x}_Q)\rangle^{-1},$$

which implies

$$\left|\Lambda_j a(x)\right| \lesssim_{L_0} \frac{1}{\langle 2^j(x-\mathbf{x}_Q)\rangle^{L_0}}\left(2^j\ell(Q)\right)^{M+n+1}\ell(Q)^{-n/p}.$$

This proves Equation (4.8).

Moreover, using the estimate (4.8), we have

$$\begin{aligned}
\left\|\Lambda_j a\right\|_{L^r(\mathbb{R}^n)} &\le \left\|\Lambda_j a\right\|_{L^r(Q^*)} + \left\|\Lambda_j a\right\|_{L^r((Q^*)^c)} \\
&\lesssim \ell(Q)^{-n/p}\min\left\{1,\left(2^j l(Q)\right)^{M+n+1}\right\}\left(|Q|^{1/r} + \left\|\frac{1}{\langle 2^j(\cdot-\mathbf{x}_Q)\rangle^{n+1}}\right\|_{L^r(\mathbb{R}^n)}\right) \\
&\lesssim \ell(Q)^{-n/p+n/r}\min\left\{1,\left(2^j l(Q)\right)^{M+n+1}\right\}\left(1+\left(2^j\ell(Q)\right)^{-n/r}\right) \\
&\le \ell(Q)^{-n/p+n/r}\min\left\{1,\left(2^j l(Q)\right)^{M+n-n/r+1}\right\}.
\end{aligned}$$

This concludes the proof of Equation (4.10). □

5. Proof of Proposition 3.1: Reduction

5.1. Reduction via paraproduct

Without loss of generality, we may assume

$$\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1 \quad \text{and} \quad \mathcal{L}_s^2[\sigma] = 1.$$

We first note that $T_\sigma(f_1,f_2,f_3)$ can be written as

$$T_\sigma(f_1,f_2,f_3) = \sum_{j_1,j_2,j_3\in\mathbb{Z}} T_\sigma\left(\Lambda_{j_1}f_1,\Lambda_{j_2}f_2,\Lambda_{j_3}f_3\right).$$

We shall work with only the case $j_1 \ge j_2 \ge j_3$ as other cases follow from a symmetric argument. When $j_1 \ge j_2 \ge j_3$, it is easy to verify that

$$T_\sigma\left(\Lambda_{j_1}f_1,\Lambda_{j_2}f_2,\Lambda_{j_3}f_3\right) = T_{\sigma_{j_1}}\left(\Lambda_{j_1}f_1,\Lambda_{j_2}f_2,\Lambda_{j_3}f_3\right)$$

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where $\sigma_j(\vec{\boldsymbol{\xi}}) := \sigma(\vec{\boldsymbol{\xi}})\widehat{\Theta}(\vec{\boldsymbol{\xi}}/2^j)$ and $\widehat{\Theta}(\vec{\boldsymbol{\xi}}) := \sum_{l=-2}^{2}\widehat{\Psi}(2^l\vec{\boldsymbol{\xi}})$ so that $\widehat{\Theta}(\vec{\boldsymbol{\xi}}) = 1$ for $2^{-2} \le |\vec{\boldsymbol{\xi}}| \le 2^2$ and $\mathrm{supp}(\widehat{\Theta}) \subset \{\vec{\boldsymbol{\xi}} \in (\mathbb{R}^n)^3 : 2^{-3} \le |\vec{\boldsymbol{\xi}}| \le 2^3\}$. Then we observe that

$$\sup_{k\in\mathbb{Z}}\big\|\sigma_k(2^k\vec{\cdot}\,)\big\|_{L^2_s((\mathbb{R}^n)^3)} \lesssim \mathcal{L}^2_s[\sigma] = 1 \tag{5.1}$$

by virtue of the triangle inequality. Moreover, using the fact that $\Gamma_j f = \sum_{k\le j}\Lambda_k f$, we write

$$\begin{aligned}
&\sum_{j_1\in\mathbb{Z}}\ \sum_{j_2,j_3\in\mathbb{Z}:j_3\le j_2\le j_1} T_{\sigma_{j_1}}\big(\Lambda_{j_1}f_1,\Lambda_{j_2}f_2,\Lambda_{j_3}f_3\big)\\
&\quad= \sum_{j\in\mathbb{Z}} T_{\sigma_j}\big(\Lambda_j f_1,\Gamma_{j-10}f_2,\Gamma_{j-10}f_3\big) + \sum_{k=0}^{9}\sum_{j\in\mathbb{Z}} T_{\sigma_j}\big(\Lambda_j f_1,\Lambda_{j-k}f_2,\Gamma_{j-k}f_3\big)\\
&\quad=: T^{(1)}_\sigma(f_1,f_2,f_3) + \sum_{k=0}^{9} T^{(2),k}_\sigma(f_1,f_2,f_3),
\end{aligned}$$

and especially, let $T^{(2)}_\sigma := T^{(2),0}_\sigma$. Then it is enough to prove that

$$\big\|T^{(\mu)}_\sigma(f_1,f_2,f_3)\big\|_{H^p(\mathbb{R}^n)} \lesssim 1, \qquad \mu = 1,2 \tag{5.2}$$

since the operator $T^{(2),k}_\sigma$, $1 \le k \le 9$, can be handled in the same way as $T^{(2)}_\sigma$.

It should be remarked that the vanishing moment condition (1.16) now implies

$$\int_{\mathbb{R}^n} x^\alpha T_{\sigma_j}\big(f_1,f_2,f_3\big)(x)\,dx = 0 \quad \text{for all multi-indices } |\alpha| \le \frac{n}{p} - n. \tag{5.3}$$

5.2. Proof of (5.2) for $\mu = 1$

In this case, we may simply follow the arguments used in the proof of Theorems B and D. The proof is based on the fact that if $\widehat{g_k}$ is supported in $\{\xi \in \mathbb{R}^n : C_0^{-1}2^k \le |\xi| \le C_0 2^k\}$ for $C_0 > 1$, then

$$\Big\|\Big\{\Lambda_j\Big(\sum_{k\in\mathbb{Z}} g_k\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^q)} \lesssim_{C_0} \big\|\{g_j\}_{j\in\mathbb{Z}}\big\|_{L^p(\ell^q)}. \tag{5.4}$$

The proof of Equation (5.4) is elementary and standard, simply using the estimate

$$\Big|\Lambda_j\Big(\sum_{k\in\mathbb{Z}} g_k\Big)(x)\Big| = \Big|\Lambda_j\Big(\sum_{k=j-h}^{j+h} g_k\Big)(x)\Big| \lesssim_{r,h} \mathcal{M}_r g_j(x)$$

for all $0 < r < \infty$ and for some $h \in \mathbb{N}$, depending on C_0, and the maximal inequality (2.4). We refer to [34, Theorem 3.6] for details.

By using the equivalence in Equation (2.2),

$$\big\|T^{(1)}_\sigma(f_1,f_2,f_3)\big\|_{H^p(\mathbb{R}^n)} \sim \Big\|\Big\{\Lambda_j\Big(\sum_{k\in\mathbb{Z}} T_{\sigma_k}\big(\Lambda_k f_1,\Gamma_{k-10}f_2,\Gamma_{k-10}f_3\big)\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^2)}.$$

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We see that the Fourier transform of $T_{\sigma_k}\big(\Lambda_k f_1,\Gamma_{k-10}f_2,\Gamma_{k-10}f_3\big)$ is supported in $\big\{\xi\in\mathbb{R}^n : 2^{k-2}\le|\xi|\le 2^{k+2}\big\}$ and thus the estimate (5.4) yields that

$$\big\|T_\sigma^{(1)}(f_1,f_2,f_3)\big\|_{H^p(\mathbb{R}^n)}\lesssim\Big\|\Big(\sum_{j\in\mathbb{Z}}\big|T_{\sigma_j}\big(\Lambda_j f_1,\Gamma_{j-10}f_2,\Gamma_{j-10}f_3\big)\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}.$$

Then it is already proved in [14, (3.14)] that the preceding expression is dominated by the right-hand side of Equation (5.2) for $s>n/\min\{p_1,p_2,p_3\}+n/2$, where we remark that $\min\{p_1,p_2,p_3\}\le 1$. This proves Equation (5.2) for $\mu=1$.

5.3. Proof of Equation (5.2) for $\mu=2$

Recall that

$$T_\sigma^{(2)}(f_1,f_2,f_3)=\sum_{j\in\mathbb{Z}}T_{\sigma_j}\big(\Lambda_j f_1,\Lambda_j f_2,\Gamma_j f_3\big)\tag{5.5}$$

and observe that

$$T_{\sigma_j}\big(\Lambda_j f_1,\Lambda_j f_2,\Gamma_j f_3\big)(x)=\sigma_j^\vee *_{3n}\big(\Lambda_j f_1\otimes\Lambda_j f_2\otimes\Gamma_j f_3\big)(x,x,x),$$

where $*_{3n}$ means the convolution on $\mathbb{R}^{3n}$.

It suffices to consider the case when $(1/p_1,1/p_2,1/p_3)$ belongs to $\mathcal{R}_1$ or $\mathcal{R}_3$, as the remaining case is symmetrical to the case $(1/p_1,1/p_2,1/p_3)\in\mathcal{R}_1$, in view of Equation (5.5). We will mainly focus on the case $(1/p_1,1/p_2,1/p_3)\in\mathcal{R}_1$, while simply providing a short description for the case $(1/p_1,1/p_2,1/p_3)\in\mathcal{R}_3$ in the remark below as almost same arguments will be applied in that case.

Therefore, we now assume $0<p_1\le 1$ and $2<p_2,p_3<\infty$, and in turn, suppose that $s>n/p_1+n/2$. By using the atomic decomposition in Equation (2.3), the function $f_1\in H^{p_1}(\mathbb{R}^n)$ can be written as $f_1=\sum_{k=1}^\infty\lambda_k a_k$, where a_k's are H^{p_1}-atoms associated with cubes Q_k, and

$$\Big(\sum_{k=1}^\infty|\lambda_k|^{p_1}\Big)^{1/p_1}\lesssim 1.\tag{5.6}$$

As mentioned before, we may assume that M is sufficiently large and $\int x^\gamma a_k(x)dx=0$ holds for all multi-indices $|\gamma|\le M$.

By the definition in Equation (1.2), we have

$$\big\|T_\sigma^{(2)}(f_1,f_2,f_3)\big\|_{H^p(\mathbb{R}^n)}\sim\Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=0}^\infty\lambda_k\phi_l*\Big(\sum_{j\in\mathbb{Z}}T_{\sigma_j}(\Lambda_j a_k,\Lambda_j f_2,\Gamma_j f_3)\Big)\Big|\Big\|_{L^p(\mathbb{R}^n)}$$

and thus we need to prove that

$$\Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=0}^\infty\lambda_k\phi_l*\Big(\sum_{j\in\mathbb{Z}}T_{\sigma_j}(\Lambda_j a_k,\Lambda_j f_2,\Gamma_j f_3)\Big)\Big|\Big\|_{L^p(\mathbb{R}^n)}\lesssim 1.\tag{5.7}$$

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The left-hand side is less than the sum of

$$\mathcal{I} := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{Q_k^{***}} \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Lambda_j a_k, \Lambda_j f_2, \Gamma_j f_3) \Big) \Big| \Big\|_{L^p(\mathbb{R}^n)}$$

and

$$\mathcal{J} := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c} \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Lambda_j a_k, \Lambda_j f_2, \Gamma_j f_3) \Big) \Big| \Big\|_{L^p(\mathbb{R}^n)},$$

recalling that Q_k^{***} is the dilate of Q_k by a factor $(10\sqrt{n})^3$. The two terms $\mathcal{I}$ and $\mathcal{J}$ will be treated separately in the next two sections.

Remark. When $(1/p_1, 1/p_2, 1/p_3) \in \mathcal{R}_3$ (that is, $0 < p_3 \le 1$, $2 < p_1, p_2 < \infty$), we need to prove

$$\Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Lambda_j f_1, \Lambda_j f_2, \Gamma_j \widetilde{a_k}) \Big) \Big| \Big\|_{L^p(\mathbb{R}^n)} \lesssim 1,$$

where $\widetilde{a_k}$ is the H^{p_3}-atom associated with f_3. This is actually, via symmetry, equivalent to the estimate that for $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$,

$$\Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Gamma_j a_k, \Lambda_j f_2, \Lambda_j f_3) \Big) \Big| \Big\|_{L^p(\mathbb{R}^n)} \lesssim 1, \tag{5.8}$$

where a_k is the H^{p_1}-atom for f_1. The proof of Equation (5.8) is almost same as that of Equation (5.7) which will be discussed in Sections 6 and 7. So this will not be pursued in this paper, just saying that Equation (4.9) will be needed rather than Equation (4.8), and the estimate $\big\| \{\Gamma_j a_k\}_{j\in\mathbb{Z}} \big\|_{L^r(\ell^\infty)} \sim \|a_k\|_{H^r(\mathbb{R}^n)}$ will be required in place of the equivalence $\big\| \{\Lambda_j a_k\}_{j\in\mathbb{Z}} \big\|_{L^r(\ell^2)} \sim \|a_k\|_{H^r(\mathbb{R}^n)}$.

6. Proof of Proposition 3.1: estimate for $\mathcal{I}$

For the estimation of $\mathcal{I}$, we need the following lemma whose proof will be given in Section 9.

Lemma 6.1. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \ \ \mathrm{i} = 1,2,3,$$

and for $x \in \mathbb{R}^n$

$$\sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{Q_k^{***}}(x) \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Lambda_j a_k, \Lambda_j f_2, \Gamma_j f_3) \Big)(x) \Big| \lesssim u_1(x) u_2(x) u_3(x). \tag{6.1}$$

This lemma, together with Hölder's inequality, clearly shows that

$$\mathcal{I} \lesssim \|u_1\|_{L^{p_1}(\mathbb{R}^n)} \|u_2\|_{L^{p_2}(\mathbb{R}^n)} \|u_3\|_{L^{p_3}(\mathbb{R}^n)} \lesssim 1.$$

7. Proof of Proposition 3.1: estimate for $\mathcal{J}$

Recall that for each Q_k and $l \in \mathbb{Z}$, $B^l_{\mathbf{x}_{Q_k}} = B(\mathbf{x}_{Q_k}, 100n2^{-l})$ stands for the ball of radius $100n2^{-l}$ and center $\mathbf{x}_{Q_k}$. Simply writing $B^l_k := B^l_{\mathbf{x}_{Q_k}}$, we bound $\mathcal{J}$ by the sum of

$$\mathcal{J}_1 := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c} \chi_{(B_k^l)^c} \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Lambda_j a_k, \Lambda_j f_2, \Gamma_j f_3) \Big) \Big| \Big\|_{L^p(\mathbb{R}^n)}$$

and

$$\mathcal{J}_2 := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c} \chi_{B_k^l} \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Lambda_j a_k, \Lambda_j f_2, \Gamma_j f_3) \Big) \Big| \Big\|_{L^p(\mathbb{R}^n)}$$

and treat them separately.

7.1. Estimate for $\mathcal{J}_1$

Using the representations in Equations (2.7) and (2.9), we write

$$\Lambda_j f_2(x) := \sum_{P\in\mathcal{D}_j} b_P^2 \psi^P(x), \qquad \Gamma_j f_3(x) := \sum_{R\in\mathcal{D}_j} b_R^3 \theta^R(x),$$

where we recall $\psi^P(x) = |P|^{1/2}\psi_j(x-\mathbf{x}_P)$ and $\theta^R(x) = |R|^{1/2}\theta_j(x-\mathbf{x}_R)$ for $P, R \in \mathcal{D}_j$. Then it follows from Equations (2.8), (2.10), (2.1) and (2.2) that

$$\big\| \{b_P^2\}_{P\in\mathcal{D}} \big\|_{\dot f^{p_2,2}} \sim \big\| \{\Lambda_j f_2\}_{j\in\mathbb{Z}} \big\|_{L^{p_2}(\ell^2)} \sim \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = 1 \tag{7.1}$$

and

$$\big\| \{b_R^3\}_{R\in\mathcal{D}} \big\|_{\dot f^{p_3,\infty}} \sim \big\| \{\Gamma_j f_3\}_{j\in\mathbb{Z}} \big\|_{L^{p_3}(\ell^\infty)} \sim \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1. \tag{7.2}$$

We write

$$\phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\sigma_j}(\Lambda_j a_k, \Lambda_j f_2, \Gamma_j f_3) \Big)(x) = \sum_{\nu=1}^{4} \phi_l * \big(\mathcal{U}_\nu(x,\cdot) \big)(x),$$

where

$$\Omega_\nu(P,R) := \begin{cases} P\cap R & \nu = 1 \\ P^c \cap R & \nu = 2 \\ P \cap R^c & \nu = 3 \\ P^c \cap R^c & \nu = 4 \end{cases} \tag{7.3}$$

and

$$\mathcal{U}_\nu(x,y) := \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\Lambda_j a_k, \psi^P, \theta^R\big)(y) \chi_{\Omega_\nu(P,R)}(x), \quad \nu = 1,2,3,4. \tag{7.4}$$

Then we have

$$\mathcal{J}_1 \lesssim_p \mathcal{J}_1^1 + \mathcal{J}_1^2 + \mathcal{J}_1^3 + \mathcal{J}_1^4,$$

where

$$\mathcal{J}_1^{\nu} := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\phi_l * \big(\mathcal{U}_{\nu}(x,\cdot)\big)(x)\Big|\Big\|_{L^p(x)}, \qquad \nu = 1,2,3,4.$$

Now, we will show that

$$\mathcal{J}_1^{\nu} \lesssim 1, \qquad \nu = 1,2,3,4. \tag{7.5}$$

7.1.1. Proof of Equation (7.5) for $\nu = 1$.

We further decompose $\mathcal{U}_1(x,y)$ as

$$\mathcal{U}_1(x,y) = \mathcal{U}_1^{\mathrm{in}}(x,y) + \mathcal{U}_1^{\mathrm{out}}(x,y),$$

where

$$\begin{aligned} \mathcal{U}_1^{\mathrm{in}}(x,y) &:= \sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{Q_k^*}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{P\cap R}(x), \\ \mathcal{U}_1^{\mathrm{out}}(x,y) &:= \sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{(Q_k^*)^c}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{P\cap R}(x), \end{aligned} \tag{7.6}$$

and accordingly, we define

$$\mathcal{J}_1^{1,\mathrm{in/out}} := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\phi_l * \big(\mathcal{U}_1^{\mathrm{in/out}}(x,\cdot)\big)(x)\Big|\Big\|_{L^p(x)}.$$

Then we claim the following lemma.

