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Hecke L-Functions and the Distribution of Totally Positive Integers

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Abstract. Let K be a totally real number field of degree n. We show that the number of totally positive integers (or more generally the number of totally positive elements of a given fractional ideal) of given trace is evenly distributed around its expected value, which is obtained from geometric considerations. This result depends on unfolding an integral over a compact torus.

Introduction

In 1921 Hecke [3] showed (among other things) that the Fourier coefficients of certain functions on a non-split torus in $\mathrm{GL}(2)/\mathbb{Q}$ were given by (what we now call) the L-functions of certain Hecke characters of a real quadratic field. Interestingly, the Hecke characters which arise here are not of type A.

Hecke's key idea in that portion of his paper is to unfold the integral that computes the Fourier coefficient in a manner foreshadowing the Rankin–Selberg method. Siegel [8, Ch. II, Section 4] observed that the same method could be used to compute the hyperbolic Fourier expansion of the nonholomorphic Eisenstein series on $\mathrm{GL}(2)/\mathbb{Q}$, and to express these coefficients in terms of Hecke L-functions.

Hecke used his Fourier expansion to study the distribution of the fractional parts of $m\alpha$ where m runs over the rational integers and α is a fixed real quadratic irrationality. Siegel used his in conjunction with Kronecker's limit formula to derive some relationships between Dedekind's η-function and invariants of real quadratic fields.

Our main goal in this paper is to generalize Hecke's result to an arbitrary totally real field K. To do so, we first explain the generalization of Siegel's Fourier expansion computation to $\mathrm{GL}(n)/\mathbb{Q}$. Where Siegel worked with a real quadratic field, we let K be any number field, $[K:\mathbb{Q}] = n$. We can view K as the $\mathbb{Q}$-points of a non-split torus T in $\mathrm{GL}(n)/\mathbb{Q}$. Let X denote the totally geodesic subspace of the symmetric space of $\mathrm{GL}(n)/\mathbb{Q}$ defined by T. We show that an automorphic form on $\mathrm{GL}(n,\mathbb{Q})$, restricted to X, has a Fourier expansion. We then consider this Fourier expansion for the standard maximal parabolic Eisenstein series $E(g,s)$ of type $(n-1,1)$. We prove that the Fourier coefficients, which are again functions of s, are in fact certain Hecke L-functions associated to K (more accurately, partial Hecke L-functions with respect

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to a certain ideal class in K). This observation is not new (see Piatetski-Shapiro and Rallis [6, (2.4)]), although here we carry out the computation in a classical, non-adelic, fashion.

Next we assume that K is totally real and use the Fourier expansion to generalize Hecke's work. Hecke studied the distribution of the fractional parts of $m\alpha$ for α a fixed real quadratic irrationality. The correct generalization of the fractional part of $m\alpha$ turns out to be the error term in the natural geometric estimate for the number of integers of K of a given trace. We form the Dirichlet series $\varphi(s)$ whose coefficients are these errors.

More specifically, let $\mathfrak{a}$ denote a fractional ideal in K, and let $\mathrm{Tr}(\mathfrak{a})$ be generated by $k > 0$. If a is a positive integral multiple of k, let N_a denote the number of totally positive elements of $\mathfrak{a}$ with trace a. There is the natural geometric estimate r_a of N_a derived from the volume of the intersection in $\mathfrak{a} \otimes \mathbb{R}$ of the cone of totally positive elements with the hyperplane defined by Trace $= a$. Denote the difference between N_a and its estimate as a volume by E_a: $E_a = N_a - r_a$. Note that E_a may be positive or negative. If a is not a positive multiple of k, we set $E_a = 0$. Then we define the Dirichlet series

$$\varphi(s) = \sum_{a>0} \frac{E_a}{a^s}.$$

Though $\varphi(s)$ depends on $\mathfrak{a}$, we shall suppress this dependence from the notation.

A comparison with Hecke's paper may be helpful. Hecke works with $K = \mathbb{Q}(\sqrt{D})$ with $D > 0$ squarefree and D congruent to 2 or 3 modulo 4. He writes $R(x) \in [0, 1)$ for the fractional part of the real number x, and defines the Dirichlet series

$$\psi(s, 1/\sqrt{D}) = \sum_{m=1}^{\infty} \frac{R\left(\frac{m}{\sqrt{D}}\right) - \frac{1}{2}}{m^s}.$$

This series measures the distribution of the deviations of the multiples of $1/\sqrt{D}$ from being rational integers. Setting $\mathfrak{a} = \mathbb{Z} + \mathbb{Z}\sqrt{D}$, so $k = 2$, a simple calculation shows that

$$\psi(s, 1/\sqrt{D}) = -2^{s-1}\,\varphi(s),$$

so that his series is essentially the series $\varphi(s)$. Hecke also studies the series

$$\psi(s, \sqrt{D}) = \sum_{m=1}^{\infty} \frac{R(m\sqrt{D}) - \frac{1}{2}}{m^s}.$$

If one chooses the fractional ideal $\mathfrak{a} = \mathbb{Z} + \frac{1}{\sqrt{D}}\mathbb{Z}$, then this series is related to the function $\varphi(s)$ by a similar relation.

