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Covering the Unit Sphere of Certain Banach Spaces by Sequences of Slices and Balls

Dedicated to Professor Pierluigi Papini on the occasion of his retirement

Vladimir P. Fonf and Clemente Zanco

Abstract. We prove that, given any covering of any infinite-dimensional Hilbert space H by countably many closed balls, some point exists in H which belongs to infinitely many balls. We do that by characterizing isomorphically polyhedral separable Banach spaces as those whose unit sphere admits a point-finite covering by the union of countably many slices of the unit ball.

1 Introduction

A famous theorem by H. H. Corson [Co] states that no covering of any infinite-dimensional reflexive space X by closed bounded convex sets can be locally finite, *i.e.*, for any such covering τ there exists a compact subset of X that meets infinitely many members of τ. Such a theorem has been improved in several directions, weakening the assumption both of reflexivity of X (see [Fo2]) and of convexity of the members of the covering (see [FZ1]).

A further subject of investigation was suggested by Corson himself: when is it possible to guarantee that even a (algebraically) finite-dimensional compact set exists that meets infinitely members of the covering? This question has been recently answered in [FZ2], where we show that this is the case provided that X contains some infinite-dimensional separable dual space and τ is a covering of X by closed bounded convex sets with nonempty interior ("bodies" in the sequel).

So it was natural to inquire about the lowest dimension of the compact sets above: in [FZ3] in particular we show that, when X contains some infinite-dimensional separable dual space and τ is a covering of X by smooth or rotund bodies, then there is a segment in X that meets infinitely many members of τ.

Clearly we are led to the following question: under which assumptions on the space X and on the members of the covering τ we can guarantee that some point exists that meets infinitely many members of τ, *i.e.*, that τ is not point-finite? Now, even considering coverings by bodies, the answer is different: in [MZ] it is proved

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that any real Banach space X can be covered by bodies in such a way that no point of X belongs to more than two of them. The general construction provided there leads to bodies that are very far from being balls in the original norm of X, so the following final very natural question arises:

> Which infinite-dimensional Banach spaces admit a point-finite covering by closed balls (each of positive radius)?

Some classical Banach spaces do. For instance, it is easy to check that the covering of c_0 that can be obtained by translating the unit ball without overlapping interiors is even locally finite. V. Klee proved in [Kl2] that the space $l_1(\Gamma)$ for suitable (uncountable) Γ can be covered by translates of its unit ball without overlapping them at all.

In this paper we show (Section 3) that no infinite-dimensional Hilbert space can be covered by countably many closed balls in a point-finite way. To do that, we characterize isomorphically polyhedral Banach spaces among the separable ones as those whose unit sphere admits point-finite countable coverings by slices. Our argument requires some preparatory work (Section 2) that can be of some interest by itself.

Throughout the paper we use standard notation for the Geometry of Banach spaces as in [JL] (in particular, the balls we consider are closed balls).

All the normed spaces we are dealing with in the present paper are considered as real spaces.

2 Auxiliary Result

We recall that a *slice* of a ball B in a normed space X is the intersection (when nonempty) of B with some closed half-space of X.

Proposition 2.1 *Let X be a separable Banach space, $\{f_i\}_{i=1}^{\infty}$ a sequence of norm-one linear functionals on X and $\{\alpha_i\}_{i=1}^{\infty}$ a sequence of nonnegative numbers converging to* 1*. Then the sequence $\{S_i\}_{i=1}^{\infty}$ of slices of B_X defined by*

$$S_i = S(f_i, \alpha_i) = \{x \in B_X : f_i(x) \geq 1 - \alpha_i\}, i = 1, 2, \dots$$

is not point-finite.

Proof Since X is separable, B_{X^*} is sequentially compact in the w^*-topology, so there exist a sequence $\{i_k\}_{k=1}^{\infty}$ of integers and $f \in B_{X^*}$ such that

$$f_{i_k} \overset{w^*}{\to} f.$$

Let us consider two cases.