Lemma 7.1. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_1^{\mathrm{in/out}}$ be defined as in Equation (7.6). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1^{in}, u_1^{out}, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_1^{\mathrm{in/out}}\|_{L^{p_1}(\mathbb{R}^n)} \lesssim 1, \qquad \|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for } \ \mathrm{i} = 2,3,$$

and for $x \in \mathbb{R}^n$

$$\sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\phi_l * \big(\mathcal{U}_1^{\mathrm{in/out}}(x,\cdot)\big)(x)\Big| \lesssim u_1^{\mathrm{in/out}}(x)u_2(x)u_3(x). \tag{7.7}$$

The proof of Lemma 7.1 will be given in Section 9. Taking the lemma for granted and using Hölder's inequality, we can easily show that

$$\mathcal{J}_1^1 \lesssim_p \mathcal{J}_1^{1,\mathrm{in}} + \mathcal{J}_1^{1,\mathrm{out}} \lesssim 1.$$

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7.1.2. Proof of Equation (7.5) for $\nu = 2$. For $P \in \mathcal{D}$ and $l \in \mathbb{Z}$ let $B_P^l := B_{\mathbf{x}_P}^l = B(\mathbf{x}_P, 100n2^{-l})$. By introducing

$$\begin{aligned}
\mathcal{U}_2^{1,\mathrm{in}}(x,y) &:= \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{Q_k^*}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{P^c\cap(B_P^l)^c\cap R}(x),\\
\mathcal{U}_2^{1,\mathrm{out}}(x,y) &:= \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{(Q_k^*)^c}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{P^c\cap(B_P^l)^c\cap R}(x),\\
\mathcal{U}_2^{2,\mathrm{in}}(x,y) &:= \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{Q_k^*}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{P^c\cap B_P^l\cap R}(x),\\
\mathcal{U}_2^{2,\mathrm{out}}(x,y) &:= \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{(Q_k^*)^c}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{P^c\cap B_P^l\cap R}(x),
\end{aligned} \tag{7.8}$$

we write $\mathcal{U}_2 = \mathcal{U}_2^{1,\mathrm{in}} + \mathcal{U}_2^{1,\mathrm{out}} + \mathcal{U}_2^{2,\mathrm{in}} + \mathcal{U}_2^{2,\mathrm{out}}$ and consequently,

$$\mathcal{J}_1^2 \lesssim_p \mathcal{J}_1^{2,1,\mathrm{in}} + \mathcal{J}_1^{2,1,\mathrm{out}} + \mathcal{J}_1^{2,2,\mathrm{in}} + \mathcal{J}_1^{2,2,\mathrm{out}},$$

where

$$\mathcal{J}_1^{2,\eta,\mathrm{in/out}} := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\, \phi_l * \big(\mathcal{U}_2^{\eta,\mathrm{in/out}}(x,\cdot)\big)(x)\Big|\Big\|_{L^p(x)}, \quad \eta = 1,2.$$

Then we apply the following lemma that will be proved in Section 9.

Lemma 7.2. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_2^{\eta,\mathrm{in/out}}$ be defined as in Equation (7.8). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1^{in}, u_1^{out}, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\big\|u_1^{\mathrm{in/out}}\big\|_{L^{p_1}(\mathbb{R}^n)} \lesssim 1, \qquad \|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \ \ \mathrm{i} = 2,3,$$

and for each $\eta = 1,2$

$$\sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\phi_l * \big(\mathcal{U}_2^{\eta,\mathrm{in/out}}(x,\cdot)\big)(x)\Big| \lesssim u_1^{\mathrm{in/out}}(x)u_2(x)u_3(x). \tag{7.9}$$

Then Lemma 7.2 and Hölder's inequality yield that $\mathcal{J}_1^2$ is controlled by the sum of four terms in the form

$$\big\|u_1^{\mathrm{in/out}}\big\|_{L^{p_1}(\mathbb{R}^n)} \|u_2\|_{L^{p_2}(\mathbb{R}^n)} \|u_3\|_{L^{p_3}(\mathbb{R}^n)},$$

which is obviously less than a constant. This proves Equation (7.5) for $\nu = 2$.

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7.1.3. Proof of Equation (7.5) for $\nu=3$. This case is essentially symmetrical to the case $\nu=2$. For $R\in\mathcal{D}$ and $l\in\mathbb{Z}$, let $B_R^l:=B_{\mathbf{x}_R}^l=B(\mathbf{x}_R,100n2^{-l})$. Let

$$\begin{aligned}\mathcal{U}_3^{1,\mathrm{in}}(x,y)&:=\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j}b_P^2b_R^3T_{\sigma_j}\big(\chi_{Q_k^*}\Lambda_ja_k,\psi^P,\theta^R\big)(y)\chi_{P\cap R^c\cap(B_R^l)^c}(x),\\ \mathcal{U}_3^{1,\mathrm{out}}(x,y)&:=\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j}b_P^2b_R^3T_{\sigma_j}\big(\chi_{(Q_k^*)^c}\Lambda_ja_k,\psi^P,\theta^R\big)(y)\chi_{P\cap R^c\cap(B_R^l)^c}(x),\\ \mathcal{U}_3^{2,\mathrm{in}}(x,y)&:=\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j}b_P^2b_R^3T_{\sigma_j}\big(\chi_{Q_k^*}\Lambda_ja_k,\psi^P,\theta^R\big)(y)\chi_{P\cap R^c\cap B_R^l}(x),\\ \mathcal{U}_3^{2,\mathrm{out}}(x,y)&:=\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j}b_P^2b_R^3T_{\sigma_j}\big(\chi_{(Q_k^*)^c}\Lambda_ja_k,\psi^P,\theta^R\big)(y)\chi_{P\cap R^c\cap B_R^l}(x),\end{aligned}\tag{7.10}$$

and then we write

$$\mathcal{J}_1^3\lesssim_p\mathcal{J}_1^{3,1,\mathrm{in}}+\mathcal{J}_1^{3,1,\mathrm{out}}+\mathcal{J}_1^{3,2,\mathrm{in}}+\mathcal{J}_1^{3,2,\mathrm{out}},$$

where

$$\mathcal{J}_1^{3,\eta,\mathrm{in/out}}:=\Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=0}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\,\phi_l*\big(\mathcal{U}_3^{\eta,\mathrm{in/out}}(x,\cdot)\big)(x)\Big|\Big\|_{L^p(x)},\qquad\eta=1,2.$$

Now, Equation (7.5) for $\nu=3$ follows from the lemma below.

Lemma 7.3. *Let $0<p_1\le1$ and $2<p_2,p_3<\infty$, and let $\mathcal{U}_3^{\eta,\mathrm{in/out}}$ be defined as in Equation (7.10). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)}=\|f_2\|_{H^{p_2}(\mathbb{R}^n)}=\|f_3\|_{H^{p_3}(\mathbb{R}^n)}=1$ and $\mathcal{L}_s^2[\sigma]=1$ for $s>n/p_1+n/2$. Then there exist nonnegative functions u_1^{in}, u_1^{out}, u_2, and u_3 on $\mathbb{R}^n$ such that*

$$\|u_1^{\mathrm{in/out}}\|_{L^{p_1}(\mathbb{R}^n)}\lesssim1,\qquad\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)}\lesssim1\quad\text{for }\ \mathrm{i}=2,3,$$

and for each $\eta=1,2$

$$\sup_{l\in\mathbb{Z}}\Big|\sum_{k=0}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\phi_l*\big(\mathcal{U}_3^{\eta,\mathrm{in/out}}(x,\cdot)\big)(x)\Big|\lesssim u_1^{\mathrm{in/out}}(x)u_2(x)u_3(x).$$

The proof of the lemma will be provided in Section 9.

7.1.4. Proof of Equation (7.5) for $\nu=4$. In this case, we divide $\mathcal{U}_4$ into eight types depending on whether x belongs to each of B_P^l and B_R^l and whether Λ_ja_k is supported in Q_k^*. Indeed, let

$$\Xi_\eta(P,R,l):=\begin{cases}P^c\cap R^c\cap(B_P^l)^c\cap(B_R^l)^c, & \eta=1\\ P^c\cap R^c\cap(B_P^l)^c\cap B_R^l, & \eta=2\\ P^c\cap R^c\cap B_P^l\cap(B_R^l)^c, & \eta=3\\ P^c\cap R^c\cap B_P^l\cap B_R^l, & \eta=4,\end{cases}\tag{7.11}$$

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and we define

$$\begin{aligned}\mathcal{U}_4^{\eta,\mathrm{in}}(x,y) &:= \sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{Q_k^*}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{\Xi_\eta}(x),\\ \mathcal{U}_4^{\eta,\mathrm{out}}(x,y) &:= \sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\chi_{(Q_k^*)^c}\Lambda_j a_k, \psi^P, \theta^R\big)(y)\chi_{\Xi_\eta}(x)\end{aligned} \tag{7.12}$$

for $\eta = 1,2,3,4$.

Then we use the following lemma to obtain the desired result.

Lemma 7.4. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_4^{\eta,\mathrm{in/out}}$ be defined as in Equation (7.12). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1^{in}, u_1^{out}, u_2, and u_3 on $\mathbb{R}^n$ such that*

$$\big\|u_1^{\mathrm{in/out}}\big\|_{L^{p_1}(\mathbb{R}^n)} \lesssim 1, \qquad \|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \;\; \mathrm{i} = 2,3, \tag{7.13}$$

and for each $\eta = 1,2,3,4$,

$$\sup_{l\in\mathbb{Z}}\Big|\sum_{k=0}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\phi_l*\big(\mathcal{U}_4^{\eta,\mathrm{in/out}}(x,\cdot)\big)(x)\Big| \lesssim u_1^{\mathrm{in/out}}(x)u_2(x)u_3(x). \tag{7.14}$$

We will prove the lemma in Section 9.

7.2. Estimate for $\mathcal{J}_2$

Let $x \in (Q_k^{***})^c \cap B_{k,l}$. For $\nu = 1,2,3,4$, let $\Omega_\nu(P,R)$ be defined as in Equation (7.3). Then as in the proof of the estimate for $\mathcal{J}_1$, we consider the four cases: $x \in \Omega_1(P,R)$, $x \in \Omega_2(P,R)$, $x \in \Omega_3(P,R)$ and $x \in \Omega_4(P,R)$. That is, for each $\nu = 1,2,3,4$, let $\mathcal{U}_\nu$ be defined as in Equation (7.4) and

$$\mathcal{J}_2^\nu := \Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=0}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}(x)\chi_{B_k^l}(x)\phi_l*\big(\mathcal{U}_\nu(x,\cdot)\big)(x)\Big|\Big\|_{L^p(x)}.$$

Then it suffices to show that for each $\nu = 1,2,3,4$,

$$\mathcal{J}_2^\nu \lesssim 1. \tag{7.15}$$

7.2.1. Proof of Equation (7.15) for $\nu = 1$. In this case, the proof can be simply reduced to the following lemma, which will be proved in Section 9.

Lemma 7.5. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_1$ be defined as in Equation (7.4). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \;\; \mathrm{i} = 1,2,3,$$

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and for $x \in \mathbb{R}^n$

$$\sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{B_k^l}(x)\phi_l * \big(\mathcal{U}_1(x,\cdot)\big)(x)\Big| \lesssim u_1(x)u_2(x)u_3(x). \tag{7.16}$$

Then it follows from Hölder's inequality that

$$\mathcal{J}_2^1 \lesssim \|u_1\|_{L^{p_1}(\mathbb{R}^n)}\|u_2\|_{L^{p_2}(\mathbb{R}^n)}\|u_3\|_{L^{p_3}(\mathbb{R}^n)} \lesssim 1.$$

7.2.2. Proof of Equation (7.15) for $\nu = 2$. For $P \in \mathcal{D}$ and $l \in \mathbb{Z}$, let $B_P^l := B_{\mathbf{x}_P}^l$ be the ball of center $\mathbf{x}_P$ and radius $100n2^{-l}$ as before. We define

$$\begin{aligned}\mathcal{U}_2^1(x,y) &:= \sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\Lambda_j a_k,\psi^P,\theta^R\big)(y)\chi_{P^c\cap(B_P^l)^c\cap R}(x),\\ \mathcal{U}_2^2(x,y) &:= \sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\Lambda_j a_k,\psi^P,\theta^R\big)(y)\chi_{P^c\cap B_P^l\cap R}(x)\end{aligned} \tag{7.17}$$

and write

$$\mathcal{J}_2^2 \lesssim \mathcal{J}_2^{2,1} + \mathcal{J}_2^{2,2}$$

where

$$\mathcal{J}_2^{2,\eta} := \Big\| \sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{B_k^l}(x)\phi_l * \big(\mathcal{U}_2^\eta(x,\cdot)\big)(x)\Big| \Big\|_{L^p(x)}, \quad \eta = 1,2.$$

Then we need the following lemmas.

Lemma 7.6. *Let $0 < p_1 \le 1$ and $2 < p_2,p_3 < \infty$ and let $\mathcal{U}_2^1$ be defined as in (7.17). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \;\; \mathrm{i} = 1,2,3,$$

and

$$\sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{B_k^l}(x)\phi_l * \big(\mathcal{U}_2^1(x,\cdot)\big)(x)\Big| \lesssim u_1(x)u_2(x)u_3(x). \tag{7.18}$$

Lemma 7.7. *Let $0 < p_1 \le 1$ and $2 < p_2,p_3 < \infty$, and let $\mathcal{U}_2^2$ be defined as in Equation (7.17). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \;\; \mathrm{i} = 1,2,3,$$

and

$$\sup_{l\in\mathbb{Z}} \Big| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x)\chi_{B_k^l}(x)\phi_l * \big(\mathcal{U}_2^2(x,\cdot)\big)(x)\Big| \lesssim u_1(x)u_2(x)u_3(x).$$

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The above lemmas will be proved in Section 9. Using Lemmas 7.6 and 7.7, we obtain

$$\mathcal{J}_2^{2,\eta} \lesssim \|u_1\|_{L^{p_1}(\mathbb{R}^n)} \|u_2\|_{L^{p_2}(\mathbb{R}^n)} \|u_3\|_{L^{p_3}(\mathbb{R}^n)} \lesssim 1, \qquad \eta = 1,2,$$

which finishes the proof of Equation (7.15) for $\nu = 2$.

7.2.3. Proof of Equation (7.15) for $\nu = 3$. We use the notation $B_R^l := B_{\mathbf{x}_R}^l$ for $R \in \mathcal{D}$ and $l \in \mathbb{Z}$ as before and write

$$\mathcal{J}_2^3 \lesssim \mathcal{J}_2^{3,1} + \mathcal{J}_2^{3,2},$$

where

$$\begin{aligned} \mathcal{U}_3^1(x,y) &:= \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\Lambda_j a_k, \psi^P, \theta^R\big)(y) \chi_{P\cap R^c \cap (B_R^l)^c}(x), \\ \mathcal{U}_3^2(x,y) &:= \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\Lambda_j a_k, \psi^P, \theta^R\big)(y) \chi_{P\cap R^c \cap B_R^l}(x), \end{aligned} \tag{7.19}$$

and

$$\mathcal{J}_2^{3,\eta} := \left\| \sup_{l\in\mathbb{Z}} \left| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x) \chi_{B_k^l}(x) \phi_l * \big(\mathcal{U}_3^\eta(x,\cdot)\big)(x) \right| \right\|_{L^p(x)}, \quad \eta = 1,2.$$

As in the proof of the case $\nu = 2$, it suffices to prove the following two lemmas.

Lemma 7.8. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_3^1$ be defined as in Equation (7.19). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \;\; \mathrm{i} = 1,2,3, \tag{7.20}$$

and

$$\sup_{l\in\mathbb{Z}} \left| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x) \chi_{B_k^l}(x) \phi_l * \big(\mathcal{U}_3^1(x,\cdot)\big)(x) \right| \lesssim u_1(x) u_2(x) u_3(x). \tag{7.21}$$

Lemma 7.9. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_3^2$ be defined as in Equation (7.19). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \;\; \mathrm{i} = 1,2,3, \tag{7.22}$$

and

$$\sup_{l\in\mathbb{Z}} \left| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x) \chi_{B_k^l}(x) \phi_l * \big(\mathcal{U}_3^2(x,\cdot)\big)(x) \right| \lesssim u_1(x) u_2(x) u_3(x). \tag{7.23}$$

The proof of Lemmas 7.8 and 7.9 will be provided in Section 9.

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7.2.4. Proof of Equation (7.15) for $\nu = 4$. Let $B_P^l := B_{\mathbf{x}^P}^l$ and $B_R^l := B_{\mathbf{x}_R}^l$ for $P,R \in \mathcal{D}$ and $l \in \mathbb{Z}$, and let $\Xi_\eta(P,R,l)$ be defined as in Equation (7.11). Now, we write

$$\mathcal{U}_4 = \mathcal{U}_4^1 + \mathcal{U}_4^2 + \mathcal{U}_4^3 + \mathcal{U}_4^4,$$

where

$$\mathcal{U}_4^\eta(x,y) := \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} b_P^2 b_R^3 T_{\sigma_j}\big(\Lambda_j a_k, \psi^P, \theta^R\big)(y) \chi_{\Xi_\eta(P,R,l)}(x), \quad \eta = 1,2,3,4. \tag{7.24}$$

Accordingly, we define

$$\mathcal{J}_2^{4,\eta} := \Bigg\| \sup_{l\in\mathbb{Z}} \Bigg| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x) \chi_{B_k^l}(x) \phi_l * \big(\mathcal{U}_4^\eta(x,\cdot)\big)(x) \Bigg| \Bigg\|_{L^p(x)}, \quad \eta = 1,2,3,4.$$

Then we obtain the desired result from the following lemmas.