Returning to the general case, the analytic properties of $\varphi(s)$ are related to the distribution of the errors E_a in the usual way of analytic number theory. The natural estimate on the size of the errors, obtained from standard results concerning the counting of lattice points in a homogeneously expanding domain, gives us that $\varphi(s)$ is (absolutely) convergent for $\Re(s) > n - 1$. It follows that

$$\sum_{a<X} E_a \ll X^{n-1}.$$

Any improvement of this abscissa of convergence implies a corresponding improvement in the estimate for $\sum_{a<X} E_a$, which measures the lack of bias in the distribution of these errors about 0.

Following Hecke, we express $\varphi(s)$ in terms of a zeta-function and a function whose Fourier series over the non-split torus we can compute. The Fourier coefficients are once again related to Hecke L-functions not of type A. From this expression, we deduce that $\varphi(s)$ can be continued to a regular function in the right half plane $\Re(s)>0$. Here we notice an important difference with Hecke's case $n = 2$. Namely, the Gamma factors, which in Hecke's case caused $\varphi(s)$ to be meromorphic with known poles, in our case give dense sets of poles which prevent the continuation of $\varphi(s)$ to the left of the line $\Re(s) = 0$. That is, each term in the Fourier expansion is an L-function times a Gamma factor with infinitely many poles, but the poles become dense along certain vertical lines when we sum up all the terms. Nevertheless, we can use the functional equation of each term separately plus Stirling's formula to study the behavior of $\varphi(s)$ in vertical strips to the right of $\Re(s) = 0$. This enables us to use a theorem of Schnee and Landau to conclude that $\varphi(s)$ converges for $\Re(s)>n - 1 - (2n - 2)/(2n + 1)$.

Our main result then follows by summation by parts.

Main Theorem *For $\epsilon > 0$,*

$$\sum_{a<X} E_a = \mathrm{O}\left(X^{n-1-\frac{2n-2}{2n+1}+\epsilon}\right),$$

with the implied constant depending on ϵ and $\mathfrak{a}$.

When $n = 2$, Hecke gets the bound $\mathrm{O}(X^{\epsilon})$, because he is able to exploit (in a complicated way) the fact that his $\varphi(s)$ has a meromorphic continuation to the whole s-plane. Although our bound is not as good — one might have expected $\mathrm{O}(X^{n-2+\epsilon})$ as the correct generalization — note that the exponent in our bound does approach $n - 2$ as n goes to infinity.

Another recent generalization of these ideas of Hecke, in this case to certain elliptical cones, may be found in Duke–Imamoḡlu [2]. Also, Chinta and Goldfeld [1] have used Siegel's computation in the case of Eisenstein series involving a modular symbol to continue a Dirichlet series which is the twist of a Hecke L-function by a modular symbol. Though we do not pursue it here, it is reasonable to expect that the computations in this paper may allow one to generalize this construction to higher rank groups.

The remainder of this paper is organized as follows. Section 1 explains how one may generalize the classical hyperbolic Fourier expansion to a toroidal Fourier expansion of an automorphic form on $\mathrm{GL}(n)$. Section 2 introduces the maximal parabolic Eisenstein series on $\mathrm{GL}(n)$ of type $(n - 1, 1)$; in Section 3 the toroidal Fourier coefficients of this Eisenstein series are expressed in terms of Hecke L-functions for K. For the remaining sections, we suppose that K is totally real. In Section 4, we introduce a family of Dirichlet series which are functions of $n - 1$ positive variables y_i as well as the complex parameter s. We show that these functions have Fourier expansions in the y_i similar to the toroidal Fourier expansion above, compute their Fourier coefficients, and show that they have continuation in s. In Section 5 we obtain a formula

(Proposition 5.1) for the quantity r_a, the natural estimate for the number of totally positive integers in a given fractional ideal $\mathfrak{a}$ which are of trace a, using geometric methods. Section 6 begins our study of the Dirichlet series $\varphi(s)$. Combining the geometry and the analysis of the previous two sections, we show that $\varphi(s)$ is holomorphic for $\Re(s) > 0$. In the last two sections we carry out an analysis of the growth of $\varphi(s)$ in vertical strips in $\Re(s) > 0$. Section 7 contains some geometric preliminaries concerning the number of points in the intersection of a lattice with a certain family of compact polyhedra. In Section 8 we use the Fourier expansion of Section 4 together with Stirling's formula and the geometric information of Section 7 in order to obtain an estimate on the growth of $\varphi(s)$. From this information we obtain the main theorem above.

1 Toroidal Fourier Expansions on GL(n)

In this section we develop a Fourier expansion on GL(n) which is the analogue of the classical hyperbolic expansion on GL(2). We describe this as a toroidal Fourier expansion to indicate that it is separate from the usual Fourier expansion which arises in the theory of automorphic forms.