First case: $f \neq 0$. Let $v \in B_X$ be such that $f(v) > 0$. Since $f_{i_k}(v) \to f(v) > 0$ and $\alpha_{i_k} \to 1$ as $k \to \infty$, for every k big enough we get $f_{i_k}(v) > 1 - \alpha_{i_k}$, so $v \in S_i$ for infinitely many i's.

Second case: $f = 0$. In this situation it is well known that, for some subsequence $\{i_{k_m}\}_{m=1}^{\infty}$ of $\{i_k\}$, the sequence $\{f_{i_{k_m}}\}_{m=1}^{\infty}$ is a basic sequence in X^*. Let $\{u_m\}_{m=1}^{\infty}$ be a

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bounded sequence in X^{**} such that $\{(f_{i_{k_m}}, u_m)\}_{m=1}^{\infty}$ is a biorthogonal sequence (*i.e.*, $u_m(f_{i_{k_n}}) = \delta_{m,n}$ for any m, n in $\mathbb{N}$).

Put $C = \sup_m \|u_m\|$ (of course $C \geq 1$) and

$$\beta_m = 3C(1 - \alpha_{i_{k_m}}), \quad m \in \mathbb{N}. \tag{2.1}$$

Passing to a subsequence if necessary, without loss of generality we may assume that

$$\sum_{m=1}^{\infty} \beta_m < \frac{1}{4C}. \tag{2.2}$$

In order to simplify notation, without loss of generality from now on we assume that all the requirements the subsequence $\{f_{i_{k_m}}\}_{m=1}^{\infty}$ has been selected for are fulfilled by the sequence $\{f_i\}_{i=1}^{\infty}$ itself.

Let $v_1 \in CB_X$ be such that $f_1(v_1) \geq 3/4$. If it happens that $|f_m(v_1)| \geq \beta_m$ for infinitely many m's, then we are done (in fact for such m's it must be true that $f_m(v_1/C) > 1 - \alpha_m$ or $f_m(-v_1/C) > 1 - \alpha_m$ with $\pm v_1/C \in B_X$, so either $v_1/C \in S_m$ or $-v_1/C \in S_m$). Otherwise put $m_1 = 1$ and let $m_2 \in \mathbb{N}$ be such that $|f_m(v_1)| < \beta_m$ for every $m \geq m_2$. Because of the w^*-density of B_E into $B_{E^{**}}$ for $E = \operatorname{Ker} f_{m_1}$ (where as usual, we identify E with its image in E^{**} under the canonical map), there exists $v_2 \in \operatorname{Ker} f_{m_1} \cap CB_X$ such that $f_{m_2}(v_2) \geq 3/4$. Let us argue as we have done for v_1: if $|f_m(v_2)| \geq \beta_m$ for infinitely many m's we are done; otherwise let us proceed inductively in the following way. Having determined m_i and v_i for $i \geq 1$, choose $m_{i+1} > m_i$ such that

$$f_m(v_j) < \beta_m, \quad m \geq m_{i+1}, \; j = 1, \dots, i \tag{2.3}$$

and choose $v_{i+1} \in \bigcap_{j=1,\dots,i} \operatorname{Ker} f_j \cap CB_X$ such that

$$f_{m_{i+1}}(v_{i+1}) \geq 3/4.$$

If we can stop in a finite number of steps we are done, otherwise let us put

$$v = \sum_{i=1}^{\infty} \beta_{m_i} v_i.$$

Clearly $\|v\| \leq C \sum_{i=1}^{\infty} \beta_{m_i} < 1/4$ because of (2.2). We claim that

$$f_{m_n}(v) \geq 1 - \alpha_{m_n}, \quad n \in \mathbb{N},$$

so $v \in S_{m_n}$ for any $n \in \mathbb{N}$. In fact, according to our construction, for any $n \in \mathbb{N}$, taking into account (2.1), (2.2) and (2.3), we have