Lemma 7.10. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_4^\eta$, $\eta = 1,2,3$, be defined as in Equation (7.24). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_\mathrm{i}\|_{L^{p_\mathrm{i}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \ \ \mathrm{i} = 1,2,3,$$

and for each $\eta = 1,2,3$

$$\sup_{l\in\mathbb{Z}} \Bigg| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x) \chi_{B_k^l}(x) \phi_l * \big(\mathcal{U}_4^\eta(x,\cdot)\big)(x) \Bigg| \lesssim u_1(x) u_2(x) u_3(x). \tag{7.25}$$

Lemma 7.11. *Let $0 < p_1 \le 1$ and $2 < p_2, p_3 < \infty$, and let $\mathcal{U}_4^4$ be defined as in Equation (7.24). Suppose that $\|f_1\|_{H^{p_1}(\mathbb{R}^n)} = \|f_2\|_{H^{p_2}(\mathbb{R}^n)} = \|f_3\|_{H^{p_3}(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p_1 + n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_\mathrm{i}\|_{L^{p_\mathrm{i}}(\mathbb{R}^n)} \lesssim 1 \quad \textit{for} \ \ \mathrm{i} = 1,2,3, \tag{7.26}$$

and

$$\sup_{l\in\mathbb{Z}} \Bigg| \sum_{k=0}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x) \chi_{B_k^l}(x) \phi_l * \big(\mathcal{U}_4^4(x,\cdot)\big)(x) \Bigg| \lesssim u_1(x) u_2(x) u_3(x). \tag{7.27}$$

The proof of the lemmas will be given in Section 9.

8. Proof of Proposition 3.2

We need to deal only with, via symmetry, the case when $0 < p_1 = p \le 1$ and $p_2 = p_3 = \infty$. As before, we assume that $\|f_1\|_{H^p(\mathbb{R}^n)} = \|f_2\|_{L^\infty(\mathbb{R}^n)} = \|f_3\|_{L^\infty(\mathbb{R}^n)} = 1$ and $\mathcal{L}_s^2[\sigma] = 1$ for $s > n/p + n/2$. In this case, we do not decompose the frequencies of f_2, f_3 and only make use of the atomic decomposition on f_1. Let a_k's be H^p-atoms associated with Q_k so that $f_1 = \sum_{k=1}^\infty \lambda_k a_k$ and $\big(\sum_{k=1}^\infty |\lambda_k|^p\big)^{1/p} \lesssim 1$. Then we will prove that

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$$\Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=1}^{\infty}\lambda_k\chi_{Q_k^{***}}\,\phi_l*T_\sigma(a_k,f_2,f_3)\Big|\Big\|_{L^p(\mathbb{R}^n)}\lesssim 1 \tag{8.1}$$

and

$$\Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=1}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}\,\phi_l*T_\sigma(a_k,f_2,f_3)\Big|\Big\|_{L^p(\mathbb{R}^n)}\lesssim 1. \tag{8.2}$$

8.1. Proof of Equation (8.1)

Since

$$\big|\phi_l*T_\sigma(a_k,f_2,f_3)(x)\big|\lesssim \mathcal{M}T_\sigma(a_k,f_2,f_3)(x),$$

the left-hand side of Equation (8.1) is controlled by

$$\Big(\sum_{k=1}^{\infty}|\lambda_k|^p\big\|\mathcal{M}T_\sigma(a_k,f_2,f_3)\big\|_{L^p(Q^{***})}^p\Big)^{1/p}.$$

Using Hölder's inequality, the L^2 boundedness of $\mathcal{M}$ and Theorem D, we have

$$\big\|\mathcal{M}T_\sigma(a_k,f_2,f_3)\big\|_{L^p(Q^{***})}\lesssim |Q_k|^{1/p-1/2}\big\|T_\sigma(a_k,f_2,f_3)\big\|_{L^2(\mathbb{R}^n)}\lesssim |Q_k|^{1/p-1/2}\|a_k\|_{L^2(\mathbb{R}^n)}\lesssim 1$$

and thus Equation (8.1) follows from $\big(\sum_{k=1}^{\infty}|\lambda_k|^p\big)^{1/p}\lesssim 1$.

8.2. Proof of Equation (8.2)

Let $B_k^l=B(\mathbf{x}_{Q_k},100n2^{-l})$ as before. We now decompose the left-hand side of Equation (8.2) as the sum of

$$\mathcal{V}_1:=\Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=1}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}\chi_{(B_k^l)^c}\,\phi_l*T_\sigma(a_k,f_2,f_3)\Big|\Big\|_{L^p(\mathbb{R}^n)},$$

$$\mathcal{V}_2:=\Big\|\sup_{l\in\mathbb{Z}}\Big|\sum_{k=1}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}\chi_{B_k^l}\,\phi_l*T_\sigma(a_k,f_2,f_3)\Big)\Big|\Big\|_{L^p(\mathbb{R}^n)},$$

and thus we need to show that

$$\mathcal{V}_1,\mathcal{V}_2\lesssim 1.$$

Actually, the proof of these estimates will be complete once we have verified the following lemmas.

Lemma 8.1. *Let $0<p\le 1$. Suppose that $\|f_1\|_{H^p(\mathbb{R}^n)}=\|f_2\|_{H^\infty(\mathbb{R}^n)}=\|f_3\|_{H^\infty(\mathbb{R}^n)}=1$ and $\mathcal{L}_s^2[\sigma]=1$ for $s>n/p+n/2$. Then there exist nonnegative functions u_1, u_2 and u_3 on $\mathbb{R}^n$ such that*

$$\|u_1\|_{L^p(\mathbb{R}^n)}\lesssim 1,\qquad \|u_{\mathrm{i}}\|_{L^\infty(\mathbb{R}^n)}\lesssim 1\quad \textit{for}\ \ \mathrm{i}=2,3,$$

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and

$$\sup_{l\in\mathbb{Z}}\Big|\sum_{k=1}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}(x)\chi_{(B_k^l)^c}(x)\,\phi_l*T_\sigma\big(a_k,f_2,f_3\big)(x)\Big|\lesssim u_1(x)u_2(x)u_3(x). \tag{8.3}$$

Lemma 8.2. *Let* $0<p\le 1$*. Suppose that* $\|f_1\|_{H^p(\mathbb{R}^n)}=\|f_2\|_{H^\infty(\mathbb{R}^n)}=\|f_3\|_{H^\infty(\mathbb{R}^n)}=1$ *and* $\mathcal{L}_s^2[\sigma]=1$ *for* $s>n/p+n/2$*. Then there exist nonnegative functions* u_1*,* u_2 *and* u_3 *on* $\mathbb{R}^n$ *such that*

$$\|u_1\|_{L^p(\mathbb{R}^n)}\lesssim 1,\qquad \|u_{\mathrm{i}}\|_{L^\infty(\mathbb{R}^n)}\lesssim 1\quad \textit{for}\ \ \mathrm{i}=2,3,$$

and

$$\sup_{l\in\mathbb{Z}}\Big|\sum_{k=1}^{\infty}\lambda_k\chi_{(Q_k^{***})^c}(x)\chi_{B_k^l}(x)\,\phi_l*T_\sigma\big(a_k,f_2,f_3\big)(x)\Big|\lesssim u_1(x)u_2(x)u_3(x). \tag{8.4}$$

The proof of the two lemmas will be given in Section 9.

9. Proof of the key lemmas

9.1. Proof of Lemma 6.1

Let $1<r<2$ such that $s>3n/r>3n/2$, and we claim the pointwise estimate

$$\big|T_{\sigma_j}(\Lambda_j a_k,\Lambda_j f_2,\Gamma_j f_3)(y)\big|\lesssim \mathcal{M}_r\Lambda_j a_k(y)\mathcal{M}_r\Lambda_j f_2(y)\mathcal{M}_r\Gamma_j f_3(y). \tag{9.1}$$

Indeed, choosing t so that $3n/r<3t<s$, we apply Hölder's inequality to bound the left-hand side of Equation (9.1) by

$$\begin{aligned}&\int_{(\mathbb{R}^n)^3}\langle 2^j\vec{\boldsymbol{z}}\rangle^{3t}|\sigma_j^\vee(\vec{\boldsymbol{z}})|\frac{|\Lambda_j a_k(y-z_1)|}{\langle 2^j z_1\rangle^t}\frac{|\Lambda_j f_2(y-z_2)|}{\langle 2^j z_2\rangle^t}\frac{|\Gamma_j f_3(y-z_3)|}{\langle 2^j z_3\rangle^t}\,d\vec{\boldsymbol{z}}\\ &\quad\le \big\|\langle 2^j\vec{\cdot}\rangle^{3t}\sigma_j^\vee\big\|_{L^{r'}((\mathbb{R}^n)^3)}\Big\|\frac{\Lambda_j a_k(y-\cdot)}{\langle 2^j\cdot\rangle^t}\Big\|_{L^r(\mathbb{R}^n)}\Big\|\frac{\Lambda_j f_2(y-\cdot)}{\langle 2^j\cdot\rangle^t}\Big\|_{L^r(\mathbb{R}^n)}\Big\|\frac{\Gamma_j f_3(y-\cdot)}{\langle 2^j\cdot\rangle^t}\Big\|_{L^r(\mathbb{R}^n)}.\end{aligned}$$

We observe that

$$\big\|\langle 2^j\vec{\cdot}\rangle^{3t}\sigma_j^\vee\big\|_{L^{r'}((\mathbb{R}^n)^3)}\lesssim 2^{3jn/r}\|\sigma(2^j\vec{\cdot})\|_{L^r_{3t}((\mathbb{R}^n)^3)}\lesssim 2^{3jn/r}\|\sigma(2^j\vec{\cdot})\|_{L^2_s((\mathbb{R}^n)^3)}\lesssim 2^{3jn/r}$$

using the Hausdorff–Young inequality, Equation (5.1) and the inclusion

$$L^{t_0}_{s_0}(A)\hookrightarrow L^{t_1}_{s_1}(A)\qquad \text{for}\ \ s_0\ge s_1,\ t_0\ge t_1,$$

where A is a ball of a constant radius, whose proof is contained in [19, (1.8)]. Applying Equation (2.5) to the remaining three L^r norms, we finally obtain Equation (9.1).

Now, we choose $\widetilde{r}$ and q such that $2<\widetilde{r}<p_2,p_3$ and $1/q+2/\widetilde{r}=1$. Finally, using the estimate (9.1) and Hölder's inequality, we have

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$$
\begin{aligned}
&\Big|\sum_{k=0}^{\infty}\lambda_k\chi_{Q_k^{***}}(x)\phi_l*\Big(\sum_{j\in\mathbb{Z}}T_{\sigma_j}(\Lambda_j a_k,\Lambda_j f_2,\Gamma_j f_3)\Big)(x)\Big| \\
&\quad\lesssim \sum_{k=0}^{\infty}\lambda_k\chi_{Q_k^{***}}(x)2^{ln}\int_{|x-y|\le 2^{-l}}\big\|\{\mathcal{M}_r\Lambda_j a_k(y)\}_{j\in\mathbb{Z}}\big\|_{\ell^2}\big\|\{\mathcal{M}_r\Lambda_j f_2(y)\}_{j\in\mathbb{Z}}\big\|_{\ell^2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\big\|\{\mathcal{M}_r\Gamma_j f_3(y)\}_{j\in\mathbb{Z}}\big\|_{\ell^\infty}\,dy \\
&\quad\lesssim u_1(x)u_2(x)u_3(x),
\end{aligned}
$$

where we choose

$$
\begin{aligned}
u_1(x)&:=\sum_{k=0}^{\infty}\lambda_k\chi_{Q_k^{***}}(x)\mathcal{M}_q\big(\big\|\{\mathcal{M}_r\Lambda_j a_k\}_{j\in\mathbb{Z}}\big\|_{\ell^2}\big)(x),\\
u_2(x)&:=\mathcal{M}_{\widetilde{r}}\big(\big\|\{\mathcal{M}_r\Lambda_j f_2\}_{j\in\mathbb{Z}}\big\|_{\ell^2}\big)(x),\\
u_3(x)&:=\mathcal{M}_{\widetilde{r}}\big(\big\|\{\mathcal{M}_r\Gamma_j f_3\}_{j\in\mathbb{Z}}\big\|_{\ell^\infty}\big)(x)
\end{aligned}
$$

and this proves (6.1). Moreover,

$$
\|u_1\|_{L^{p_1}(\mathbb{R}^n)}\le\Big(\sum_{k=0}^{\infty}|\lambda_k|^{p_1}\Big\|\mathcal{M}_q\big(\big\|\{\mathcal{M}_r\Lambda_j a_k\}_{j\in\mathbb{Z}}\big\|_{\ell^2}\big)\Big\|_{L^{p_1}(Q_k^{***})}^{p_1}\Big)^{1/p_1}\lesssim 1,
$$

where the last inequality follows from Equation (5.6) and the estimate

$$
\begin{aligned}
\Big\|\mathcal{M}_q\big(\big\|\{\mathcal{M}_r\Lambda_j a_k\}_{j\in\mathbb{Z}}\big\|_{\ell^2}\big)\Big\|_{L^{p_1}(Q_k^{***})}&\lesssim|Q_k|^{1/p_1-1/r_0}\Big\|\mathcal{M}_q\big(\big\|\{\mathcal{M}_r\Lambda_j a_k\}_{j\in\mathbb{Z}}\big\|_{\ell^2}\big)\Big\|_{L^{r_0}(\mathbb{R}^n)}\\
&\lesssim|Q_k|^{1/p_1-1/r_0}\big\|\{\Lambda_j a_k\}_{j\in\mathbb{Z}}\big\|_{L^{r_0}(\ell^2)}\sim|Q_k|^{1/p_1-1/r_0}\|a_k\|_{L^{r_0}(\mathbb{R}^n)}\lesssim 1
\end{aligned}
$$

for $q<r_0<\infty$. Here, we applied Hölder's inequality, the maximal inequality (2.4), the equivalence in (2.2) and properties of the H^{p_1}-atom a_k. It is also easy to verify

$$
\|u_2\|_{L^{p_2}(\mathbb{R}^n)}\lesssim\big\|\{\Lambda_j f_2\}_{j\in\mathbb{Z}}\big\|_{L^{p_2}(\ell^2)}\sim 1
$$

and

$$
\|u_3\|_{L^{p_3}(\mathbb{R}^n)}\lesssim\big\|\{\Gamma_j f_3\}_{j\in\mathbb{Z}}\big\|_{L^{p_3}(\ell^\infty)}\sim 1
$$

using Equations (2.4), (2.1) and (2.2).

9.2. Proof of Lemma 7.1

Since

$$
s>n/p_1+n/2=\big(n/p_1-n/2\big)+n/2+n/2,
$$

we can choose s_1,s_2,s_3 such that $s_1>n/p_1-n/2$, $s_2,s_3>n/2$, and $s=s_1+s_2+s_3$.

Using the estimates

$$
\big\|\psi^P\big\|_{L^\infty(\mathbb{R}^n)}\le|P|^{-1/2}\quad\text{and}\quad\big\|\theta^R\big\|_{L^\infty(\mathbb{R}^n)}\le|R|^{-1/2},
$$

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we have

$$\begin{aligned}\big|\mathcal{U}_1^{\rm in}(x,y)\big| &\lesssim \sum_{j\in\mathbb{Z}}\Big(\sum_{P\in\mathcal{D}_j}|b_P^2||P|^{-1/2}\chi_P(x)\Big)\Big(\sum_{R\in\mathcal{D}_j}|b_R^3||R|^{-1/2}\chi_R(x)\Big)\\ &\qquad\times\int_{(\mathbb{R}^n)^3}\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\big|\Lambda_j a_k(z_1)\big|\chi_{Q_k^*}(z_1)\,d\vec{\boldsymbol{z}}\\ &\le g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x)\\ &\qquad\times\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{(\mathbb{R}^n)^3}\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\big|\Lambda_j a_k(z_1)\big|\chi_{Q_k^*}(z_1)\,d\vec{\boldsymbol{z}}\Big)^2\Big)^{1/2}. \end{aligned}\tag{9.2}$$

We observe that for $|x-y|\le 2^{-l}$, $x\in (Q_k^{***})^c\cap(B_k^l)^c$ and $z_1\in Q_k^*$,

$$|x-\mathbf{x}_{Q_k}|\lesssim|y-z_1|\tag{9.3}$$

and thus, by using Lemma 4.5,

$$\begin{aligned}&\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1}\int_{(\mathbb{R}^n)^3}\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\big|\Lambda_j a_k(z_1)\big|\chi_{Q_k^*}(z_1)\,d\vec{\boldsymbol{z}}\\ &\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_1)\rangle^{s_1}\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\chi_{Q_k^*}(z_1)\,d\vec{\boldsymbol{z}}\\ &\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\big\|\langle 2^j\cdot\rangle^{-s_2}\big\|_{L^2(\mathbb{R}^n)}\big\|\langle 2^j\cdot\rangle^{-s_3}\big\|_{L^2(\mathbb{R}^n)}2^{jn}I_{k,j,s}^{\rm in}(y)\\ &\sim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I_{k,j,s}^{\rm in}(y)\end{aligned}$$

for sufficiently large M, where

$$I_{k,j,s}^{\rm in}(y):=2^{-jn}\int_{Q_k^*}\Big\|\langle 2^j(y-z_1),2^jz_2,2^jz_3\rangle^s\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\Big\|_{L^2(z_2,z_3)}\,dz_1.\tag{9.4}$$

This proves that

$$\begin{aligned}&\int_{(\mathbb{R}^n)^3}\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\big|\Lambda_j a_k(z_1)\big|\chi_{Q_k^*}(z_1)\,d\vec{\boldsymbol{z}}\\ &\qquad\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-s_1}I_{k,j,s}^{\rm in}(y),\end{aligned}\tag{9.5}$$

and therefore, we obtain

$$\begin{aligned}\big|\mathcal{U}_1^{\rm in}(x,y)\big| &\lesssim g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x)\ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\\ &\qquad\times\Big(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I_{k,j,s}^{\rm in}(y)\Big)^2\Big)^{1/2}.\end{aligned}\tag{9.6}$$

Similar to Equation (9.2), we write

$$\begin{aligned}\big|\mathcal{U}_1^{\rm out}(x,y)\big| &\lesssim g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x)\\ &\qquad\times\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{(\mathbb{R}^n)^3}\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\big|\Lambda_j a_k(z_1)\big|\chi_{(Q_k^*)^c}(z_1)d\vec{\boldsymbol{z}}\Big)^2\Big)^{1/2}.\end{aligned}$$