Let K be a number field, $[K:\mathbb{Q}] = n$, with r_1 real embeddings $\sigma_k\colon K \to \mathbb{R}$, $1 \le k \le r_1$, and $2r_2$ complex embeddings $\sigma_{r_1+k}\colon K \to \mathbb{C}$, $1 \le k \le 2r_2$. Set $r = r_1 + r_2 - 1$, and suppose $r > 0$. Order the complex embeddings so that $\sigma_{r_1+2j} = \overline{\sigma}_{r_1+2j-1}$ for $1 \le j \le r_2$; then the archimedean places of K are indexed by the set $\mathfrak{I} = \{k \mid 1 \le k \le r_1 \text{ or } k = r_1 + 2j - 1, 1 \le j \le r_2\}$. For $\alpha \in K$ and $1 \le k \le n$, let $\alpha^{(k)} = \sigma_k(\alpha)$, and let

$$\widehat{\alpha}^{(k)} = \begin{cases} \sigma_k(\alpha) & 1 \le k \le r_1, \\ (a_k, b_k) & \text{where } \sigma_k(\alpha) = a_k + b_k i, \quad k = r_1+1, r_1+3, \dots, n-1, \end{cases}$$

$$\widetilde{\alpha}^{(k)} = \begin{cases} \sigma_k(\alpha) & 1 \le k \le r_1, \\ \begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix} & \text{where } \sigma_k(\alpha) = a_k + b_k i, \quad k = r_1+1, r_1+3, \dots, n-1. \end{cases}$$

Let $\epsilon_1, \dots, \epsilon_r$ be units which together with the roots of unity in K generate the full group of units in the ring of integers of K. Let $\mathfrak{b}$ be a fractional ideal of K with $\mathbb{Z}$-basis $\omega_1, \dots, \omega_n$, and let ϕ denote the $n \times n$ matrix $\phi = (\widehat{\omega}_i^{(j)})$, $1 \le i \le n$, $j \in \mathfrak{I}$. Then

$$|\det \phi| = (\operatorname{disc} \mathfrak{b})^{1/2}\, 2^{-r_2}, \tag{1.1}$$

where $\operatorname{disc}\mathfrak{b}$ denotes the absolute discriminant. There are matrices $\rho_\ell \in \mathrm{GL}(n, \mathbb{Z})$, $1 \le \ell \le r$, such that $\rho_\ell \phi = \phi E_\ell$ where $E_\ell \in \mathrm{GL}(n, \mathbb{R})$ is given by

$$E_\ell = \operatorname{diag}\big(\widetilde{\epsilon}_\ell^{(1)}, \dots, \widetilde{\epsilon}_\ell^{(r_1)}, \widetilde{\epsilon}_\ell^{(r_1+1)}, \widetilde{\epsilon}_\ell^{(r_1+3)}, \dots, \widetilde{\epsilon}_\ell^{(n-1)}\big).$$

Indeed, ϕE_ℓ is of the form $(\widehat{\beta}_i^{(j)})$, $1 \le i \le n$, $j \in \mathfrak{I}$ with $\beta_i = \omega_i \epsilon_\ell \in \mathfrak{b}$, and the matrix ρ_ℓ gives the β_i in terms of the basis $\{\omega_j\}$.

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Let $F(g)$ be an automorphic form on $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, not necessarily cuspidal. For simplicity let us suppose that in fact $F(g)$ has trivial central character, is K-fixed, and is of level 1. Let $H = \mathrm{GL}(n, \mathbb{R})/ZO(n)$ be the symmetric space of $\mathrm{GL}(n, \mathbb{R})$; here Z denotes the center consisting of scalar matrices and $O(n)$ is the real orthogonal group. By the Iwasawa decomposition, each coset is represented by an upper triangular matrix τ with (n, n)-entry 1. We write the natural action of $\mathrm{GL}(n, \mathbb{R})$ on H as $\gamma \circ \tau$. Then F corresponds to a function $f \colon H \to \mathbb{C}$ such that $f(\gamma \circ \tau) = f(\tau)$ for all $\gamma \in \mathrm{GL}(n, \mathbb{Z})$.

Let T be the torus in H consisting of $y = \mathrm{diag}(y_1, \dots, y_{n-1}, 1)$ such that (if $r_2 > 0$) $y_{r_1+2j} = y_{r_1+2j-1}$ for $1 \le j \le r_2$. Let $X = \phi \circ T$. Then X is a totally geodesic subspace of H. Moreover, the projection $X_{\mathfrak{b}}$ of X to $\mathrm{GL}(n, \mathbb{Z})\backslash H$ is independent of the choice of $\mathbb{Z}$-basis for $\mathfrak{b}$, since a change of basis corresponds to left translation of ϕ by an element of $\mathrm{GL}(n, \mathbb{Z})$. From now on we choose a $\mathbb{Z}$-basis for $\mathfrak{b}$ with $\omega_n = 1$. Write $x \in X$ as $x = \phi \circ y$ with $y = \mathrm{diag}(y_1, y_2, \dots, y_{n-1}, 1) \in T$. Let $x^*_\ell = \rho_\ell \circ x$. Then $x^*_\ell = \rho_\ell \phi \circ y = \phi E_\ell \circ y = \phi \circ y^*_\ell$ where $y^*_\ell = E_\ell \circ y$. Note that if $a + bi = re^{i\theta}$, then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} t & \\ & t \end{pmatrix} = \begin{pmatrix} rt & \\ & rt \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Since H consists of left $O(n)$ cosets, y^*_ℓ is again in T, so that $x^*_\ell \in X$, and this action is given by multiplicative translation:

$$y^*_\ell = E_\ell \circ y = \mathrm{diag}\left(\left| \frac{\epsilon_\ell^{(1)}}{\epsilon_\ell^{(n)}} \right| y_1, \left| \frac{\epsilon_\ell^{(2)}}{\epsilon_\ell^{(n)}} \right| y_2, \dots, \left| \frac{\epsilon_\ell^{(n-1)}}{\epsilon_\ell^{(n)}} \right| y_{n-1}, 1 \right).$$

As we shall now explain, this implies that f restricted to $X_{\mathfrak{b}}$ has a Fourier expansion.