$$\begin{aligned} f_{m_n}(v) &= \sum_{i=1}^{\infty} \beta_{m_i} f_{m_n}(v_i) = \sum_{i=1}^{n} \beta_{m_i} f_{m_n}(v_i) = \beta_{m_n} f_{m_n}(v_n) + \sum_{i=1}^{n-1} \beta_{m_i} f_{m_n}(v_i) \\ &\geq \frac{3}{4}\beta_{m_n} - \sum_{i=1}^{n-1} \beta_{m_n}\beta_{m_i} = \beta_{m_n}\Big(\frac{3}{4} - \sum_{i=1}^{n-1} \beta_{m_i}\Big) > \frac{3}{2}C(1 - \alpha_{m_n}) > 1 - \alpha_{m_n}. \end{aligned}$$

The proof is complete. ■

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Remark 2.2 If a point z different from the origin belongs to some slice, clearly that slice contains the point $z/\|z\|$ on the unit sphere. Hence we can actually guarantee that, under the assumption of Proposition 2.1, some point exists in S_X that belongs to infinitely many slices.

The following immediate consequence of Proposition 2.1 may be of interest.

Corollary 2.3 *Suppose that X is a separable Banach space and $\{F_i\}_{i=1}^{\infty}$ is a sequence of closed half-spaces of X. If for some point $z \in X$ it happens that*

$$z \in \operatorname{cl} \bigcup_{i=1}^{\infty} F_i \quad \text{and} \quad z \notin \bigcup_{i=1}^{\infty} F_i,$$

then there are points in X arbitrarily close to z that belong to F_i for infinitely many i's.

3 Main Results

In order to prove our main results we need some notions the reader might not be familiar with, so a short introduction is needed.

According to Klee [Kl1], we say that a Banach space is *isomorphically polyhedral* whenever it admits an equivalent norm under which any finite-dimensional section of the unit ball is a polytope (*i.e.*, the convex hull of finitely many points). It is known that any such space contains some isomorphic copy of the space c_0 (in fact it is c_0-saturated; see [Fo1]), so that it cannot be reflexive.

Let U be a body in a Banach space X. A subset $\mathcal{U}$ of S_{X^*} is said to be a *boundary for U* if, for every point $x \in \partial U$, there is $f_x \in \mathcal{U}$ such that

$$f_x(x) = \sup_{z \in U} f_x(z).$$

Clearly, by the Hahn–Banach theorem, any body U has some nonempty boundary and S_{X^*} always is a boundary for B_X. We pay special attention to the class of separable Banach spaces X such that B_X has a countable boundary; in fact, by Theorem 1 in [Fo1], this is exactly the class of isomorphically polyhedral separable Banach spaces.

We start with the following characterization of polyhedrality via slices in the context of separable Banach spaces.

Theorem 3.1 *For a separable Banach space Y the following conditions are equivalent.*

(a) *There exists a renorming X of Y under which S_X admits a point-finite covering by countably many slices*

$$S_i = S(f_i, \alpha_i) = \{x \in B_X : f_i(x) \geq 1 - \alpha_i\}, \quad f_i \in S_{X^*},\ 0 \leq \alpha_i < 1,\ i = 1, 2, \dots.$$

(b) *Y is isomorphically polyhedral.*

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Proof (a) $\Rightarrow$ (b). Since the family of slices $\{S_i\}$ is point-finite, Proposition 2.1 applied to the space X implies $\sup_i \alpha_i < 1$. Put

$$U = \bigcap_{i=1}^{\infty} \{x \in X : f_i(x) \le 1 - \alpha_i\}.$$

We claim that $U \subset B_X$. In fact, if $x \notin B_X$, then there exists i_0 such that $x/\|x\| \in S(f_{i_0}, \alpha_{i_0})$, *i.e.*, $f_{i_0}(x/\|x\|) \ge 1 - \alpha_{i_0}$ that implies $f_{i_0}(x) > 1 - \alpha_{i_0}$, so $x \notin U$.