Instead of Equation (9.3), we make use of the estimate

$$\langle 2^j(x-\mathbf{x}_{Q_k})\rangle \lesssim \langle 2^j(y-\mathbf{x}_{Q_k})\rangle \le \langle 2^j(y-z_1)\rangle\langle 2^j(z_1-\mathbf{x}_{Q_k})\rangle \tag{9.7}$$

for $|x-y|\le 2^{-l}$ and $x\in (Q_k^{***})^c\cap (B_k^l)^c$. Then, using the argument that led to Equation (9.5), we have

$$\begin{aligned}\int_{(\mathbb{R}^n)^3} &\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\big|\Lambda_j a_k(z_1)\big|\chi_{(Q_k^*)^c}(z_1)\,d\vec{\boldsymbol{z}}\\ &\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-s_1} I_{k,j,s}^{\mathrm{out}}(y),\end{aligned} \tag{9.8}$$

where M, L_0 are sufficiently large numbers and

$$\begin{aligned}I_{k,j,s}^{\mathrm{out}}(y) := 2^{-jn}\int_{(Q_k^*)^c} &\frac{(2^j\ell(Q_k))^n}{\langle 2^j(z_1-\mathbf{x}_{Q_k})\rangle^{L_0-s_1}}\\ &\times\Big\|\langle 2^j(y-z_1),2^jz_2,2^jz_3\rangle^s\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|\Big\|_{L^2(z_2,z_3)}\,dz_1.\end{aligned} \tag{9.9}$$

Now, we deduce

$$\begin{aligned}\big|\mathcal{U}_1^{\mathrm{out}}(x,y)\big| &\lesssim g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x)\ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\\ &\times\bigg(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I_{k,j,s}^{\mathrm{out}}(y)\Big)^2\bigg)^{1/2}.\end{aligned} \tag{9.10}$$

According to Equations (9.6) and (9.10), the estimate (7.7) follows from taking

$$\begin{aligned}u_1^{\mathrm{in/out}}(x) &:= \sum_{k=0}^{\infty}|\lambda_k|\ell(Q_k)^{-n/p_1}\chi_{(Q_k^{***})^c}(x)|x-\mathbf{x}_{Q_k}|^{-s_1}\\ &\qquad\times\mathcal{M}\Bigg[\bigg(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\{1,(2^j\ell(Q_k))^M\}I_{k,j,s}^{\mathrm{in/out}}(\cdot)\Big)^2\bigg)^{1/2}\Bigg](x),\\ u_2(x) &:= g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x),\\ u_3(x) &:= g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x).\end{aligned}$$

It is clear that

$$\|u_2\|_{L^{p_2}(\mathbb{R}^n)} = \big\|\{b_P^2\}_{P\in\mathcal{D}}\big\|_{\dot{f}^{p_2,2}} \sim 1 \tag{9.11}$$

$$\|u_3\|_{L^{p_3}(\mathbb{R}^n)} = \big\|\{b_R^3\}_{R\in\mathcal{D}}\big\|_{\dot{f}^{p_3,\infty}} \sim 1 \tag{9.12}$$

in view of Equations (7.1) and (7.2). To estimate u_1^{in} and u_1^{out}, we note that

$$\begin{aligned}\big\|I_{k,j,s_1}^{\mathrm{in}}\big\|_{L^2(\mathbb{R}^n)} &\le 2^{-jn}\int_{Q_k}\bigg(\int_{\mathbb{R}^n}\Big\|\langle 2^jy,2^jz_2,2^jz_3\rangle^s\big|\sigma_j^\vee(y,z_2,z_3)\big|\Big\|_{L^2(z_2,z_3)}^2\,dy\bigg)^{1/2}\,dz_1\\ 0 &= 2^{-jn}\ell(Q_k)^n\big\|\langle 2^j\vec{\cdot}\rangle^s\sigma_j^\vee\big\|_{L^2((\mathbb{R}^n)^3)} \le 2^{jn/2}\ell(Q_k)^n,\end{aligned} \tag{9.13}$$

where we applied Minkowski's inequality and a change of variables, and similarly,

$$\begin{aligned}\big\|I^{\mathrm{out}}_{k,j,s}\big\|_{L^2(\mathbb{R}^n)} &\lesssim 2^{-jn}\int_{(Q_k^*)^c}\frac{(2^j\ell(Q_k))^n}{\langle 2^j(z_1-\mathbf{x}_{Q_k})\rangle^{L_0-s_1}}\,dz_1\big\|\langle 2^j\vec{\cdot}\rangle^s\sigma_j^\vee\big\|_{L^2((\mathbb{R}^n)^3)}\\ &\lesssim 2^{jn/2}\ell(Q_k)^n\big(2^j\ell(Q_k)\big)^{-(L_0-s_1-n)} \end{aligned}\tag{9.14}$$

for $L_0>s+n$. Now, we have

$$\begin{aligned}\big\|u_1^{\mathrm{in}}\big\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \le \sum_{k=0}^\infty |\lambda_k|^{p_1}\ell(Q_k)^{-n}\int_{(Q_k^{***})^c}|x-\mathbf{x}_{Q_k}|^{-s_1p_1}\\ \times\bigg(\mathcal{M}\bigg[\bigg(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I^{\mathrm{in}}_{k,j,s_1}(\cdot)\Big)^2\bigg)^{1/2}\bigg](x)\bigg)^{p_1}dx\end{aligned}$$

and the integral is dominated by

$$\begin{aligned}&\big\||\cdot-\mathbf{x}_{Q_k}|^{-s_1p_1}\big\|_{L^{(2/p_1)'}((Q_k^{***})^c)}\\ &\times\bigg\|\bigg(\mathcal{M}\bigg[\bigg(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I^{\mathrm{in}}_{k,j,s_1}(\cdot)\Big)^2\bigg)^{1/2}\bigg]\bigg)^{p_1}\bigg\|_{L^{2/p_1}(\mathbb{R}^n)}.\end{aligned}$$

The first term is no more than a constant times $\ell(Q_k)^{-p_1(s_1-(n/p_1-n/2))}$, and the second one is bounded by

$$\begin{aligned}&\bigg(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\big\{1,\big(2^j\ell(Q_k)\big)^N\big\}\big\|I^{\mathrm{in}}_{k,j,s_1}\big\|_{L^2(\mathbb{R}^n)}\Big)^2\bigg)^{p_1/2}\\ &\quad\lesssim \ell(Q_k)^{p_1n}\bigg(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}2^{jn/2}\Big)^2\bigg)^{p_1/2}\lesssim \ell(Q_k)^{s_1p_1+p_1n/2},\end{aligned}$$

due to Equation (9.13). This proves

$$\big\|u_1^{\mathrm{in}}\big\|_{L^{p_1}(\mathbb{R}^n)}\lesssim\bigg(\sum_{k=0}^\infty|\lambda_k|^{p_1}\bigg)^{1/p_1}\lesssim 1.\tag{9.15}$$

In a similar way, together with Equation (9.14), we can also prove

$$\big\|u_1^{\mathrm{out}}\big\|_{L^{p_1}(\mathbb{R}^n)}\lesssim\bigg(\sum_{k=0}^\infty|\lambda_k|^{p_1}\bigg)^{1/p_1}\lesssim 1,\tag{9.16}$$

choosing $M>L_0-3n/2$.

9.3. Proof of Lemma 7.2

As in the proof of Lemma 7.1, we pick s_1,s_2,s_3 satisfying $s_1>n/p_1-n/2$, $s_2,s_3>n/2$, and $s=s_1+s_2+s_3>n/p_1+n/2$.

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We first consider the case $\eta = 1$. For $x \in P^c \cap (B_P^l)^c$ and $|x-y| \le 2^{-l}$, we have

$$\langle 2^j(x-\mathbf{x}_P)\rangle \lesssim \langle 2^j(y-\mathbf{x}_P)\rangle \le \langle 2^j(y-z_2)\rangle \langle 2^j(z_2-\mathbf{x}_P)\rangle. \tag{9.17}$$

By using

$$\left\|\theta^R\right\|_{L^\infty(\mathbb{R}^n)} \le |R|^{-1/2},$$

we have

$$\begin{aligned} \left|\mathcal{U}_2^{1,\mathrm{in}}(x,y)\right| &\lesssim \sum_{j\in\mathbb{Z}} \Big(\sum_{R\in\mathcal{D}_j} |b_R^3||R|^{-1/2}\chi_R(x)\Big) \int_{(\mathbb{R}^n)^3} \left|\sigma_j^\vee(y-z_1,y-z_2,z_3)\right| \\ &\quad \times \left|\Lambda_j a_k(z_1)\right| \chi_{Q_k^*}(z_1) \Big(\sum_{P\in\mathcal{D}_j} |b_P^2| \chi_{P^c}(x)\chi_{(B_P^l)^c}(x) \left|\psi^P(z_2)\right|\Big)\, d\vec{z}. \end{aligned} \tag{9.18}$$

Using Equations (9.3) and (9.17) and Lemma 4.5, the integral in the preceding expression is bounded by

$$\begin{aligned} &\ell(Q_k)^{-n/p_1} \min\left\{1, \left(2^j\ell(Q_k)\right)^M\right\} \langle 2^j(x-c_{Q_k})\rangle^{-s_1} \int_{(\mathbb{R}^n)^3} \langle 2^j(y-z_1)\rangle^{s_1} \langle 2^j(y-z_2)\rangle^{s_2} \\ &\quad \times \left|\sigma_j^\vee(y-z_1,y-z_2,z_3)\right| \chi_{Q_k^*}(z_1) \left(\sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}} \left|\widetilde{\psi^P}(z_2)\right| \right) d\vec{z} \\ &\lesssim \ell(Q_k)^{-n/p_1} \min\left\{1, \left(2^j\ell(Q_k)\right)^M\right\} \langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-s_1} I_{k,j,s}^{\mathrm{in}}(y) \\ &\quad \times 2^{jn/2} \left\| \sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}} \left|\widetilde{\psi^P}(\cdot)\right| \right\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

for sufficiently large $M > 0$, where $\widetilde{\psi^P}(z_2) := \langle 2^j(z_2-\mathbf{x}_P)\rangle^{s_2}\psi^P(z_2)$ for $P \in \mathcal{D}_j$ and $I_{k,j,s}^{\mathrm{in}}$ is defined as in Equation (9.4). Note that

$$|b_P^2| \lesssim \mathcal{B}_P^2(f_2) := \left\langle \left|\widetilde{\Lambda}_j f_2\right|, \frac{2^{jn/2}}{\langle 2^j(\cdot-\mathbf{x}_P)\rangle^L} \right\rangle \quad \text{for} \quad L > n,s, \tag{9.19}$$

and thus it follows from Lemma 4.2 that the L^2 norm in the last displayed expression is dominated by

$$2^{-jn/2} \left(\sum_{P\in\mathcal{D}_j} \left(\left|\mathcal{B}_P^2(f_2)\right| |P|^{-1/2} \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}} \right)^2 \right)^{1/2}.$$

This yields that

$$\begin{aligned} \left|\mathcal{U}_2^{1,\mathrm{in}}(x,y)\right| &\lesssim \ell(Q_k)^{-n/p_1} |x-c_{Q_k}|^{-s_1} \left(\sum_{j\in\mathbb{Z}} \left(2^{-js_1} \min\left\{1, \left(2^j\ell(Q_k)\right)^M\right\} I_{k,j,s}^{\mathrm{in}}(y)\right)^2 \right)^{1/2} \\ &\quad \times \left(\sum_{P\in\mathcal{D}} \left(\left|\mathcal{B}_P^2(f_2)\right| |P|^{-1/2} \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}} \right)^2 \right)^{1/2} g^\infty\left(\{b_R^3\}_{R\in\mathcal{D}}\right)(x). \end{aligned} \tag{9.20}$$

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Similarly, using Equations (9.7) and (9.17), Lemma 4.5 and Lemma 4.2, we have

$$
\begin{aligned}
\big|\mathcal{U}_2^{1,\mathrm{out}}(x,y)\big| &\lesssim \sum_{j\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{D}_j}|b_R^3||R|^{-1/2}\chi_R(x)\Big)\int_{(\mathbb{R}^n)^3}\big|\sigma_j^{\vee}(y-z_1,y-z_2,z_3)\big| \\
&\qquad\times\big|\Lambda_j a_k(z_1)\big|\chi_{(Q_k^*)^c}(z_1)\Big(\sum_{P\in\mathcal{D}_j}|b_P^2|\chi_{P^c}(x)\chi_{(B_P^l)^c}(x)\big|\psi^P(z_2)\big|\Big)\,d\vec{z} \\
&\lesssim \ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\Big(\sum_{j\in\mathbb{Z}}\big(2^{-js_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I_{k,j,s}^{\mathrm{out}}(y)\big)^2\Big)^{1/2} \\
&\qquad\times\bigg(\sum_{P\in\mathcal{D}}\Big(\big|\mathcal{B}_P^2(f_2)\big||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Big)^2\bigg)^{1/2} g^{\infty}\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x),
\end{aligned}
\tag{9.21}
$$

where $I_{k,j,s}^{\mathrm{out}}$ is defined as in Equation (9.9).

When $\eta=2$, we use the inequality

$$
\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-1}\le\langle 2^j(x-\mathbf{x}_P)\rangle^{-1} \tag{9.22}
$$

for $x\in(B_k^l)^c\cap B_P^l$. Then, similar to Equation (9.18), we have

$$
\begin{aligned}
\big|\mathcal{U}_2^{2,\mathrm{in}}(x,y)\big| &\lesssim \sum_{j\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{D}_j}|b_R^3||R|^{-1/2}\chi_R(x)\Big)\int_{(\mathbb{R}^n)^3}\big|\sigma_j^{\vee}(y-z_1,y-z_2,z_3)\big| \\
&\qquad\times\big|\Lambda_j a_k(z_1)\big|\chi_{Q_k^*}(z_1)\Big(\sum_{P\in\mathcal{D}_j}|b_P^2|\chi_{P^c}(x)\chi_{B_P^l}(x)\big|\psi^P(z_2)\big|\Big)\,d\vec{z},
\end{aligned}
$$

and the integral is dominated by a constant times

$$
\begin{aligned}
&\ell(Q_k)^{-n/p_1}\min\{1,\big(2^j\ell(Q_k)\big)^M\}\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-(s_1+s_2)}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_1)\rangle^{s_1+s_2} \\
&\qquad\times\big|\sigma_j^{\vee}(y-z_1,y-z_2,z_3)\big|\chi_{Q_k^*}(z_1)\Big(\sum_{P\in\mathcal{D}_j}|b_P^2|\chi_{P^c}(x)\chi_{B_P^l}(x)\big|\psi^P(z_2)\big|\Big)\,d\vec{z} \\
&\le \ell(Q_k)^{-n/p_1}\min\{1,\big(2^j\ell(Q_k)\big)^M\}\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-s_1}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_1)\rangle^{s_1+s_2} \\
&\qquad\times\big|\sigma_j^{\vee}(y-z_1,y-z_2,z_3)\big|\chi_{Q_k^*}(z_1)\Big(\sum_{P\in\mathcal{D}_j}|b_P^2|\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\big|\psi^P(z_2)\big|\Big)\,d\vec{z} \\
&\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-s_1}I_{k,j,s}^{\mathrm{in}}(y) \\
&\qquad\times\bigg(\sum_{P\in\mathcal{D}}\Big(\big|\mathcal{B}_P^2(f_2)\big||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Big)^2\bigg)^{1/2}
\end{aligned}
$$

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due to Equations (9.3) and (9.22), Lemma 4.5 and Lemma 4.2, where $I_{k,j,s}^{\mathrm{in}}$ and $\mathcal{B}_P^2(f_2)$ are defined as in Equations (9.4) and (9.19). Therefore,

$$\begin{aligned}\big|\mathcal{U}_2^{2,\mathrm{in}}(x,y)\big| &\lesssim \ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\Big(\sum_{j\in\mathbb{Z}}\big(2^{-js_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I_{k,j,s}^{\mathrm{in}}(y)\big)^2\Big)^{1/2}\\ &\times\Big(\sum_{P\in\mathcal{D}}\Big(\big|\mathcal{B}_P^2(f_2)\big||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Big)^2\Big)^{1/2} g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x). \quad (9.23)\end{aligned}$$

Similarly, we can also prove that

$$\begin{aligned}\big|\mathcal{U}_2^{2,\mathrm{out}}(x,y)\big| &\lesssim \ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\Big(\sum_{j\in\mathbb{Z}}\big(2^{-js_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I_{k,j,s}^{\mathrm{out}}(y)\big)^2\Big)^{1/2}\\ &\times\Big(\sum_{P\in\mathcal{D}}\Big(\big|\mathcal{B}_P^2(f_2)\big||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Big)^2\Big)^{1/2} g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x).\end{aligned} \quad (9.24)$$

Combining Equations (9.20), (9.21), (9.23) and (9.24), the estimate (7.9) holds with

$$\begin{aligned}u_1^{\mathrm{in/out}}(x) &:= \sum_{k=0}^{\infty}|\lambda_k|\ell(Q_k)^{-n/p_1}\chi_{(Q_k^{***})^c}(x)|x-\mathbf{x}_{Q_k}|^{-s_1}\\ &\qquad\times\mathcal{M}\Big[\Big(\sum_{j\in\mathbb{Z}}\Big(2^{-s_1j}\min\{1,(2^j\ell(Q_k))^M\}I_{k,j,s}^{\mathrm{in/out}}(\cdot)\Big)^2\Big)^{1/2}\Big](x),\\ u_2(x) &:= \Big(\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\Big(\big|\mathcal{B}_P^2(f_2)\big||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Big)^2\Big)^{1/2},\\ u_3(x) &:= g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x).\end{aligned}$$

Clearly, as in Equations (9.15), (9.16) and (9.12),

$$\big\|u_1^{\mathrm{in/out}}\big\|_{L^{p_1}(\mathbb{R}^n)}\lesssim 1, \qquad \|u_3\|_{L^{p_3}(\mathbb{R}^n)}\lesssim 1$$

and Lemma 4.3 proves that

$$\|u_2\|_{L^{p_2}(\mathbb{R}^n)}\lesssim 1.$$

9.4. Proof of Lemma 7.3

The proof is almost same as that of Lemma 7.2. By letting $M>0$ be sufficiently large and exchanging the role of terms associated with f_2 and f_3 in the estimate (9.18), we may obtain

$$\begin{aligned}\big|\mathcal{U}_3^{1,\mathrm{in}}(x,y)\big| &\lesssim \sum_{j\in\mathbb{Z}}\Big(\sum_{P\in\mathcal{D}_j}|b_P^2||P|^{-1/2}\chi_P(x)\Big)\int_{(\mathbb{R}^n)^3}\big|\sigma_j^\vee(y-z_1,z_2,y-z_3)\big|\\ &\qquad\times\big|\Lambda_j a_k(z_1)\big|\chi_{Q_k^*}(z_1)\Big(\sum_{R\in\mathcal{D}_j}|b_R^3|\chi_{R^c}(x)\chi_{(B_R^l)^c}(x)\big|\theta^R(z_3)\big|\Big)\,d\vec{z}\end{aligned}$$

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$$\lesssim \ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\Big(\sum_{j\in\mathbb{Z}}\Big(2^{-js_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I^{\mathrm{in}}_{k,j,s}(y)\Big)^2\Big)^{1/2}$$
$$\times g^2\big(\{b^2_P\}_{P\in\mathcal{D}}\big)(x)\sup_{j\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{D}_j}\Big(\big|\mathcal{B}^3_R(f_3)\big||R|^{-1/2}\frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\Big)^2\Big)^{1/2}, \tag{9.25}$$

where $I^{\mathrm{in}}_{k,j,s}$ is defined as in Equation (9.4), $\widetilde{\theta^R}(z_3):=\langle 2^j(z_3-\mathbf{x}_R)\rangle^{s_3}\theta^R(z_3)$ for $R\in\mathcal{D}_j$ and

$$\mathcal{B}^3_R(f_3):=\Big\langle\big|\widetilde{\Gamma_j}f_3\big|,\frac{2^{jn/2}}{\langle 2^j(\cdot-\mathbf{x}_R)\rangle^L}\Big\rangle\quad \text{for}\quad L>s,n. \tag{9.26}$$

Similarly,

$$\big|\mathcal{U}^{1,\mathrm{out}}_3(x,y)\big|\lesssim \ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\Big(\sum_{j\in\mathbb{Z}}\Big(2^{-js_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I^{\mathrm{out}}_{k,j,s}(y)\Big)^2\Big)^{1/2}$$
$$\times g^2\big(\{b^2_P\}_{P\in\mathcal{D}}\big)(x)\sup_{j\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{D}_j}\Big(\big|\mathcal{B}^3_R(f_3)\big||R|^{-1/2}\frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\Big)^2\Big)^{1/2}, \tag{9.27}$$

where $I^{\mathrm{out}}_{k,j,s}$ is defined as in Equation (9.9).