Let $V \cong \mathbb{R}^r$ be the vector space of $1 \times n$ vectors with last entry 0 and (if $r_2 > 0$) equal entries in positions $r_1 + 2j - 1$ and $r_1 + 2j$ for $1 \le j \le r_2$. Then there is an isomorphism of V with T given by $e^v := \mathrm{diag}(e^{v_1}, \dots, e^{v_n})$, and $X = \{\phi \circ e^v \mid v \in V\}$. Let $\eta_\ell \in V$ be the vectors

$$\eta_\ell = \left(\log|\epsilon_\ell^{(1)}| - \log|\epsilon_\ell^{(n)}|, \log|\epsilon_\ell^{(2)}| - \log|\epsilon_\ell^{(n)}|, \dots, \log|\epsilon_\ell^{(n-1)}| - \log|\epsilon_\ell^{(n)}|, 0 \right).$$

Let $\Lambda = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2 + \cdots + \mathbb{Z}\eta_r$, and let $\langle\ ,\ \rangle$ be the inner product on V induced from the usual Euclidean inner product on $\mathbb{R}^n$. Thus for $\mu, \nu \in V$

$$\langle \mu, \nu \rangle = \sum_{j=1}^{r_1} \mu_j \nu_j + 2 \sum_{j=1}^{r_2 - 1} \mu_{r_1+2j-1} \nu_{r_1+2j-1}. \tag{1.2}$$

The volume of V/Λ with respect to the measure induced by this inner product is given by nR, where R is the regulator of K. Let

$$\Lambda^* = \{\mu \in V \mid \langle \eta, \mu \rangle \in \mathbb{Z} \text{ for all } \eta \in \Lambda\}$$

be the dual lattice of Λ with respect to $\langle\ ,\ \rangle$.

Now the function $f(\tau)$ is invariant under $\tau \mapsto \gamma \circ \tau$. Apply this when $\gamma = \rho_\ell$, $1 \le \ell \le n - 1$, and with τ restricted to X. Then the computation above shows that $f(\phi \circ e^v)$, regarded as a function of v, is periodic with period lattice Λ. Thus it has a Fourier expansion in v. We record this as the following proposition.

Proposition 1.1 *Let $f(\tau)$ be automorphic with respect to* $\mathrm{GL}(n,\mathbb{Z})$*. Then f restricted to $X_{\mathfrak{b}}$ has a Fourier expansion*

$$f(\phi \circ e^{\nu}) = \sum_{\mu \in \Lambda^*} a_\mu \, e^{2\pi i \langle \nu, \mu \rangle},$$

where the Fourier coefficients are given by

$$a_\mu = \frac{1}{nR} \int_{V/\Lambda} f(\phi \circ e^{\nu}) \, e^{-2\pi i \langle \nu, \mu \rangle} \, d\nu.$$

2 Maximal Parabolic Eisenstein Series

We will compute the Fourier expansion of Proposition 1.1 for the maximal parabolic Eisenstein series. To define these, let P be the standard maximal parabolic subgroup of $\mathrm{GL}(n)$ with Levi decomposition $\mathrm{GL}(n-1) \times \mathrm{GL}(1)$. Let $E(\tau, s)$ be the Eisenstein series given for $\mathrm{Re}(s) \gg 0$ by

$$E(\tau, s) = \sum_{\gamma \in P \cap \mathrm{GL}(n,\mathbb{Z}) \backslash \mathrm{GL}(n,\mathbb{Z})} \det(\gamma \circ \tau)^s.$$

Let $\widetilde{E}(\tau, s) = 2\zeta(ns) E(\tau, s)$. The map sending a coset of $P \cap \mathrm{GL}(n,\mathbb{Z}) \backslash \mathrm{GL}(n,\mathbb{Z})$ to its bottom row establishes a one-to-one correspondence between this quotient space and the set of relatively prime n-tuples of integers whose first non-zero entry is positive. Moreover, a computation with the scalar matrices (or see (3.3) below) shows that

$$\det(\mathrm{diag}(1, \dots, 1, m) \circ \tau) = m^{1-n} \det \tau.$$

For a relatively prime n-tuple of integers $a_1, \dots, a_n$, let $\gamma_{a_1,\dots,a_n}$ denote a matrix in $\mathrm{GL}(n,\mathbb{Z})$ with this bottom row. Then we may write

$$\widetilde{E}(\tau, s) = \sum_{m>0} \sum_{a} m^{-s} \det(\mathrm{diag}(1, \dots, 1, m) \gamma_{a_1,\dots,a_n} \circ \tau)^s \tag{2.1}$$

where the sum is over positive integers m and over all relatively prime n-tuples of integers $a = (a_1, \dots, a_n)$.