Clearly U is closed and (since $\sup_i \alpha_i < 1$) contains the origin as an interior point.

We claim that the sequence $\{f_i\}_{i=1}^{\infty}$ is a (countable) boundary for U, *i.e.*, that for every $x \in \partial U$ there exists an index i_x such that $f_{i_x}(x) = 1 - \alpha_{i_x}$ $(= \max_{z \in U} f_{i_x}(z))$.

Assume first that $\|x\| = 1$. Since S_X is covered by $\bigcup S_i$, there is an index i_x such that $f_{i_x}(x) \ge 1 - \alpha_{i_x}$. On the other hand, we have $f_i(x) \le 1 - \alpha_i$ for every i, since $x \in U$; it follows that $f_{i_x}(x) = 1 - \alpha_{i_x} = \sup_{z \in U} f_{i_x}(z)$, so we are done.

Assume now that $\|x\| < 1$. Since $\|f_i\| = 1$ for every i and x is not an interior point of U, it is clear that

$$\sup_i \{f_i(x)/(1 - \alpha_i)\} = 1.$$

We just need to show that such a supremum is attained. Since $x/\|x\| \in S_i$ for only finitely many i's, there exists i_x such that $x/\|x\| \in S_{i_x}$ and $x/\|x\| \notin S_i$ for $i > i_x$, *i.e.*,

$$f_i(x/\|x\|) < 1 - \alpha_i, \quad i > i_x,$$

that implies

$$f_i(x)/(1 - \alpha_i) < \|x\| < 1, \quad i > i_x.$$

Hence again we have $f_i(x)/(1 - a_i) = 1$ for some $i \le i_x$.

Now consider the centrally symmetric set $V = U \cap -U$. Since any of its boundary points must be a boundary point of U or of $-U$, it follows that V too has a countable boundary. Let us pass to the renorming of X under which V is the unit ball. It is clear that V still has a countable boundary in the new norm, so that from [Fo1, Theorem 1] we get that X, so Y too, is isomorphically polyhedral.

(b) $\Rightarrow$ (a). Trivially, any finite-dimensional Banach space Y satisfies (a), so we assume that Y is infinite-dimensional. By [Fo2, Theorem A], there is a (polyhedral) renorming X of Y such that the union of countably many faces

$$\{x \in S_X : f_i(x) = 1, f_i \in S_{X^*}\}_{i=0}^{\infty}$$

covers S_X and, for any $x \in S_X$, there exists $\varepsilon_x > 0$ such that for only finitely many i's it happens that $f_i(x) > 1 - \varepsilon_x$. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a null positive sequence. Of course the union of the slices $\{x \in B_X : f_i(x) \ge 1 - \alpha_i\}_{i=1}^{\infty}$ also covers S_X and turns out to be point-finite (even locally finite).

The proof is complete. ■

We pass now to the theorem that is in fact the main goal of the paper.

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Theorem 3.2 *Let the union of countably many balls cover the infinite-dimensional Hilbert space H. Then there is a point in H that belongs to infinitely many balls.*

Proof Let $\{B_i\}_{i=1}^{\infty}$ be a sequence of closed balls whose union covers H. Let H' be an infinite-dimensional separable closed subspace of H. Because of the inner product structure, when $B_i \cap H'$ is nonempty, it is a closed ball in H' (we are allowed to consider also balls of radius 0). So, without loss of generality, we may assume that H itself is separable.

We start by observing that, for any $\rho > 0$, when

$$\rho S_H \not\subset B_i \quad \text{and} \quad \rho S_H \cap B_i \neq \varnothing, \tag{3.1}$$

the set $\rho S_H \cap B_i$ determines in a natural way a slice of ρB_H. In fact, it is an easy exercise to verify that the intersection among two spheres in any Hilbert space lies in some hyperplane. For any index i and ρ as in (3.1), let F_i be the hyperplane containing $\rho S_H \cap \partial B_i$: consider the slice of ρB_H determined by F_i that is contained in B_i. Clearly ρS_H is covered by the union of such slices.