For the case $\eta=2$, we use the fact that for $x\in(B^l_k)^c\cap B^l_R$,

$$\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{-1}\le\langle 2^j(x-\mathbf{x}_R)\rangle^{-1},$$

instead of Equation (9.22). Then we have

$$\big|\mathcal{U}^{2,\mathrm{in}}_3(x,y)\big|\lesssim \ell(Q_k)^{-n/p_1}|x-\mathbf{x}_{Q_k}|^{-s_1}\Big(\sum_{j\in\mathbb{Z}}\big(2^{-js_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}I^{\mathrm{in}}_{k,j,s}(y)\big)^2\Big)^{1/2}$$
$$\times g^2\big(\{b^2_P\}_{P\in\mathcal{D}}\big)(x)\sup_{j\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{D}_j}\Big(\big|\mathcal{B}^3_R(f_3)\big||R|^{-1/2}\frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\Big)^2\Big)^{1/2}$$

and

$$\big|\mathcal{U}^{2,\mathrm{out}}_3(x,y)\big|\lesssim \ell(Q_k)^{-n/p_1}\Big(\sum_{j\in\mathbb{Z}}\Big(\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\langle 2^j(x-\mathbf{x}_Q)\rangle^{-s_1}I^{\mathrm{out}}_{k,j,s}(y)\Big)^2\Big)^{1/2}$$
$$\times g^2\big(\{b^2_P\}_{P\in\mathcal{D}}\big)\sup_{j\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{D}_j}\Big(\big|\mathcal{B}^3_R(f_3)\big||R|^{-1/2}\frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\Big)^2\Big)^{1/2}$$

which are analogous to Equations (9.25) and (9.27).

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Then Lemma 7.3 follows from Equations (9.15), (9.16) and (9.11) and Lemma 4.4 by choosing

$$u_1^{\mathrm{in/out}}(x) := \sum_{k=0}^{\infty} |\lambda_k| \ell(Q_k)^{-n/p_1} \chi_{(Q_k^{***})^c}(x) |x - \mathbf{x}_{Q_k}|^{-s_1}$$
$$\times \mathcal{M}\left[\left(\sum_{j\in\mathbb{Z}} \left(2^{-s_1 j} \min\{1,(2^j \ell(Q_k))^M\} I_{k,j,s}^{\mathrm{in/out}}(\cdot)\right)^2\right)^{1/2}\right](x)$$
$$u_2(x) := g^2\left(\{b_P^2\}_{P\in\mathcal{D}}\right)(x)$$
$$u_3(x) := \sup_{j\in\mathbb{Z}} \left(\sum_{R\in\mathcal{D}_j} \left(|\mathcal{B}_R^3(f_3)||R|^{-1/2} \frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\right)^2\right)^{1/2}.$$

9.5. Proof of Lemma 7.4

Let $I_{k,j,s}^{\mathrm{in}}$, $I_{k,j,s}^{\mathrm{out}}$, $\mathcal{B}_P^2(f_2)$ and $\mathcal{B}_R^3(f_3)$ be defined as before. Let $M > 0$ be a sufficiently large number. We claim the pointwise estimates that for each $\eta = 1,2,3,4$,

$$|\mathcal{U}_4^{\eta,\mathrm{in/out}}(x,y)| \lesssim \ell(Q_k)^{-n/p_1} |x - x_{Q_k}|^{-s_1} \left(\sum_{j\in\mathbb{Z}} \left(2^{-js_1} \min\{1,(2^j\ell(Q_k))^M\} I_{k,j,s}^{\mathrm{in/out}}(y)\right)^2\right)^{1/2}$$
$$\times \left(\sum_{P\in\mathcal{D}} \left(|\mathcal{B}_P^2(f_2)||P|^{-1/2} \frac{\chi_{P_c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\right)^2\right)^{1/2}$$
$$\times \sup_{j\in\mathbb{Z}} \left(\sum_{P\in\mathcal{D}_j} \left(|\mathcal{B}_R^3(f_3)||R|^{-1/2} \frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\right)^2\right)^{1/2}.$$

The proof of the above claim is a repetition of the arguments used in the proof of Lemmas 7.2 and 7.3, so we omit the details. We now take

$$u_1^{\mathrm{in/out}}(x) := \sum_{k=0}^{\infty} |\lambda_k| \ell(Q_k)^{-n/p_1} \chi_{(Q_k^{***})^c}(x) |x - \mathbf{x}_{Q_k}|^{-s_1}$$
$$\times \mathcal{M}\left[\left(\sum_{j\in\mathbb{Z}} \left(2^{-s_1 j} \min\{1,(2^j \ell(Q_k))^M\} I_{k,j,s}^{\mathrm{in/out}}(\cdot)\right)^2\right)^{1/2}\right](x)$$
$$u_2(x) := \left(\sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \left(|\mathcal{B}_P^2(f_2)||P|^{-1/2} \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\right)^2\right)^{1/2}$$
$$u_3(x) := \sup_{j\in\mathbb{Z}} \left(\sum_{R\in\mathcal{D}_j} \left(|\mathcal{B}_R^3(f_3)||R|^{-1/2} \frac{1}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\right)^2\right)^{1/2}.$$

Then it is obvious that Equations (7.13) and (7.14) hold.

9.6. Proof of Lemma 7.5

We choose $0 < \epsilon < 1$ such that

$$N_{p_1} := [n/p_1 - n] \le n/p_1 - n < [n/p_1 - n] + \epsilon < s - 3n/2. \tag{9.28}$$

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We note that

$$2^l \lesssim |x - \mathbf{x}_{Q_k}|^{-1} \quad \text{for} \quad x \in B_k^l.$$

By using Lemma 4.1 with the vanishing moment condition (5.3), we have

$$\begin{aligned}
\big|\phi_l * \big(\mathcal{U}_1(x,\cdot)\big)(x)\big| &\lesssim \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} |b_P^2||b_R^3|\chi_{P\cap R}(x) 2^{l(n+N_{p_1}+\epsilon)} \\
&\quad \times \int_{\mathbb{R}^n} |y - \mathbf{x}_{Q_k}|^{N_{p_1}+\epsilon} \big|T_{\sigma_j}\big(\Lambda_j a_k, \psi^P, \theta^R\big)(y)\big|\, dy \\
&\lesssim \frac{1}{|x - \mathbf{x}_{Q_k}|^{n+N_{p_1}+\epsilon}} \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} |b_P^2||b_R^3|\chi_{P\cap R}(x) \\
&\quad \times \Big(\mathcal{K}^{j,\mathrm{in}}_{N_{p_1}+\epsilon}(Q_k,P,R) + \mathcal{K}^{j,\mathrm{out}}_{N_{p_1}+\epsilon}(Q_k,P,R)\Big),
\end{aligned}$$

where

$$\mathcal{K}^{j,\mathrm{in}}_{N_{p_1}+\epsilon}(Q_k,P,R) := \int_{Q_k^{**}} |y - \mathbf{x}_{Q_k}|^{N_{p_1}+\epsilon} \big|T_{\sigma_j}\big(\Lambda_j a_k, \psi^P, \theta^R\big)(y)\big|\, dy,$$

and

$$\mathcal{K}^{j,\mathrm{out}}_{N_{p_1}+\epsilon}(Q_k,P,R) := \int_{(Q_k^{**})^c} |y - \mathbf{x}_{Q_k}|^{N_{p_1}+\epsilon} \big|T_{\sigma_j}\big(\Lambda_j a_k, \psi^P, \theta^R\big)(y)\big|\, dy.$$

Now, the left-hand side of Equation (7.16) is dominated by $\mathcal{J}^{in}(x) + \mathcal{J}^{out}(x)$

$$\begin{aligned}
\mathcal{J}^{\mathrm{in/out}}(x) := \sum_{k=0}^{\infty} |\lambda_k| \chi_{(Q_k^{***})^c}(x) \frac{1}{|x - \mathbf{x}_{Q_k}|^{n+N_{p_1}+\epsilon}} \\
\times \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \sum_{R\in\mathcal{D}_j} |b_P^2||b_R^3|\chi_{P\cap R}(x) \mathcal{K}^{j,\mathrm{in/out}}_{N_{p_1}+\epsilon}(Q_k,P,R).
\end{aligned}$$

To estimate $\mathcal{J}^{\mathrm{in}}$, we first see that

$$\begin{aligned}
&\mathcal{K}^{j,\mathrm{in}}_{N_{p_1}+\epsilon}(Q_k,P,R) \\
&\quad \lesssim \ell(Q_k)^{N_{p_1}+\epsilon} |P|^{-1/2}|R|^{-1/2} \int_{y\in Q_k^{**}} \int_{(\mathbb{R}^n)^3} \big|\sigma_j^{\vee}(y - z_1, z_2, z_3)\big| \big|\Lambda_j a_k(z_1)\big|\, d\vec{\boldsymbol{z}}dy \\
&\quad \lesssim \ell(Q_k)^{N_{p_1}+\epsilon} |P|^{-1/2}|R|^{-1/2} \|\Lambda_j a_k\|_{L^1(\mathbb{R}^n)},
\end{aligned}$$

using the Cauchy–Schwarz inequality with $s > 3n/2$, and thus

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$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j}&|b_P^2||b_R^3|\chi_{P\cap R}(x)\mathcal{K}_{N_{p_1}+\epsilon}^{j,\mathrm{in}}(Q_k,P,R)\\ &\lesssim \ell(Q_k)^{N_{p_1}+\epsilon}\big\|\big\{\Lambda_j a_k\big\}_{j\in\mathbb{Z}}\big\|_{L^1(\ell^2)}g^2\big(\big\{b_P^2\big\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\big\{b_R^3\big\}_{R\in\mathcal{D}}\big)(x)\\ &\lesssim |Q_k|^{-1/p_1}\ell(Q_k)^{n+N_{p_1}+\epsilon}g^2\big(\big\{b_P^2\big\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\big\{b_R^3\big\}_{R\in\mathcal{D}}\big)(x)\end{aligned}\tag{9.29}$$

by using the fact that

$$\big\|\big\{\Lambda_j a_k\big\}_{j\in\mathbb{Z}}\big\|_{L^1(\ell^2)}\sim\|a_k\|_{H^1(\mathbb{R}^n)}\lesssim\ell(Q_k)^{-n/p_1+n}.$$

For the other term $\mathcal{J}^{out}$, we choose s_1 such that

$$N_{p_1}+n/2+\epsilon<s_1<s-n,\tag{9.30}$$

which is possible due to Equation (9.28), and $s_2,s_3>n/2$ such that

$$s_1+n<s_1+s_2+s_3=s.\tag{9.31}$$

We observe that, for $y\in(Q_k^{**})^c$,

$$\begin{aligned}&\langle 2^j(y-\mathbf{x}_{Q_k})\rangle^{s_1}\big|\Lambda_j a_k(z_1)\big|\\ &\quad\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\langle 2^j(y-\mathbf{x}_{Q_k})\rangle^{s_1}\\ &\qquad\times\Big(\chi_{Q_k^*}(z_1)+\chi_{(Q_k^*)^c}(z_1)\frac{(2^j\ell(Q_k))^n}{\langle 2^j(z_1-\mathbf{x}_{Q_k})\rangle^{L_0}}\Big)\\ &\quad\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\langle 2^j(y-z_1)\rangle^{s_1}\\ &\qquad\times\Big(\chi_{Q_k^*}(z_1)+\chi_{(Q_k^*)^c}(z_1)\frac{(2^j\ell(Q_k))^n}{\langle 2^j(z_1-\mathbf{x}_{Q_k})\rangle^{L_0-s_1}}\Big),\end{aligned}$$

where Lemma 4.5 is applied in the first inequality. Here, M and L_0 are sufficiently large numbers such that $L_0-s_1>n$ and $M-L_0+3n/2>0$. By letting

$$A_{j,Q_k}(z_1):=\chi_{Q_k^*}(z_1)+\chi_{(Q_k^*)^c}(z_1)\frac{(2^j\ell(Q_k))^n}{\langle 2^j(z_1-\mathbf{x}_{Q_k})\rangle^{L_0-s_1}},\tag{9.32}$$

we have

$$\begin{aligned}&\big|T_{\sigma_j}\big(\Lambda_j a_k,\psi^P,\theta^R\big)(y)\big|\chi_{(Q_k^{**})^c}(y)\\ &\quad\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\frac{1}{\langle 2^j(y-\mathbf{x}_{Q_k})\rangle^{s_1}}|P|^{-1/2}|R|^{-1/2}\\ &\qquad\times\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_1)\rangle^{s_1}\big|\sigma_j^\vee(y-z_1,z_2,z_3)\big|A_{j,Q_k}(z_1)\,d\vec{\boldsymbol{z}}\end{aligned}\tag{9.33}$$

and the integral is, via the Cauchy-Schwarz inequality, less than

$$2^{-jn}\int_{\mathbb{R}^n}|A_{j,Q_k}(z_1)|\big\|\langle 2^j(y-z_1,z_2,z_3)\rangle^s\sigma_j^\vee(y-z_1,z_2,z_3)\big\|_{L^2(z_2,z_3)}\,dz_1.$$

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This deduces that

$$\begin{aligned}
&\mathcal{K}^{j,\mathrm{out}}_{N_{p_1}+\epsilon}(Q_k,P,R)\\
&\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}|P|^{-1/2}|R|^{-1/2}2^{-js_1}2^{-jn}\int_{\mathbb{R}^n}\big|A_{j,Q_k}(z_1)\big|\\
&\quad\times\Big(\int_{(Q_k^{**})^c}\frac{1}{|y-\mathbf{x}_{Q_k}|^{s_1-(N_{p_1}+\epsilon)}}\big\|\langle 2^j(y-z_1,z_2,z_3)\rangle^s\sigma_j^\vee(y-z_1,z_2,z_3)\big\|_{L^2(z_2,z_3)}\,dy\Big)\,dz_1\\
&\lesssim \ell(Q_k)^{-n/p_1}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}|P|^{-1/2}|R|^{-1/2}2^{-js_1}2^{-jn}\|A_{j,Q_k}\|_{L^1(\mathbb{R}^n)}\\
&\quad\times\Big\|\frac{1}{|\cdot-\mathbf{x}_{Q_k}|^{s_1-(N_{p_1}+\epsilon)}}\Big\|_{L^2((Q_k^{**})^c)}\big\|\langle 2^j\vec{\cdot}\rangle^s\sigma_j^\vee\big\|_{L^2((\mathbb{R}^n)^3)}\\
&\lesssim \ell(Q_k)^{-n/p_1+n+N_{p_1}+\epsilon}\big(2^j\ell(Q_k)\big)^{-(s_1-n/2)}\min\big\{1,\big(2^j\ell(Q_k)\big)^{M-L_0+s_1+n}\big\}|P|^{-1/2}|R|^{-1/2}
\end{aligned}$$

since

$$\|A_{j,Q_k}\|_{L^1(\mathbb{R}^n)}\lesssim \ell(Q_k)^n\Big(1+(2^j\ell(Q_k))^{-(L_0-s_1-n)}\Big)\quad \text{for}\ \ L_0-s_1>n.$$

Therefore, by using the Cauchy–Schwarz inequality

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j}|b_P^2||b_R^3|\chi_{P\cap R}(x)\mathcal{K}^{j,\mathrm{out}}_{N_{p_1}+\epsilon}(Q_k,P,R)\\
&\quad\lesssim |Q_k|^{-1/p_1}\ell(Q_k)^{n+N_{p_1}+\epsilon}g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x)\\
&\qquad\times\Big(\sum_{j\in\mathbb{Z}}\Big(\big(2^j\ell(Q_k)\big)^{-(s_1-n/2)}\min\big\{1,\big(2^j\ell(Q_k)\big)^{M-L_0+s_1+n}\big\}\Big)^2\Big)^{1/2}\\
&\quad\lesssim |Q_k|^{-1/p_1}\ell(Q_k)^{n+N_{p_1}+\epsilon}g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x)g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x),
\end{aligned}\tag{9.34}$$

where the last inequality holds due to $s_1>n/2$ and $M-L_0+3n/2>0$.