3 The Toroidal Fourier Expansion of the Maximal Parabolic Eisenstein Series

Let N denote the absolute value of the norm map from K to $\mathbb{Q}$, d denote the absolute discriminant of $K/\mathbb{Q}$, w denote the number of roots of unity in K, and recall that R denotes the regulator of $K/\mathbb{Q}$. Let $\mu = (\mu_j) \in \Lambda^*$. Define a Hecke character χ_μ as follows. On the principal ideals (β), define

$$\chi_\mu((\beta)) = \prod_{j=1}^{n-1} \left| \frac{\beta^{(n)}}{\beta^{(j)}} \right|^{-2\pi i \mu_j}.$$

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Note that since $\mu \in \Lambda^*$, this definition is independent of the choice of generator for the ideal (β). Then χ_μ may be extended to all ideals. Such an extension is obtained by writing the ideal class group as a direct product of cyclic subgroups, choosing generators for these subgroups, and if the generator $\mathfrak{m}$ has order ℓ in the ideal class group, $\mathfrak{m}^\ell = (\beta)$, choosing $\chi_\mu(\mathfrak{m})$ to be an ℓ-th root of $\chi_\mu((\beta))$. Let A be the integral ideal class of $\mathfrak{b}^{-1}$ in the wide sense. Then the partial Hecke L-function attached to χ_μ and the ideal class A is given by

$$L(s, \chi_\mu, A) = \sum_{\mathfrak{a} \in A} \chi_\mu(\mathfrak{a})\, N(\mathfrak{a})^{-s} = \frac{(N\mathfrak{b})^s}{\chi_\mu(\mathfrak{b})} \sum_{\mathfrak{b} | (\beta) \neq (0)} \chi_\mu((\beta)) N(\beta)^{-s}. \tag{3.1}$$

(The adjective "partial" here is used to indicate that the sum is restricted to ideals in a fixed ideal class, not that a finite number of places are removed from an Euler product.) In this section we shall prove that each Fourier coefficient of the Eisenstein series $\widetilde{E}(z, s)$ is such a partial Hecke L-function. More precisely, we have the following.

Theorem 3.1 *Let A be the integral ideal class of $\mathfrak{b}^{-1}$ in the wide sense, and let $\mu \in \Lambda^*$. Then the Fourier coefficient $a_\mu(s)$ of $\widetilde{E}(z, s)$ restricted to $X_\mathfrak{b}$ is given by*

$$a_\mu(s) = \frac{w2^{-r}}{nR}\, d^{s/2}\, 2^{-r_2 s}\, \Gamma_\mu(s) \Gamma\Big(\frac{ns}{2}\Big)^{-1} \chi_\mu(\mathfrak{b}) L(s, \chi_\mu, A),$$

where

$$\Gamma_\mu(s) = \Gamma\Big(\frac{s}{2} + \pi i \sum_{j=1}^{n-1} \mu_j\Big) \prod_{j=1}^{n-1} \Gamma\Big(\frac{s}{2} - \pi i \mu_j\Big)$$

if K is totally real, and

$$\Gamma_\mu(s) = \Gamma\Big(s + \pi i \sum_{j=1}^{n-2} \mu_j\Big) \prod_{j=1}^{r_1} \Gamma\Big(\frac{s}{2} - \pi i \mu_j\Big) \prod_{j=1}^{r_2 - 1} \Gamma(s - 2\pi i \mu_{r_1 + 2j - 1})$$

otherwise.

Proof To compute the Fourier expansion, we begin as follows. Let $\gamma \in \mathrm{GL}(n, \mathbb{R})$ have bottom row $(b_1, \dots, b_n)$, and let $y = \operatorname{diag}(y_1, y_2, \dots, y_{n-1}, 1)$. Then $\gamma y = \tau k(rI_n)$ for some $\tau \in H$, $k \in O(n)$, and scalar $r > 0$, where I_n denotes the $n \times n$ identity matrix. Comparing norms of the bottom rows, we see that

$$b_1^2 y_1^2 + \cdots + b_{n-1}^2 y_{n-1}^2 + b_n^2 = r^2. \tag{3.2}$$

It follows that

$$\det(\gamma \circ y) = \det(\tau) = |\det(\gamma)| \det(y) r^{-n}. \tag{3.3}$$

Note that this quantity depends in γ only on the bottom row of γ and on $|\det(\gamma)|$.