Now assume that the origin of H belongs to B_i for only finitely many i's (otherwise we are done). Then there exist $\rho > 0$ and an index i_0 such that ρS_H is covered by $\bigcup_{i=i_0}^{\infty} B_i$, which union does not cover the origin. In the present situation, infinitely many balls are needed in order to cover ρS_H (in fact the complement of the union of finitely many balls, not containing the origin, always contains a weak neighborhood of the origin), so countably many slices of ρB_H are determined whose union covers ρS_H. Without loss of generality we may assume that $\rho = 1$: we are done by Theorem 3.1, (a) $\Rightarrow$ (b), since no infinite-dimensional Hilbert space is isomorphically polyhedral. ■

It is a trivial fact that, if a separable normed space X is covered by the union of uncountably many balls, each of positive radius, such a covering cannot be point-finite. (In fact, let $\{x_n\}$ be any sequence dense in X. Since each ball has nonempty interior, for some n_0 it must happen that x_{n_0} belongs to uncountably many balls.) Hence, as a consequence of Theorem 3.2 we get the following corollary.

Corollary 3.3 *No covering of the infinite-dimensional separable Hilbert space by closed balls, each of positive radius, can be point-finite.*

Remark 3.4 It is worthwhile to notice that the special case of Theorem 3.2 when H is separable and the balls B_i have the same positive radius can be handled in a much simpler way. In fact, if no point of H were to belong to B_i for infinitely many i's, then the set of the centers of such balls would provide a countable proximinal subset Γ of H with uniformly bounded distances from h to Γ when h ranges in H. It is known (see [FL, Proposition 2.1]) that H contains no such countable proximinal subset.

In the framework of covering sets by balls, part of the literature is devoted to the study of coverings of the unit sphere of a Banach space X by balls that do not cover the origin. This topic has a wide interest even in the finite-dimensional setting. We refer to [Pa] also an exhaustive list of references; see also [FR] and [FZ4]. The following

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theorem states a further result in that direction. Its proof can be carried on like the proof of Theorem 3.2, taking into account Remark 2.2.

Theorem 3.5 *Let H be an infinite-dimensional Hilbert space and $\{B_i\}_{i=1}^{\infty}$ be a sequence of balls in H whose union covers S_H without covering the origin. Then there is some point in S_H that belongs to B_i for infinitely many i's.*

4 More About Sequences of Balls in Hilbert Space

Our previous results do not allow us to establish at which points a sequence of balls in a Hilbert space H can or cannot be point-finite. A partial answer to this question is given by the following proposition and corollary.

Proposition 4.1 *Let $B(x_n, R_n)_{n=1}^{\infty}$ any sequence of balls in H such that*

(i) *there exist two positive numbers α, β such that $\alpha < R_n < \beta$ for every n;*
(ii) *$\|x_n\| > R_n$ for every n;*
(iii) *$\|x_n\| - R_n$ converges to 0 when n goes to infinity* (i.e., *none of the balls $B(x_n, R_n)$ contains the origin, but the balls approach the origin when n goes to infinity).*

Then the following are equivalent:

(a) *no point in H belongs to $B(x_n, R_n)$ for infinitely many n's;*
(b) *the sequence $\{x_n\}$ of the centers of the balls converges weakly to the origin.*