In conclusion, the estimate (7.16) can be derived from Equations (9.29) and (9.34), using the choices of

$$\begin{aligned}
u_1(x)&:=\sum_{k=0}^\infty|\lambda_k||Q_k|^{-1/p_1}\chi_{(Q^{**})^c}\frac{\ell(Q_k)^{n+N_{p_1}+\epsilon}}{|x-\mathbf{x}_{Q_k}|^{n+N_{p_1}+\epsilon}},\\
u_2(x)&:=g^2\big(\{b_P^2\}_{P\in\mathcal{D}}\big)(x),\\
u_3(x)&:=g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x).
\end{aligned}$$

It is obvious from Equations (7.1) and (7.2) that $\|u_2\|_{L^{p_2}(\mathbb{R}^n)},\|u_3\|_{L^{p_3}(\mathbb{R}^n)}\lesssim 1$. Furthermore,

$$\begin{aligned}
\|u_1\|_{L^{p_1}(\mathbb{R}^n)}&\lesssim\Big(\sum_{k=0}^\infty|\lambda_k|^{p_1}|Q_k|^{-1}\int_{(Q_k^{***})^c}\frac{\ell(Q_k)^{(n+N_{p_1}+\epsilon)p_1}}{|x-\mathbf{x}_{Q_k}|^{(n+N_{p_1}+\epsilon)p_1}}dx\Big)^{1/p_1}\\
&\lesssim\Big(\sum_{k=0}^\infty|\lambda_k|^{p_1}\Big)^{1/p_1}\lesssim 1.
\end{aligned}\tag{9.35}$$

This completes the proof.

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9.7. Proof of Lemma 7.6

Choose s_1, s_2, and s_3 such that $s_1 > n/p_1 - n/2$, $s_2 > n/2$, $s_3 > n/2$ and $s = s_1 + s_2 + s_3$. For $x \in B_k^l \cap (B_P^l)^c$ and $|x-y| \le 2^{-l}$, we have

$$|x - \mathbf{x}_{Q_k}| \le |x - \mathbf{x}_P| \lesssim |y - \mathbf{x}_P|. \tag{9.36}$$

This implies

$$\begin{aligned}
&\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1}\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}\left|\phi_l * T_{\sigma_j}\left(\Lambda_j a_k, \psi^P, \theta^R\right)(x)\right| \\
&\lesssim |R|^{-1/2}2^{ln}\int_{|x-y|\le 2^{-l}}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-\mathbf{x}_P)\rangle^{s_1+s_2} \\
&\quad \times \left|\sigma_j^\vee(y-z_1, y-z_2, z_3)\right|\left|\Lambda_j a_k(z_1)\right|\left|\psi^P(z_2)\right| \, d\vec{\boldsymbol{z}}\, dy \\
&\le |R|^{-1/2}2^{ln}\int_{|x-y|\le 2^{-l}}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_2)\rangle^{s_1+s_2} \\
&\quad \times \left|\sigma_j^\vee(y-z_1, y-z_2, z_3)\right|\left|\Lambda_j a_k(z_1)\right|\left|\widetilde{\psi^P}(z_2)\right| \, d\vec{\boldsymbol{z}}\, dy,
\end{aligned}$$

where

$$\widetilde{\psi^P}(z_2) := \langle 2^j(z_2 - \mathbf{x}_P)\rangle^{s_1+s_2}\psi^P(z_2).$$

By using the Cauchy–Schwarz inequality and Lemma 4.2, we obtain

$$\begin{aligned}
&\left|\phi_l * \left(\mathcal{U}_2^1(x,\cdot)\right)(x)\right| \\
&\lesssim \sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\sum_{R\in\mathcal{D}_j}|b_P^2||b_R^3|\chi_{P^c}(x)\chi_R(x)\frac{1}{\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1}}\frac{1}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}|R|^{-1/2} \\
&\quad \times 2^{ln}\int_{|x-y|\le 2^{-l}}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_2)\rangle^{s_1+s_2}\left|\sigma_j^\vee(y-z_1, y-z_2, z_3)\right|\left|\Lambda_j a_k(z_1)\right| \, d\vec{\boldsymbol{z}}dy \\
&\lesssim g^\infty\left(\{b_R^3\}_{R\in\mathcal{D}}\right)(x)\sum_{j\in\mathbb{Z}}\frac{1}{\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1}}2^{ln}\int_{|x-y|\le 2^{-l}}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_2)\rangle^{s_1+s_2} \\
&\quad \times \left|\sigma_j^\vee(y-z_1, y-z_2, z_3)\right|\left|\Lambda_j a_k(z_1)\right|\Bigg(\sum_{P\in\mathcal{D}_j}|b_P^2|\frac{\chi_{(P)^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\left|\widetilde{\psi^P}(z_3)\right|\Bigg) \, d\vec{\boldsymbol{z}}dy \\
&\lesssim g^\infty\left(\{b_R^3\}_{R\in\mathcal{D}}\right)(x)\frac{1}{|x-\mathbf{x}_{Q_k}|^{s_1}}\Bigg(\sum_{j\in\mathbb{Z}}\left(2^{-js_1}\mathcal{M}J_{k,j,s}^1(x)\right)^2\Bigg)^{1/2} \\
&\quad \times \Bigg(\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\Bigg(\left|\mathcal{B}_P^2(f_2)\right||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Bigg)^2\Bigg)^{1/2},
\end{aligned} \tag{9.37}$$

where $\mathcal{B}_P^2(f_2)$ is defined as in Equation (9.19) for some $L > n, s_2$, and

$$J_{k,j,s}^1(y) := 2^{-jn}\int_{\mathbb{R}^n}\left|\Lambda_j a_k(z_1)\right|\left\|\langle 2^j(y-z_1,z_2,z_3)\rangle^s\sigma_j^\vee(y-z_1,z_2,z_3)\right\|_{L^2(z_2,z_3)}dz_1. \tag{9.38}$$

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Now, we choose

$$u_1(x) := \sum_{k=0}^{\infty} |\lambda_k| \chi_{(Q_k^{***})^c}(x) \frac{1}{|x-\mathbf{x}_{Q_k}|^{s_1}} \Big(\sum_{j\in\mathbb{Z}} \big(2^{-js_1}\mathcal{M}J^1_{k,j,s}(x)\big)^2 \Big)^{1/2},$$

$$u_2(x) := \Big(\sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2} \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}} \Big)^2 \Big)^{1/2},$$

$$u_3(x) := g^{\infty}\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x).$$

Clearly, Equation (7.18) holds and $\|u_2\|_{L^{p_2}(\mathbb{R}^n)}, \|u_3\|_{L^{p_3}(\mathbb{R}^n)} \lesssim 1$ due to Lemma 4.3 and Equation (7.2). In addition,

$$\|u_1\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} |\lambda_k|^{p_1} \int_{(Q_k^{***})^c} |x-\mathbf{x}_{Q_k}|^{-s_1p_1} \Big(\sum_{j\in\mathbb{Z}} \big(2^{-s_1j}\mathcal{M}J^1_{k,j,s}(x)\big)^2 \Big)^{p_1/2} dx$$

and the integral is controlled by

$$\begin{aligned} &\big\| |\cdot - \mathbf{x}_{Q_k}|^{-s_1p_1} \big\|_{L^{(2/p_1)'}((Q_k^{***})^c)} \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{-s_1j}\mathcal{M}J^1_{k,j,s}(x)\big)^2 \Big)^{p_1/2} \Big\|_{L^{2/p_1}(\mathbb{R}^n)} \\ &\lesssim \ell(Q_k)^{-p_1(s_1-(n/p_1-n/2))} \Big(\sum_{j\in\mathbb{Z}} 2^{-2js_1} \|J^1_{k,j,s}\|^2_{L^2(\mathbb{R}^n)} \Big)^{p_1/2} \end{aligned}$$

by using Hölder's inequality and the L^2 boundedness of $\mathcal{M}$. It follows from Minkowski's inequality and Lemma 4.5 that

$$\begin{aligned} \|J^1_{k,j,s}\|_{L^2(\mathbb{R}^n)} &\lesssim 2^{-jn} \|\Lambda_j a_k\|_{L^1(\mathbb{R}^n)} \Big\| \langle 2^j \vec{\cdot}\rangle^s |\sigma_j^\vee| \Big\|_{L^2((\mathbb{R}^n)^3)} \\ &\lesssim \ell(Q_k)^{-n/p_1+n} 2^{jn/2} \min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\}, \end{aligned}$$

and this finally yields that

$$\|u_1\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{p_1} \Big)^{1/p_1} \lesssim 1. \tag{9.39}$$

9.8. Proof of Lemma 7.7

For $x \in B_k^l \cap B_P^l$,

$$2^l \lesssim |x-\mathbf{x}_{Q_k}|^{-1}, |x-\mathbf{x}_P|^{-1}. \tag{9.40}$$

Since $s > n/p_1 + n/2$, there exist $0 < \epsilon_0, \epsilon_1 < 1$ such that

$$n/p_1 + n/p_2 < \big[n/p_1 + n/p_2\big] + \epsilon_0 \quad \text{and} \quad \big[n/p_1 + n/p_2\big] + \epsilon_0 + \epsilon_1 < s - \big(n/2 - n/p_2\big).$$

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Choose t_1 and t_2 satisfying $t_1 > n/p_1$, $t_2 > n/p_2$ and $t_1 + t_2 = [n/p_1 + n/p_2] + \epsilon_0$, and let $N_0 := [n/p_1 + n/p_2] - n$. Then Lemma 4.1, together with the vanishing moment condition (5.3), and the estimate (9.40) yield that

$$\begin{aligned}
&\left|\phi_l * T_{\sigma_j}\left(\Lambda_j a_k, \psi^P, \theta^R\right)(x)\right| \\
&\lesssim 2^{l(N_0+n+\epsilon_0)} \int_{\mathbb{R}^n} |y - \mathbf{x}_P|^{N_0+\epsilon_0} \left|T_{\sigma_j}\left(\Lambda_j a_k, \psi^P, \theta^R\right)(y)\right| dy \\
&\lesssim |R|^{-1/2} \frac{1}{|x - \mathbf{x}_{Q_k}|^{t_1}} 2^{-j(t_1-n)} \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^3} \langle 2^j(y - z_2)\rangle^{N_0+\epsilon_0} \\
&\quad \times \left|\sigma_j^\vee(y - z_1, y - z_2, z_3)\right| \left|\Lambda_j a_k(z_1)\right| \frac{|\widetilde{\psi^P}(z_2)|}{\langle 2^j(x - \mathbf{x}_P)\rangle^{t_2}} \, d\vec{\boldsymbol{z}} dy,
\end{aligned}$$

where

$$\widetilde{\psi^P}(z_2) := \langle z_2 - \mathbf{x}_P\rangle^{N_0+\epsilon_0} \psi^P(z_2).$$

This deduces

$$\begin{aligned}
&\left|\phi_l * \left(\mathcal{U}_2^2(x,\cdot)\right)(x)\right| \\
&\lesssim g^\infty\left(\{b_R^3\}_{R\in\mathcal{D}}\right)(x) \frac{1}{|x - \mathbf{x}_{Q_k}|^{t_1}} \sum_{j\in\mathbb{Z}} 2^{-j(t_1-n)} \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^3} \langle 2^j(y - z_2)\rangle^{N_0+\epsilon_0} \\
&\quad \times \left|\sigma_j(y - z_1, y - z_2, z_3)\right| \left|\Lambda_j a_k(z_1)\right| \Big(\sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x - \mathbf{x}_P)\rangle^{t_2}} |\widetilde{\psi^P}(z_2)| \Big) \, d\vec{\boldsymbol{z}} dy. \qquad (9.41)
\end{aligned}$$

Using Hölder's inequality with $\frac{1}{2} + \frac{1}{(1/p_2' - 1/2)^{-1}} + \frac{1}{p_2} = 1$ and Lemma 4.2, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \langle 2^j(y - z_2)\rangle^{N_0+\epsilon_0} \left|\sigma_j(y - z_1, y - z_2, z_3)\right| \Big(\sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x - \mathbf{x}_P)\rangle^{t_2}} |\widetilde{\psi^P}(z_2)| \Big) \, dz_2 \\
&\le \left\|\langle 2^j z_2\rangle^{s-n-\epsilon_1} \sigma_j^\vee(y - z_1, z_2, z_3)\right\|_{L^2(z_2)} \left\|\langle 2^j \cdot\rangle^{-(s-t_1-t_2-\epsilon_1)}\right\|_{L^{(1/p_2'-1/2)^{-1}}(\mathbb{R}^n)} \\
&\quad \times \Big\| \sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x - \mathbf{x}_P)\rangle^{t_2}} |\widetilde{\psi^P}(z_2)| \Big\|_{L^{p_2}(\mathbb{R}^n)} \\
&\lesssim 2^{-\frac{jn}{2}} \left\|\langle 2^j z_2\rangle^{s-n-\epsilon_1} \sigma_j^\vee(y - z_1, z_2, z_3)\right\|_{L^2(z_2)} \\
&\quad \times \Big(\sum_{P\in\mathcal{D}_j} \Big(|\mathcal{B}_P^2(f_2)| |P|^{-1/2} \frac{\chi_{P^c}(x)}{\langle 2^j(x - \mathbf{x}_P)\rangle^{t_2}} \Big)^{p_2} \Big)^{1/p_2}
\end{aligned}$$

because $s - n - \epsilon_1 = s - t_1 - t_2 - \epsilon_1 + N_0 + \epsilon_0$, $s - t_1 - t_2 - \epsilon_1 > n(1/p_2' - 1/2)$. This shows that the integral in the right-hand side of Equation (9.41) is dominated by a constant times

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$$\|\Lambda_j a_k\|_{L^1(\mathbb{R}^n)}\Big(\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\Big)^{p_2}\Big)^{1/p_2}$$
$$\times 2^{-jn/2}\int_{(\mathbb{R}^n)^2}\big\|\langle 2^j z_2\rangle^{s-n-\epsilon_1}\sigma_j^\vee(y,z_2,z_3)\big\|_{L^2(z_2)}\,dy\,dz_3$$
$$\lesssim \ell(Q_k)^{-n/p_1+n}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\Big(\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\Big)^{p_2}\Big)^{1/p_2},$$

where $\mathcal{B}_P^2(f_2)$ is defined as in Equation (9.19) and M is sufficiently large. Consequently,

$$\begin{aligned}
&\big|\phi_l*\big(\mathcal{U}_2^2(x,\cdot)\big)(x)\big|\\
&\lesssim g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x)\sup_{j\in\mathbb{Z}}\Big(\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\Big)^{p_2}\Big)^{1/p_2}\\
&\quad\times\frac{1}{|x-\mathbf{x}_{Q_k}|^{t_1}}\ell(Q_k)^{-n/p_1+n}\sum_{j\in\mathbb{Z}}2^{-j(t_1-n)}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\\
&\lesssim |Q_k|^{-1/p_1}\frac{\ell(Q_k)^{t_1}}{|x-\mathbf{x}_{Q_k}|^{t_1}}\Big(\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\Big)^{p_2}\Big)^{1/p_2}\\
&\quad\times g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x).
\end{aligned}\tag{9.42}$$

Now, we are done with

$$\begin{aligned}
u_1(x)&:=\sum_{k=0}^\infty|\lambda_k||Q_k|^{-1/p_1}\frac{\ell(Q_k)^{t_1}}{|x-\mathbf{x}_{Q_k}|^{t_1}}\chi_{(Q_k^{***})^c}(x)\\
u_2(x)&:=\Big(\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\Big)^{p_2}\Big)^{1/p_2}\\
u_3(x)&:=g^\infty\big(\{b_R^3\}_{R\in\mathcal{D}}\big)(x)
\end{aligned}$$

as $\|u_{\mathrm{i}}\|_{L^{p_{\mathrm{i}}}(\mathbb{R}^n)}\lesssim 1$, i $=1,2,3$, follow from Lemma 4.3, Equation (7.2) and the argument that led to (9.35) with $t_1>n/p_1$.

9.9. Proof of Lemma 7.8

Let s_1, s_2 and s_3 satisfy $s_1>n/p_1-n/2$, $s_2>n/2$, $s_3>n/2$ and $s=s_1+s_2+s_3$. By mimicking the argument that led to Equation (9.37) with

$$|x-\mathbf{x}_{Q_k}|\le|x-\mathbf{x}_R|\lesssim|y-\mathbf{x}_R|$$

for $x\in B_k^l\cap(B_R^l)^c$ and $|x-y|\le 2^{-l}$, instead of Equation (9.36), we can prove

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$$\left|\phi_l*\left(\mathcal{U}_3^1(x,\cdot)\right)(x)\right|\lesssim\frac{1}{|x-\mathbf{x}_{Q_k}|^{s_1}}\Big(\sum_{j\in\mathbb{Z}}\left(2^{-js_1}\mathcal{M}J_{k,j,s}^1(x)\right)^2\Big)^{1/2}g^2\left(\left\{b_P^2\right\}_{P\in\mathcal{D}}\right)(x)$$
$$\times\sup_{j\in\mathbb{Z}}\Bigg(\sum_{R\in\mathcal{D}_j}\left(\left|\mathcal{B}_R^3(f_3)\right||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\right)^2\Bigg)^{1/2},$$

where $J_{k,j,s}^1$ and $\mathcal{B}_R^3(f_3)$ are defined as in Equations (9.38) and (9.26) for some $L>n,s_3$.
Now, let

$$u_1(x):=\sum_{k=0}^{\infty}|\lambda_k|\chi_{(Q_k^{***})^c}(x)\frac{1}{|x-\mathbf{x}_{Q_k}|^{s_1}}\Big(\sum_{j\in\mathbb{Z}}\left(2^{-js_1}\mathcal{M}J_{k,j,s}^1(x)\right)^2\Big)^{1/2}$$
$$u_2(x):=g^2\left(\left\{b_P^2\right\}_{P\in\mathcal{D}}\right)(x)$$
$$u_3(x):=\sup_{j\in\mathbb{Z}}\Bigg(\sum_{P\in\mathcal{D}_j}\left(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\right)^2\Bigg)^{1/2}.$$

Then the estimate (7.21) is clear and it follows from Equations (9.39) and (7.1) and Lemma 4.4 that Equation (7.20) holds.