We apply this computation to compute the Fourier coefficients $a_\mu(s)$. Recall that we write $y = \operatorname{diag}(y_1, \dots, y_n) \in T$ as $y = e^v$, so that v is given by

$$v = (\log y_1, \dots, \log y_{n-1}, 0).$$

It suffices to establish the result for $\operatorname{Re}(s) \gg 0$. Then, by (2.1) and (3.3), we have

$$\begin{aligned} a_\mu(s) &= \frac{1}{nR} \int_{V/\Lambda} \widetilde{E}(\phi \circ e^v, s)\, e^{-2\pi i \langle v,\mu\rangle}\, dv \\ &= \frac{1}{nR} \int_{V/\Lambda} \sum_{m>0} \sum_a m^{-s} \det(\operatorname{diag}(1, \dots, 1, m)\gamma_{a_1,\dots,a_n}\phi \circ y)^s\, e^{-2\pi i \langle v,\mu\rangle}\, dv \\ &= \frac{1}{nR} \int_{V/\Lambda} \sum_{\beta \in \mathfrak{b}-\{0\}} |\det \phi|^s \det(\gamma_\beta \circ y)^s\, e^{-2\pi i \langle v,\mu\rangle}\, dv. \end{aligned}$$

Here, for $\beta \in \mathfrak{b}$, $\gamma_\beta \in \mathrm{SL}(n, \mathbb{R})$ denotes a matrix with bottom row $(\widehat{\beta}^{(j)})$, $j \in \mathfrak{I}$, and the last equality holds since each $\beta \in \mathfrak{b}$ may be uniquely written as a sum $\beta = \sum_{i=1}^n m a_i \omega_i$ with m a positive integer and the a_i relatively prime integers. Now using (1.1), (3.2), and (3.3) we obtain

$$a_\mu(s) = \frac{1}{nR} \operatorname{disc}(\mathfrak{b})^{s/2}\, 2^{-r_2 s} \int_{V/\Lambda} \sum_{\beta \in \mathfrak{b}-\{0\}} \Big(\frac{y_1 \cdots y_{n-1}}{(\sum_{j \in \mathfrak{I}} |\beta^{(j)}|^2 y_j^2)^{n/2}} \Big)^s e^{-2\pi i \langle v,\mu\rangle}\, dv.$$

It is convenient to write the integral in terms of the y_i. If $\mu = (\mu_j) \in \Lambda^*$, then

$$e^{-2\pi i \langle v,\mu\rangle} = \prod_{j=1}^{n-1} y_j^{-2\pi i \mu_j}.$$

(The extra factor of 2 in (1.2) appearing at the complex places is accounted for by recalling that $y_{r_1+2j} = y_{r_1+2j-1}$ for $1 \le j \le r_2$.) Let $\mathfrak{J}$ be the index set $\mathfrak{I}$ with the largest index removed. Then $\mathfrak{J}$ indexes the distinct nonzero coordinates of V, and $dv_j = dy_j/y_j$ for $j \in \mathfrak{J}$. Then we obtain

$$\begin{aligned} a_\mu(s) = \frac{1}{nR} &\operatorname{disc}(\mathfrak{b})^{s/2}\, 2^{-r_2 s} \\ &\times \int_{\exp V/\exp\Lambda} \sum_{\beta \in \mathfrak{b}-\{0\}} \Big(\frac{y_1 \cdots y_{n-1}}{(\sum_{j \in \mathfrak{I}} |\beta^{(j)}|^2 y_j^2)^{n/2}} \Big)^s \prod_{j=1}^{n-1} y_j^{-2\pi i \mu_j} \prod_{j \in \mathfrak{J}} \frac{dy_j}{y_j}. \end{aligned}$$

Fixing a choice of fundamental domain for $\exp V/\exp\Lambda$ (by abuse of notation, we will continue to use the quotient notation), we may pull the sum over β outside the integral. Introduce new variables w_j by

$$w_j = \Big| \frac{\beta^{(j)}}{\beta^{(n)}} \Big| y_j. \tag{3.4}$$

Then the terms in the integrand which depend on β factor out of the integral, and we obtain

$$(3.5)\quad a_\mu(s) = \frac{1}{nR}\operatorname{disc}(\mathfrak{b})^{s/2}\,2^{-r_2 s} \times \sum_{\beta\in\mathfrak{b}-\{0\}} N(\beta)^{-s}\prod_{j=1}^{n-1}\left|\frac{\beta^{(n)}}{\beta^{(j)}}\right|^{-2\pi i\mu_j}\int_{R_\beta}\frac{\prod_{j=1}^{n-1} w_j^{s-2\pi i\mu_j}}{(1+\sum_{j\in\mathcal{J}} w_j^2)^{ns/2}}\prod_{j\in\mathcal{J}}\frac{dw_j}{w_j},$$

where R_β is a domain in $\exp V$ depending on β which we shall discuss further below.

Next, we group the terms in the sum which give the same principal ideal (β). We have $(\beta) = (\beta_1)$ if and only if $\beta_1 = \epsilon\beta$ for some unit ϵ. However, the factor in front of the integral in (3.5) is the same for β as for β_1. For it suffices to establish this when $\epsilon = \iota\epsilon_k$ for some k, $1 \le k \le r$, where ι is a root of unity in K. In that case, certainly $N(\beta)^{-s} = N(\beta_1)^{-s}$, while upon replacing β by β_1, the term $\prod_{j=1}^{n-1}\left|\frac{\beta^{(n)}}{\beta^{(j)}}\right|^{-2\pi i\mu_j}$ changes by a factor of

$$\prod_{j=1}^{n-1}\left|\frac{\epsilon_k^{(n)}}{\epsilon_k^{(j)}}\right|^{-2\pi i\mu_j} = e^{-2\pi i\left(\sum_{j=1}^{n-1}\mu_j(\log|\epsilon_k^{(n)}|-\log|\epsilon_k^{(j)}|)\right)} = e^{2\pi i\langle\eta_k,\mu\rangle},$$

and this is 1 since $\mu \in \Lambda^*$ and hence $\langle\eta_k,\mu\rangle \in \mathbb{Z}$.