Proof (a) $\Rightarrow$ (b). Assume to the contrary that (a) holds and $\{x_n\}$ doesn't converge weakly to the origin. Then there exist $f \in S_{H^*}$, $\delta > 0$ and a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that $|f(x_{n_k})| > \delta$ for every k. Without loss of generality we can assume that the sequence $\{R_{n_k}\}_{k=1}^{\infty}$ is convergent, say to $R > 0$. Put $\{y\} = f^{-1}(1) \cap S_H$, $L = \ker(f)$, $l = \operatorname{span}(y)$. Clearly $H = L \oplus_2 l$. Put $x_n = u_n + v_n$, $u_n \in L$, $v_n = f(x_n)y \in l$. We have

$$\|u_{n_k}\| = \sqrt{\|x_{n_k}\|^2 - \|v_{n_k}\|^2} \leq \sqrt{\|x_{n_k}\|^2 - \delta^2}.$$

Since $\|x_n\| \to R$, for some $\delta_1 > 0$ we have, for all k big enough, $\|u_{n_k}\| \leq R - \delta_1$. Therefore, there is a positive γ such that, for all but finitely many k's, the segment $B(x_{n_k}, R_{n_k}) \cap l$ has length bigger than γ. Since all such segments lie in a bounded region of the line l, there must be a point that belongs to infinitely many of them, contradicting (a).

(b) $\Rightarrow$ (a). Assume to the contrary that (b) holds and there is some point y that belongs to $B(x_n, R_n)$ for infinitely many n's. Of course, y is not the origin of H. Without loss of generality we can assume that $y \in \bigcap_{n=1}^{\infty} B(x_n, R_n)$ and $\{R_n\}$ is convergent, say to R. Take the functional $f \in S_{H^*}$ such that $f(y) = \|y\|$. Borrowing our notation from the proof of (a) $\Rightarrow$ (b), because of (b) we get that $\{v_n = f(x_n)y\}$ converges to the origin strongly, so that, since $\|x_n\| \to R$,

$$\|u_n\| = \sqrt{\|x_n\|^2 - \|v_n\|^2} \to R \quad \text{as } n \to \infty.$$

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Therefore the lengths of the intervals $B(x_n, R_n) \cap l$ go to 0. Since y is the common point of all these intervals, it follows that the sequence $\{v_n\}$ converges strongly to y that is different from the origin of H, a contradiction. ■

The following corollary gives a location for the w-limit of the sequence of centers of any sequence of balls, whose radii are bounded away from above and from below, provided the balls have a common point.

Corollary 4.2 *Let $\alpha < \beta$ be two positive numbers and let $\{B(x_n, R_n)\}_{n=1}^{\infty}$ be a sequence of balls in H such that $\alpha < R_n < \beta$ for every n and $\bigcap_{n=1}^{\infty} B(x_n, R_n) \neq \varnothing$. If the sequence $\{x_n\}$ weakly converges to some point z, then z belongs to all but finitely many balls.*

Proof Assume to the contrary that, for some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$, it happens that

$$z \notin \bigcup_{k=1}^{\infty} B(x_{n_k}, R_{n_k})$$

(*i.e.*, $\|x_{n_k} - z\| > R_{n_k}$ for every k). Without loss of generality we may assume that $\{\|x_{n_k} - z\|\}$ is convergent, say to R. Take any sequence $\{r_k\}_{k=1}^{\infty}$ of positive numbers such that $\|x_{n_k} - z\| > r_k > R_{n_k}$ for every k and $\{r_k\}$ converges to R when k goes to ∞. Let y be any point common to all the balls $B(x_n, R_n)$; clearly the point $y - z$ is a common point of the balls $B(x_{n_k} - z, r_k)$. Since the sequence $\{x_{n_k} - z\}$ converges weakly to the origin and the sequence $\{\|x_{n_k}\| - r_k\}$ converges to 0, part (b) ⇒ (a) of Proposition 4.1 is contradicted. ■

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Department of Mathematics, Ben-Gurion University of the Negev, 84105 Beer-Sheva, Israel
e-mail: fonf@math.bgu.ac.il

Dipartimento di Matematica, Università degli Studi, Via C. Saldini, 50, 20133 Milano MI, Italy
e-mail: clemente.zanco@unimi.it

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