9.10. Proof of Lemma 7.9

Let $0<\epsilon_0,\epsilon_1<1$ satisfy

$$n/p_1+n/p_3<\left[n/p_1+n/p_3\right]+\epsilon_0\quad\text{ and }\quad\left[n/p_1+n/p_3\right]+\epsilon_0+\epsilon_1<s-\left(n/2-n/p_3\right),$$

and select t_1,t_3 so that $t_1>n/p_1$, $t_3>n/p_3$ and $t_1+t_3=\left[n/p_1+n/p_3\right]+\epsilon_0$. Let $N_0:=\left[n/p_1+n/p_3\right]-n$ and $B_R^3(f_3)$ be defined as in Equation (9.26). Then, as the counterpart of Equation (9.42), we can get

$$\left|\phi_l*\left(\mathcal{U}_3^2(x,\cdot)\right)(x)\right|\lesssim|Q_k|^{-1/p_1}\frac{\ell(Q_k)^{t_1}}{|x-\mathbf{x}_{Q_k}|^{t_1}}g^2\left(\left\{b_P^2\right\}_{P\in\mathcal{D}}\right)(x)$$
$$\times\sup_{j\in\mathbb{Z}}\Bigg(\sum_{R\in\mathcal{D}_j}\left(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}}\right)^{p_3}\Bigg)^{1/p_3},$$

where the embedding $\ell^2\hookrightarrow\ell^\infty$ is applied. By taking

$$u_1(x):=\sum_{k=0}^{\infty}|\lambda_k||Q_k|^{-1/p_1}\frac{\ell(Q_k)^{t_1}}{|x-\mathbf{x}_{Q_k}|^{t_1}}\chi_{(Q_k^{***})^c}(x)$$
$$u_2(x):=g^2\left(\left\{b_P^2\right\}_{P\in\mathcal{D}}\right)(x),$$
$$u_3(x):=\sup_{j\in\mathbb{Z}}\Bigg(\sum_{R\in\mathcal{D}_j}\left(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}}\right)^{p_3}\Bigg)^{1/p_3},$$

we obtain the inequality (7.22) and Equation (7.23).

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9.11. Proof of Lemma 7.10

The proof is almost same as that of Lemmas 7.6 and 7.8. Let s_1, s_2 and s_3 be numbers such that $s_1 > n/p_1 - n/2$, $s_2 > n/2$, $s_3 > n/2$ and $s = s_1 + s_2 + s_3$. We claim that for $\eta = 1,2,3$,

$$\begin{aligned}\left|\phi_l * \left(\mathcal{U}_4^\eta(x,\cdot)\right)(x)\right| &\lesssim \frac{1}{|x-\mathbf{x}_{Q_k}|^{s_1}}\Big(\sum_{j\in\mathbb{Z}}\left(2^{-js_1}\mathcal{M}J^1_{k,j,s}(x)\right)^2\Big)^{1/2} \\ &\quad\times\Big(\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Big)^2\Big)^{1/2} \\ &\quad\times\sup_{j\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{D}_j}\Big(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\Big)^2\Big)^{1/2},\end{aligned} \tag{9.43}$$

where $J^1_{k,j,s}$, $\mathcal{B}_P^2(f_2)$ and $\mathcal{B}_R^3(f_3)$ are defined as in Equations (9.38), (9.19) and (9.26), respectively. Then we have Equation (7.25) with the choice

$$\begin{aligned} u_1(x) &:= \sum_{k=0}^{\infty}|\lambda_k|\chi_{(Q_k^{***})^c}(x)\frac{1}{|x-\mathbf{x}_{Q_k}|^{s_1}}\Big(\sum_{j\in\mathbb{Z}}\left(2^{-js_1}\mathcal{M}J^1_{k,j,s}(x)\right)^2\Big)^{1/2}, \\ u_2(x) &:= \Big(\sum_{j\in\mathbb{Z}}\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}}\Big)^2\Big)^{1/2}, \\ u_3(x) &:= \sup_{j\in\mathbb{Z}}\Big(\sum_{P\in\mathcal{D}_j}\Big(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}}\Big)^2\Big)^{1/2}. \end{aligned}$$

The estimates for u_1, u_2, u_3 follow from Equation (9.39), Lemma 4.3 and Lemma 4.4.

Now, we return to the proof of Equation (9.43). For $x \in B_k^l \cap (B_P^l)^c \cap (B_R^l)^c$ and $|x-y| \le 2^{-l}$, we have

$$|x-\mathbf{x}_{Q_k}| \le |x-\mathbf{x}_P| \lesssim |y-\mathbf{x}_P| \quad \text{and} \quad |x-\mathbf{x}_R| \lesssim |y-\mathbf{x}_R|. \tag{9.44}$$

Then we have

$$\begin{aligned}&\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1}\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}\left|\phi_l * T_{\sigma_j}\left(\Lambda_j a_k,\psi^P,\theta^R\right)(x)\right| \\ &\lesssim 2^{ln}\int_{|x-y|\le 2^{-l}}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_2)\rangle^{s_1+s_2}\langle 2^j(y-z_3)\rangle^{s_3}\left|\sigma_j^\vee(y-z_1,y-z_2,y-z_3)\right| \\ &\quad\times|\Lambda_j a_k(z_1)|\left|\widetilde{\psi^P}(z_2)\right|\left|\widetilde{\theta^R}(z_3)\right|\, d\vec{z}dy,\end{aligned}$$

where

$$\begin{aligned}\widetilde{\psi^P}(z_2) &:= \langle 2^j(z_2-\mathbf{x}_P)\rangle^{s_1+s_2}\psi^P(z_2), \\ \widetilde{\theta^R}(z_3) &:= \langle 2^j(z_3-\mathbf{x}_R)\rangle^{s_3}\theta^R(z_3).\end{aligned}$$

Now, using the method similar to that used in the proof of Equation (9.37), we obtain Equation (9.43) for $\eta = 1$.

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For the case $\eta = 2$, we use the fact, instead of Equation (9.44), that for $x \in B_k^l \cap (B_P^l)^c \cap B_R^l$ and $|x-y| \le 2^{-l}$,

$$|x - \mathbf{x}_{Q_k}|, |x - \mathbf{x}_R| \le |x - \mathbf{x}_P| \lesssim |y - \mathbf{x}_P|.$$

This shows that

$$\begin{aligned}&\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1}\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}\left|\phi_l * T_{\sigma_j}\left(\Lambda_j a_k,\psi^P,\theta^R\right)(x)\right|\\&\lesssim 2^{ln}\int_{|x-y|\le 2^{-l}}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_2)\rangle^s\left|\sigma_j^\vee(y-z_1,y-z_2,y-z_3)\right|\\&\quad\times|\Lambda_j a_k(z_1)|\left|\widetilde{\psi^P}(z_2)\right|\left|\theta^R(z_3)\right|\,d\vec{\boldsymbol{z}}dy,\end{aligned}$$

where

$$\widetilde{\psi^P}(z_2) := \langle 2^j(z_2 - \mathbf{x}_P)\rangle^s \psi^P(z_2),$$

and then Equation (9.43) for $\eta = 2$ follows.

Similarly, we can prove that for $x \in B_k^l \cap B_P^l \cap (B_R^l)^c$ and $|x-y| \le 2^{-l}$,

$$\begin{aligned}&\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1}\langle 2^j(x-\mathbf{x}_P)\rangle^{s_2}\langle 2^j(x-\mathbf{x}_R)\rangle^{s_3}\left|\phi_l * T_{\sigma_j}\left(\Lambda_j a_k,\psi^P,\theta^R\right)(x)\right|\\&\lesssim 2^{ln}\int_{|x-y|\le 2^{-l}}\int_{(\mathbb{R}^n)^3}\langle 2^j(y-z_3)\rangle^s\left|\sigma_j^\vee(y-z_1,y-z_2,y-z_3)\right|\\&\quad\times|\Lambda_j a_k(z_1)|\left|\psi^P(z_2)\right|\left|\widetilde{\theta^R}(z_3)\right|\,d\vec{\boldsymbol{z}}dy,\end{aligned}$$

where

$$\widetilde{\theta^R}(z_3) := \langle 2^j(z_3 - \mathbf{x}_R)\rangle^s \theta^R(z_3).$$

This proves (9.43) for $\eta = 3$.

9.12. Proof of Lemma 7.11

We first note that

$$2^l \lesssim |x - \mathbf{x}_{Q_k}|^{-1}, |x - \mathbf{x}_P|^{-1}, |x - \mathbf{x}_R|^{-1} \tag{9.45}$$

for $x \in B_k^l \cap B_P^l \cap B_R^l$. Since $n/p < s - \left(n/2 - n/p_2 - n/p_3\right)$, there exist $0 < \epsilon_0, \epsilon_1 < 1$ such that

$$n/p < \left[n/p\right] + \epsilon_0 \quad \text{ and } \quad \left[n/p\right] + \epsilon_0 + \epsilon_1 < s - \left(n/2 - n/p_2 - n/p_3\right).$$

Choose t_1, t_2, and t_3 satisfying $t_1 > n/p_1$, $t_2 > n/p_2$, $t_3 > n/p_3$, and $t_1 + t_2 + t_3 = \left[n/p\right] + \epsilon_0$ and let $N_0 := \left[n/p\right] - n$. Then it follows from Lemma 4.1 and the estimate (9.45) that

$$\begin{aligned}&\left|\phi_l * T_{\sigma_j}\left(\Lambda_j a_k,\psi^P,\theta^R\right)(x)\right|\\&\lesssim 2^{l(N_0+n+\epsilon_0)}\int_{\mathbb{R}^n}|y-\mathbf{x}_P|^{N_0+\epsilon_0}\left|T_{\sigma_j}\left(\Lambda_j a_k,\psi^P,\theta^R\right)(y)\right|dy\end{aligned}$$

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$$\lesssim \frac{1}{|x-\mathbf{x}_{Q_k}|^{t_1}} 2^{-j(t_1-n)} \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^3} \langle 2^j(y-z_2)\rangle^{N_0+\epsilon_0} \big|\sigma_j^{\vee}(y-z_1,y-z_2,y-z_3)\big| \\ \times \big|\Lambda_j a_k(z_1)\big| \frac{|\widetilde{\psi^P}(z_2)|}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}} \frac{|\theta^R(z_3)|}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}}\, d\vec{\boldsymbol{z}}dy,$$

where

$$\widetilde{\psi^P}(z_2) := \langle z_2-\mathbf{x}_P\rangle^{N_0+\epsilon_0}\psi^P(z_2).$$

This deduces that

$$\big|\phi_l * \big(\mathcal{U}_4^4(x,\cdot)\big)(x)\big| \\ \lesssim \frac{1}{|x-\mathbf{x}_{Q_k}|^{t_1}} \sum_{j\in\mathbb{Z}} 2^{-j(t_1-n)} \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^3} \langle 2^j(y-z_2)\rangle^{N_0+\epsilon_0} \big|\sigma_j(y-z_1,y-z_2,z_3)\big| \big|\Lambda_j a_k(z_1)\big| \\ \times \Bigg(\sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}} \big|\widetilde{\psi^P}(z_2)\big|\Bigg) \Bigg(\sum_{R\in\mathcal{D}_j} |b_R^3| \frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}} \big|\theta^R(z_3)\big|\Bigg)\, d\vec{\boldsymbol{z}}dy. \tag{9.46}$$

Since $s-[n/p]+n/2-\epsilon_0-\epsilon_1 > (n/2-n/p_2)+(n/2-n/p_3)$, there exist μ_2 and μ_3 such that $\mu_2 > n/2-n/p_2$, $\mu_3 > n/2-n/p_3$, and $\mu_1+\mu_2 = s-[n/p]+n/2-\epsilon_0-\epsilon_1$. Using Hölder's inequality with

$$\frac{1}{2}+\frac{1}{(1/p_2'-1/2)^{-1}}+\frac{1}{p_2} = \frac{1}{2}+\frac{1}{(1/p_3'-1/2)^{-1}}+\frac{1}{p_3} = 1,$$

we have

$$\int_{(\mathbb{R}^n)^2} \langle 2^j(y-z_2)\rangle^{N_0+\epsilon_0} \big|\sigma_j(y-z_1,y-z_2,y-z_3)\big| \\ \times \Bigg(\sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}} \big|\widetilde{\psi^P}(z_2)\big|\Bigg) \Bigg(\sum_{R\in\mathcal{D}_j} |b_R^3| \frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}} \big|\theta^R(z_3)\big|\Bigg)\, dz_2dz_3 \\ \le \big\|\langle 2^j z_2\rangle^{N_0+\epsilon_0+\mu_2}\langle 2^j z_3\rangle^{\mu_3}\sigma_j^{\vee}(y-z_1,z_2,z_3)\big\|_{L^2(z_2,z_3)} \big\|\langle 2^j\cdot\rangle^{-\mu_2}\big\|_{L^{(1/p_2'-1/2)^{-1}}(\mathbb{R}^n)} \\ \times \big\|\langle 2^j\cdot\rangle^{-\mu_3}\big\|_{L^{(1/p_3'-1/2)^{-1}}(\mathbb{R}^n)} \Bigg\| \sum_{P\in\mathcal{D}_j} |b_P^2| \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}} \big|\widetilde{\psi^P}(z_2)\big| \Bigg\|_{L^{p_2}(z_2)} \\ \times \Bigg\| \sum_{R\in\mathcal{D}_j} |b_R^3| \frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}} \big|\theta^R(z_3)\big| \Bigg\|_{L^{p_2}(z_3)},$$

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and then Lemma 4.2 yields that the preceding expression is less than a constant times

$$2^{-jn}\left\|\langle 2^j(z_2,z_3)\rangle^{N_0+\epsilon_0+\mu_1+\mu_2}\sigma_j^{\vee}(y-z_1,z_2,z_3)\right\|_{L^2(z_2,z_3)}$$
$$\times\left(\sum_{P\in\mathcal{D}_j}\left(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\right)^{p_2}\right)^{1/p_2}$$
$$\times\left(\sum_{R\in\mathcal{D}_j}\left(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}}\right)^{p_3}\right)^{1/p_3}$$

because $\mu_2>n(1/p_2'-1/2)$ and $\mu_3>n(1/p_3'-1/2)$, where $\mathcal{B}_P^2(f_2)$ and $\mathcal{B}_R(f_3)$ are defined as in Equations (9.19) and (9.26).

Now, the integral in the right-hand side of Equation (9.46) is dominated by a constant times

$$2^{-jn}\|\Lambda_j a_k\|_{L^1(\mathbb{R}^n)}\int_{\mathbb{R}^n}\left\|\langle 2^j(z_2,z_3)\rangle^{s-n/2-\epsilon_1}\sigma_j^{\vee}(y,z_2,z_3)\right\|_{L^2(z_2,z_3)}dy$$
$$\times\left(\sum_{P\in\mathcal{D}_j}\left(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\right)^{p_2}\right)^{1/p_2}$$
$$\times\left(\sum_{R\in\mathcal{D}_j}\left(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}}\right)^{p_3}\right)^{1/p_3},$$

and this is no more than

$$\ell(Q_k)^{-n/p_1+n}\min\left\{1,\left(2^j\ell(Q_k)\right)^M\right\}$$
$$\times\left(\sum_{P\in\mathcal{D}_j}\left(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\right)^{p_2}\right)^{1/p_2}$$
$$\times\left(\sum_{R\in\mathcal{D}_j}\left(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}}\right)^{p_3}\right)^{1/p_3},$$

where $N_0+\epsilon_0+\mu_2+\mu_3=s-\frac{n}{2}-\epsilon_1$. Hence, it follows that

$$\left|\phi_l*\left(\mathcal{U}_4^4(x,\cdot)\right)(x)\right|$$
$$\lesssim\frac{1}{|x-\mathbf{x}_{Q_k}|^{t_1}}\ell(Q_k)^{-n/p_1+n}\sum_{j\in\mathbb{Z}}2^{-j(t_1-n)}\min\left\{1,\left(2^j\ell(Q_k)\right)^M\right\}$$
$$\times\sup_{j\in\mathbb{Z}}\left(\sum_{P\in\mathcal{D}_j}\left(|\mathcal{B}_P^2(f_2)||P|^{-1/2}\frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}}\right)^{p_2}\right)^{1/p_2}$$
$$\times\sup_{j\in\mathbb{Z}}\left(\sum_{R\in\mathcal{D}_j}\left(|\mathcal{B}_R^3(f_3)||R|^{-1/2}\frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}}\right)^{p_3}\right)^{1/p_3}$$

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$$\lesssim |Q_k|^{-1/p_1} \frac{\ell(Q_k)^{t_1}}{|x-\mathbf{x}_{Q_k}|^{t_1}} \bigg(\sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2} \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}} \Big)^{p_2} \bigg)^{1/p_2}$$
$$\times \sup_{j\in\mathbb{Z}} \bigg(\sum_{R\in\mathcal{D}_j} \Big(|\mathcal{B}_R^3(f_3)||R|^{-1/2} \frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}} \Big)^{p_3} \bigg)^{1/p_3}.$$

Now, let

$$u_1(x) := \sum_{k=0}^{\infty} |\lambda_k||Q_k|^{-1/p_1} \frac{\ell(Q_k)^{t_1}}{|x-\mathbf{x}_{Q_k}|^{t_1}} \chi_{(Q_k^{***})^c}(x),$$
$$u_2(x) := \bigg(\sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{D}_j} \Big(|\mathcal{B}_P^2(f_2)||P|^{-1/2} \frac{\chi_{P^c}(x)}{\langle 2^j(x-\mathbf{x}_P)\rangle^{t_2}} \Big)^{p_2} \bigg)^{1/p_2},$$
$$u_3(x) := \sup_{j\in\mathbb{Z}} \bigg(\sum_{R\in\mathcal{D}_j} \Big(|\mathcal{B}_R^3(f_3)||R|^{-1/2} \frac{\chi_{R^c}(x)}{\langle 2^j(x-\mathbf{x}_R)\rangle^{t_3}} \Big)^{p_3} \bigg)^{1/p_3}.$$

Then it is easy to prove Equations (7.26) and (7.27).