We now observe that as we collect the terms which give the same principal ideal (β), the domains R_β join up without overlap to give the entire region $\mathbb{R}_+^{n-1}$ exactly w times. To check this, it is easiest to go back to the logarithmic picture. Before the variable change (3.4) the integration is taken over a region R such that $\log R$ is a fundamental domain for the lattice Λ in V (see Section 1). The region $\log R_\beta$ is obtained by shifting $\log R$ by the vector whose j-th entry is $\log|\beta^{(j)}/\beta^{(n)}|$. Now if $\beta_1 = \epsilon\beta$ with $\epsilon = \iota\prod_{j=1}^{r}\epsilon_j^{m_j}$ and ι a root of unity in K, then

$$\log R_{\beta_1} = \log R_\beta + \sum_{j=1}^{r} m_j\eta_j.$$

Since the lattice Λ is the $\mathbb{Z}$-span of the η_j, $1 \le j \le r$, we see that the domains $\log R_\beta$ which give the same principal ideal (β) do indeed fill out V. In fact, they fill out this region exactly w times, due to the roots of unity ι.

Thus we arrive at the formula

$$a_\mu(s) = \frac{w}{nR}\operatorname{disc}(\mathfrak{b})^{s/2}\,2^{-r_2 s}\sum_{\mathfrak{b}|(\beta)\neq(0)} N(\beta)^{-s}\prod_{j=1}^{n-1}\left|\frac{\beta^{(n)}}{\beta^{(j)}}\right|^{-2\pi i\mu_j} I_\mu(s),$$

where $I_\mu(s)$ is the integral

$$I_\mu(s) = \int_{\mathbb{R}_+^r}\frac{\prod_{j=1}^{n-1} w_j^{s-2\pi i\mu_j}}{(1+\sum_{j\in\mathcal{J}} w_j^2)^{ns/2}}\prod_{j\in\mathcal{J}}\frac{dw_j}{w_j}.$$

In this integral we recall that $w_{r_1+2j} = w_{r_1+2j-1}$ for $1 \leq j \leq r_2$ (so in particular if $r_2 > 0$, then $w_{n-1} = 1$). Using (3.1) and recalling that $\operatorname{disc}(\mathfrak{b}) = N(\mathfrak{b})^2 d$, we obtain the formula

$$a_\mu(s) = \frac{w}{nR} d^{s/2}\, 2^{-r_2 s} \chi_\mu(\mathfrak{b}) L(s, \chi_\mu, A) I_\mu(s).$$

To complete the proof of Theorem 3.1, it remains to evaluate $I_\mu(s)$. We have

$$\Gamma\Big(\frac{ns}{2}\Big) I_\mu(s) = \int_{\mathbb{R}_+^{r_1+r_2}} t^{ns/2} \Big(\prod_{j=1}^{n-1} w_j^{s-2\pi i \mu_j}\Big) e^{-t(1+\sum_{j\in\mathcal{J}} w_j^2)} \prod_{j\in\mathcal{J}} \frac{dw_j}{w_j} \frac{dt}{t}.$$

Changing variables $w_j \mapsto (w_j/t)^{1/2}$, $j \in \mathcal{J}$, gives

$$\Gamma\Big(\frac{ns}{2}\Big) I_\mu(s) = 2^{-r} \int_{\mathbb{R}_+^{r_1+r_2}} t^{\delta s/2 + \pi i \sum_{j=1}^{n-1} \mu_j} \Big(\prod_{j=1}^{n-1} w_j^{s/2-\pi i \mu_j}\Big) e^{-t-\sum_{j\in\mathcal{J}} w_j} \prod_{j\in\mathcal{J}} \frac{dw_j}{w_j} \frac{dt}{t},$$

where $\delta = 1$ if $r_2 = 0$ (*i.e.*, K is totally real) and $\delta = 2$ otherwise. Each integral is a Gamma function. Thus we obtain

$$\Gamma\Big(\frac{ns}{2}\Big) I_\mu(s) = 2^{-r}\Gamma\Big(\frac{s}{2} + \pi i \sum_{j=1}^{n-1} \mu_j\Big) \prod_{j=1}^{n-1} \Gamma\Big(\frac{s}{2} - \pi i \mu_j\Big)$$

if $r_2 = 0$, and

$$\Gamma\Big(\frac{ns}{2}\Big) I_\mu(s) = 2^{-r}\Gamma(s + \pi i \sum_{j=1}^{n-2} \mu_j) \prod_{j=1}^{r_1} \Gamma\Big(\frac{s}{2} - \pi i \mu_j\Big) \prod_{j=1}^{r_2-1} \Gamma(s - 2\pi i \mu_{r_1+2j-1})$$

if $r_2 > 0$ (recall that $\mu_{n-1} = 0$ in this case). This completes the proof of Theorem 3.1. ■

4 Dirichlet Series Constructed from Totally Real Number Fields

Suppose from now on that the field K is totally real. In this section we introduce and analyze the Fourier coefficients of a family of Dirichlet series constructed from K. These coefficients are once again related to Hecke L-functions, and the computation is effected using the same methods as in Section 3 above. In the next sections we shall show how these series may be used to get information about the growth of arithmetic quantities. (Similar Dirichlet series may be studied in the non-totally-real case, but as our application makes use of the totally real condition, we restrict to this case for convenience.)