9.13. Proof of Lemma 8.1

Using the fact that $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\vec{\boldsymbol{\xi}}) = 1$ for $\vec{\boldsymbol{\xi}} \neq \vec{\mathbf{0}}$, we can write

$$T_\sigma(a_k,f_2,f_3) = \sum_{j\in\mathbb{Z}} T_{\widetilde{\sigma_j}}(a_k,f_2,f_3), \tag{9.47}$$

where $\widetilde{\sigma_j}(\vec{\boldsymbol{\xi}}) := \sigma(\vec{\boldsymbol{\xi}})\widehat{\Psi}(2^{-j}\vec{\boldsymbol{\xi}})$ so that

$$\sup_{k\in\mathbb{Z}} \big\| \widetilde{\sigma_k}(2^k \vec{\cdot}) \big\|_{L^2_s((\mathbb{R}^n)^3)} = \mathcal{L}^2_s[\sigma] = 1.$$

Moreover, due to the support of $\widetilde{\sigma_j}$,

$$T_{\widetilde{\sigma_j}}(a_k,f_2,f_3) = T_{\widetilde{\sigma_j}}(\Gamma_{j+1}a_k,f_2,f_3). \tag{9.48}$$

Now, the left-hand side of Equation (8.3) is less than

$$\sup_{l\in\mathbb{Z}} \Big| \sum_{k=1}^{\infty} \lambda_k \chi_{(Q_k^{***})^c}(x) \chi_{(B_k^l)^c}(x) \phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\widetilde{\sigma_j}}\big(\Gamma_{j+1}a_k,f_2,f_3\big)(x) \Big)(x) \Big|.$$

Let s_1,s_2,s_3 be numbers such that $s_1 > n/p - n/2$, $s_2,s_3 > n/2$, and $s = s_1+s_2+s_3$. For $x \in (Q^{***})^c \cap (B_k^l)^c$ and $|x-y| \le 2^{-l}$,

$$|x-\mathbf{x}_{Q_k}| \lesssim |y-\mathbf{x}_{Q_k}|.$$

In the same argument as in the proof of Equations (9.5) and (9.8), with Equation (4.8) replaced by Equation (4.9), we can get

$$\langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1} \big| T_{\widetilde{\sigma_j}}\big(\Gamma_{j+1}a_k,f_2,f_3\big)(y) \big|$$
$$\lesssim \|f_2\|_{L^\infty(\mathbb{R}^n)} \|f_3\|_{L^\infty(\mathbb{R}^n)} \langle 2^j(x-\mathbf{x}_{Q_k})\rangle^{s_1} \int_{(\mathbb{R}^n)^3} \big| \widetilde{\sigma_j}^\vee(y-z_1,z_2,z_3) \big| |\Gamma_{j+1}a_k(z_1)| \, d\vec{\boldsymbol{z}}$$
$$\lesssim \ell(Q_k)^{-n/p} \min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\} I_{k,j,s}(y),$$

where $I^{in}_{k,j,s}$ and $I^{out}_{k,j,s}$ are defined as in Equaitons (9.4) and (9.9), respectively, and

$$I_{k,j,s}(y) := I^{in}_{k,j,s}(y) + I^{out}_{k,j,s}(y).$$

This yields that

$$\begin{aligned}&\Big|\phi_l * \Big(\sum_{j\in\mathbb{Z}} T_{\widetilde{\sigma_j}}(\Gamma_{j+1}a_k, f_2, f_3)\Big)(x)\Big|\\&\lesssim \ell(Q_k)^{-n/p}|x-\mathbf{x}_{Q_k}|^{-s_1}\mathcal{M}\Big(\sum_{j\in\mathbb{Z}} 2^{-s_1 j}\min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\} I_{k,j,s}(\cdot)\Big)(x)\end{aligned}$$

and thus Equation (8.3) follows from choosing $u_2(x) = u_3(x) := 1$ and

$$\begin{aligned}u_1(x) := \sum_{k=1}^{\infty}|\lambda_k|\chi_{(Q_k^{***})^c}(x)\ell(Q_k)^{-n/p}|x-\mathbf{x}_{Q_k}|^{-s_1}\\ \times\mathcal{M}\Big(\sum_{j\in\mathbb{Z}} 2^{-s_1 j}\min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\} I_{k,j,s}(\cdot)\Big)(x).\end{aligned}$$

Now, it is straightforward that $\|u_1\|_{L^p(\mathbb{R}^n)}$ is less than

$$\Big(\sum_{k=1}^{\infty}|\lambda_k|^p\ell(Q_k)^{-n}\Big\||\cdot-\mathbf{x}_{Q_k}|^{-s_1}\mathcal{M}\Big(\sum_{j\in\mathbb{Z}} 2^{-s_1 j}\min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\} I_{k,j,s}(\cdot)\Big)\Big\|^p_{L^p((Q_k^{***})^c)}\Big)^{1/p},$$

and the L^p-norm in the preceding expression is less than

$$\begin{aligned}&\big\||\cdot-\mathbf{x}_{Q_k}|^{-s_1}\big\|_{L^{p(2/p)'}((Q_k^{***})^c)}\Big\|\mathcal{M}\Big(\sum_{j\in\mathbb{Z}} 2^{-s_1 j}\min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\} I_{k,j,s_1}(\cdot)\Big)\Big\|_{L^2(\mathbb{R}^n)}\\&\lesssim \ell(Q_k)^{-(s_1-(n/p-n/2))}\sum_{j\in\mathbb{Z}} 2^{-s_1 j}\min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\}\|I_{k,j,s}\|_{L^2(\mathbb{R}^n)}\\&\lesssim \ell(Q_k)^{-(s_1-(n/p-n/2))}\ell(Q_k)^n\sum_{j\in\mathbb{Z}} 2^{-(s_1-n/2)j}\min\big\{1, \big(2^j\ell(Q_k)\big)^M\big\} \lesssim \ell(Q_k)^{n/p},\end{aligned}$$

where Equations (9.13) and (9.14) are applied in the penultimate inequality for sufficiently large M. This concludes that

$$\|u_1\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\sum_{k=1}^{\infty}|\lambda_k|^p\Big)^{1/p} \lesssim 1.$$

9.14. Proof of Lemma 8.2

Select $0 < \epsilon < 1$ such that

$$N_p := [n/p-n] \le n/p-n < [n/p-n]+\epsilon < s-3n/2.$$

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Then Lemma 4.1 yields that

$$\begin{aligned}\left|\phi_l * T_\sigma\big(a_k,f_2,f_3\big)(x)\right| &\lesssim 2^{l(N_p+n+\epsilon)}\int_{\mathbb{R}^n}|y-\mathbf{x}_{Q_k}|^{N_p+\epsilon}\left|T_\sigma(a_k,f_2,f_3)(y)\right|dy\\ &\lesssim \frac{1}{|x-\mathbf{x}_{Q_k}|^{N_p+n+\epsilon}}\int_{\mathbb{R}^n}|y-\mathbf{x}_{Q_k}|^{N_p+\epsilon}\left|T_\sigma(a_k,f_2,f_3)(y)\right|dy\\ &\le \frac{1}{|x-\mathbf{x}_{Q_k}|^{N_p+n+\epsilon}}\big(\mathcal{K}^{in}_{N_p+\epsilon}(a_k,f_2,f_3)+\mathcal{K}^{in}_{N_p+\epsilon}(a_k,f_2,f_3)\big),\end{aligned}$$

where we applied $2^l \lesssim |x-\mathbf{x}_{Q_k}|$ for $x\in B_k^l$ in the penultimate inequality and

$$\begin{aligned}\mathcal{K}^{\mathrm{in}}_{N_p+\epsilon}(a_k,f_2,f_3) &:= \int_{Q_k^{**}}|y-\mathbf{x}_{Q_k}|^{N_p+\epsilon}\left|T_\sigma(a_k,f_2,f_3)(y)\right|dy,\\ \mathcal{K}^{\mathrm{out}}_{N_p+\epsilon}(a_k,f_2,f_3) &:= \int_{(Q_k^{**})^c}|y-\mathbf{x}_{Q_k}|^{N_p+\epsilon}\left|T_\sigma(a_k,f_2,f_3)(y)\right|dy.\end{aligned}$$

Now, we claim that

$$\mathcal{K}^{\mathrm{in/out}}_{N_p+\epsilon}(a_k,f_2,f_3)\lesssim \ell(Q_k)^{N_p-n/p+n+\epsilon}. \tag{9.49}$$

Once Equation (9.49) holds, we obtain

$$\left|\phi_l * T_\sigma\big(a_k,f_2,f_3\big)(x)\right|\lesssim |Q_k|^{-1/p}\frac{\ell(Q_k)^{N_p+n+\epsilon}}{|x-\mathbf{x}_{Q_k}|^{N_p+n+\epsilon}},$$

which implies (8.4) with $u_2(x)=u_3(x):=1$ and

$$u_1(x):=\sum_{k=1}^\infty|\lambda_k||Q_k|^{-1/p}\frac{\ell(Q_k)^{N_p+n+\epsilon}}{|x-\mathbf{x}_{Q_k}|^{N_p+n+\epsilon}}\chi_{(Q_k^{***})^c}(x).$$

Moreover,

$$\begin{aligned}\|u_1\|_{L^p(\mathbb{R}^n)} &\le \Big(\sum_{k=1}^\infty|\lambda_k|^p|Q_k|^{-1}\ell(Q_k)^{p(N_p+n+\epsilon)}\big\||\cdot-\mathbf{x}_{Q_k}|^{-(N_p+n+\epsilon)}\big\|^p_{L^p((Q_k^{***})^c)}\Big)^{1/p}\\ &\lesssim \Big(\sum_{k=1}^\infty|\lambda_k|^p\Big)^{1/p}\lesssim 1\end{aligned}$$

because $N_p+n+\epsilon>n/p$.

Therefore, it remains to show Equation (9.49). Indeed, it follows from Theorem D that

$$\begin{aligned}\mathcal{K}^{\mathrm{in}}_{N_p+\epsilon}(a_k,f_2,f_3) &\lesssim \ell(Q_k)^{N_p+\epsilon}\big\|T_\sigma(a_k,f_2,f_3)\big\|_{L^1(\mathbb{R}^n)}\\ &\lesssim \ell(Q_k)^{N_p+\epsilon}\|a_k\|_{L^1(\mathbb{R}^n)}\lesssim \ell(Q_k)^{N_p-n/p+n+\epsilon}.\end{aligned}$$

For the other term, we use both Equations (9.47) and (9.48) to write

$$\mathcal{K}^{\mathrm{out}}_{N_p+\epsilon}(a_k,f_2,f_3)\lesssim\sum_{j\in\mathbb{Z}}2^{-j(N_p+\epsilon)}\int_{(Q_k^{**})^c}\langle 2^j(y-\mathbf{x}_{Q_k})\rangle^{N_p+\epsilon}\left|T_{\widetilde{\sigma_j}}\big(\Gamma_{j+1}a_k,f_2,f_3\big)(y)\right|dy.$$

Let s_1,s_2,s_3 be numbers satisfying

$$N_p+n/2+\epsilon<s_1<s-n,\quad s_2,s_3>n/2,\quad s_1+n<s_1+s_2+s_3=s,$$

similar to Equations (9.30) and (9.31). Then, using the argument in Equation (9.33), we have

$$\begin{aligned}\big|T_{\widetilde{\sigma_j}}\big(\Gamma_{j+1}a_k,f_2,f_3\big)(y)\big|\chi_{(Q_k^{**})^c}(y)&\lesssim \ell(Q_k)^{-n/p}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\frac{1}{\langle 2^j(y-\mathbf{x}_{Q_k})\rangle^{s_1}}\\ &\quad\times 2^{-jn}\int_{\mathbb{R}^n}|A_{j,Q_k}(z_1)|\big\|\langle 2^j(y-z_1,z_2,z_3)\rangle^s\widetilde{\sigma_j}^\vee(y-z_1,z_2,z_3)\big\|_{L^2(z_2,z_3)}dz_1,\end{aligned}$$

where A_{j,Q_k} is defined as in Equation (9.32). This finally yields that

$$\begin{aligned}&\mathcal{K}^{\mathrm{out}}_{N_p+\epsilon}(a_k,f_2,f_3)\lesssim \ell(Q_k)^{-n/p}\sum_{j\in\mathbb{Z}}2^{-j(N_p+n+\epsilon)}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\int_{\mathbb{R}^n}|A_{j,Q_k}(z_1)|\\ &\quad\times\Big(\int_{(Q_k^{**})^c}\frac{1}{\langle 2^j(y-\mathbf{x}_{Q_k})\rangle^{s_1-(N_p+\epsilon)}}\big\|\langle 2^j(y-z_1,z_2,z_3)\rangle^s\widetilde{\sigma_j}^\vee(y-z_1,z_2,z_3)\big\|_{L^2(z_2,z_3)}dy\Big)\,dz_1\\ &\lesssim \ell(Q_k)^{-n/p}\sum_{j\in\mathbb{Z}}2^{-j(s_1+n)}\min\big\{1,\big(2^j\ell(Q_k)\big)^M\big\}\|A_{j,Q_k}\|_{L^1(\mathbb{R}^n)}\\ &\qquad\times\big\|\langle 2^j\vec{\cdot}\rangle^s\widetilde{\sigma_j}^\vee\big\|_{L^2((\mathbb{R}^n)^3)}\big\||\cdot-\mathbf{x}_{Q_k}|^{-(s_1-(N_p+\epsilon))}\big\|_{L^2(\mathbb{R}^n)}\\ &\lesssim \ell(Q_k)^{-n/p+n}\ell(Q_k)^{-(s_1-(N_p+n/2+\epsilon))}\sum_{j\in\mathbb{Z}}2^{-j(s_1-n/2)}\min\big\{1,\big(2^j\ell(Q_k)\big)^{M-(L_0-s_1-n)}\big\}\\ &\lesssim \ell(Q_k)^{N_p-n/p+n+\epsilon}\end{aligned}$$

for M and L_0 satisfying $M>L_0-s_1-n$, which completes the proof of Equation (9.49).

Appendix A. Bilinear Fourier multipliers $(m=2)$

We remark that Theorem 1 still holds in the bilinear setting where all the arguments above work as well.

Theorem 2. *Let $0<p_1,p_2\le\infty$ and $0<p\le 1$ with $1/p=1/p_1+1/p_2$. Suppose that*

$$s>n\quad\text{and}\quad\frac{1}{p}-\frac{1}{2}<\frac{s}{n}+\sum_{j\in J}\Big(\frac{1}{p_j}-\frac{1}{2}\Big),$$

where J is an arbitrary subset of $\{1,2\}$. Let σ be a function on $(\mathbb{R}^n)^2$ satisfying

$$\sup_k\big\|\sigma(2^k\vec{\cdot})\widehat{\Psi^{(2)}}\big\|_{L^2_s((\mathbb{R}^n)^2)}<\infty$$

and the bilinear analogue of the vanishing moment condition (1.16). Then the bilinear Fourier multiplier T_σ, associated with σ, satisfies

$$\big\|T_\sigma(f_1,f_2)\big\|_{H^p(\mathbb{R}^n)}\lesssim\sup_k\big\|\sigma(2^k\vec{\cdot})\widehat{\Psi^{(2)}}\big\|_{L^2_s((\mathbb{R}^n)^2)}\|f_1\|_{H^{p_1}(\mathbb{R}^n)}\|f_2\|_{H^{p_2}(\mathbb{R}^n)}$$

for $f_1,f_2\in\mathcal{S}_0(\mathbb{R}^n)$.

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The proof is similar, but much simpler than that of Theorem 1. Moreover, unlike Theorem 1, Theorem 2 covers the results for $p_j=\infty$, $j=1,2$, which follow immediately from the bilinear analogue of Proposition 3.2.

Appendix B. General m-linear Fourier multipliers for $m\geq 4$

The structure of the proof of Theorem 1 is actually very similar to those of Theorems C and D, in which $T_\sigma(f_1,\dots,f_m)$ is written as a finite sum of $T^\kappa(f_1,\dots,f_m)$ for some variant operators T^κ, and then

$$\big|T^\kappa\big(f_1,\dots,f_m\big)(x)\big|\lesssim \sup_{k\in\mathbb{Z}}\big\|\sigma(2^k\vec{\cdot})\widehat{\Psi^{(m)}}\big\|_{L^2_s((\mathbb{R}^n)^m)}u_1(x)\cdots u_m(x), \tag{B.1}$$

where $\|u_j\|_{L^{p_j}(\mathbb{R}^n)}\lesssim\|f_j\|_{L^{p_j}(\mathbb{R}^n)}$ for $1\leq j\leq m$. Compared to the $H^{p_1}\times\cdots\times H^{p_m}\to L^p$ estimates in Theorems C and D, one of the obstacles to be overcome for the boundedness into Hardy space H^p is to replace the left-hand side of Equation (B.1) by

$$\sup_{l\in\mathbb{Z}}\big|\phi_l*T^\kappa\big(f_1,\dots,f_m\big)(x)\big|,$$

and we have successfully accomplished this for $m=3$ as mentioned in Equation (1.20). One of the methods we have adopted is

$$\chi_{Q_k^{***}}(x)2^{ln}\int_{|x-y|\leq 2^{-l}}F_1(y)F_2(y)F_3(y)\,dy\lesssim\chi_{Q_k^{***}}(x)\mathcal{M}_qF_1(x)\mathcal{M}_{\widetilde{r}}F_2(x)\mathcal{M}_{\widetilde{r}}F_3(x),$$

where $2<\widetilde{r}<p_2,p_3$ and $1/q+2/\widetilde{r}=1$. Then we have

$$\|\mathcal{M}_{\widetilde{r}}F_2\|_{L^{p_2}(\mathbb{R}^n)}\lesssim\|F_j\|_{L^{p_j}(\mathbb{R}^n)},\qquad j=2,3$$

by the L^{p_j} boundedness of $\mathcal{M}_{\widetilde{r}}$ with $\widetilde{r}<p_j$. Such an argument is contained in the proof of Lemma 6.1. However, if we consider m-linear operators for $m\geq 4$, then the above argument does not work for $p_2,\dots,p_m>2$. For example, it is easy to see that $1/q+3/\widetilde{r}$ exceeds 1 if $\widetilde{r}>2$ is sufficiently close to 2. That is, we are not able to obtain m-linear estimates for $0<p_1\leq 1$ and $2<p_2,\cdots,p_m<\infty$, $m\geq 4$. This is critical because our approach in this paper highly relies on interpolation between the estimates in the regions $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$, which are trilinear versions of $\{(1/p_1,\cdots,1/p_m):0<p_1\leq 1, 2<p_2,\cdots,p_m<\infty\}$.

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