Let $\mathcal{O}_K$ denote the integers of K and let U be a subgroup of the units of $\mathcal{O}_K$ which is of finite index in the full group of units $\mathcal{O}_K^\times$ and which contains -1. Let $\mathfrak{b}$ be a fractional ideal. Let p be a function on $\mathfrak{b}$ such that $p(\alpha u) = p(\alpha)$ for all $\alpha \in \mathfrak{b}$, $u \in U$, and such that for some δ, $|p(\alpha)| = O(N(\alpha)^\delta)$ for $\alpha \in \mathfrak{b}$. Let $k \geq 1$ be a fixed integer. Let $y_1, \dots, y_{n-1}$ be positive real variables, $y = \operatorname{diag}(y_1, \dots, y_{n-1}, 1)$, and let s be complex. For notational convenience let $y_n = (y_1 \cdots y_{n-1})^{-1}$. We introduce the series

$$\Phi(s, y; p, k, \mathfrak{b}) = \sum_{0 \neq \alpha \in \mathfrak{b}} \frac{p(\alpha)}{\left(\sum_{i=1}^{n-1} |\alpha^{(i)}|^k y_i^k y_n^{k/n} + |\alpha^{(n)}|^k y_n^{k/n}\right)^s}. \tag{4.1}$$

Here $y_n^{k/n}$ is the positive n-th root. Since

$$\sum_{i=1}^{n-1} |\alpha^{(i)}|^k y_i^k y_n^{k/n} + |\alpha^{(n)}|^k y_n^{k/n} \geq c(y)\, N(\alpha)^{k/n}$$

where $c(y) = n \min(y_1^k y_n^{k/n}, \ldots, y_{n-1}^k y_n^{k/n}, y_n^{k/n})$, this series converges absolutely for $\Re(s) > n(1+\delta)/k$.

Let u_ℓ, $1 \leq \ell \leq n-1$, be units which together with $\{\pm 1\}$ generate U, and for each such unit let

$$u_\ell^* = \operatorname{diag}\left(\left| \frac{u_\ell^{(1)}}{u_\ell^{(n)}} \right|, \left| \frac{u_\ell^{(2)}}{u_\ell^{(n)}} \right|, \cdots, \left| \frac{u_\ell^{(n-1)}}{u_\ell^{(n)}} \right|, 1 \right).$$

Then $\Phi(s, u_\ell^* y; p, k, \mathfrak{b}) = \Phi(s, y; p, k, \mathfrak{b})$. Thus Φ has a Fourier expansion. To give this, let V be the vector space of Section 1, and let Λ_U be the lattice in V spanned by the vectors

$$\lambda_\ell = \left(\log |u_\ell^{(1)}| - \log |u_\ell^{(n)}|, \log |u_\ell^{(2)}| - \log |u_\ell^{(n)}|, \ldots, \log |u_\ell^{(n-1)}| - \log |u_\ell^{(n)}|, 0 \right).$$

Then V/Λ_U is compact (hence the convergence in (4.1) is uniform in y), and Λ_U has covolume cnR, where c is the index of U in the full group of units. Let Λ_U^* be the dual lattice to Λ_U in V with respect to the standard inner product. For $v \in V$, recall that $e^v = \operatorname{diag}(e^{v_1}, \ldots, e^{v_{n-1}}, 1)$. Then

$$\Phi(s, e^v; p, k, \mathfrak{b}) = \sum_{\mu \in \Lambda_U^*} a_\mu(s; p, k, \mathfrak{b}) e^{2\pi i \langle v, \mu \rangle},$$

where

$$a_\mu(s; p, k, \mathfrak{b}) = \frac{1}{cnR} \int_{V/\Lambda_U} \Phi(s, e^v; p, k, \mathfrak{b})\, e^{-2\pi i \langle v, \mu \rangle}\, dv.$$

To compute these Fourier coefficients, we proceed as in Section 3 above. We have

$$a_\mu(s; p, k, \mathfrak{b}) =$$

$$\frac{1}{cnR} \int_{\exp V/\exp \Lambda_U} \sum_{0 \neq \alpha \in \mathfrak{b}} \frac{p(\alpha)}{\left(\sum_{i=1}^{n-1} |\alpha^{(i)}|^k y_i^k y_n^{k/n} + |\alpha^{(n)}|^k y_n^{k/n}\right)^s} \prod_{j=1}^{n-1} y_j^{-2\pi i \mu_j} \frac{dy_j}{y_j}.$$

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Fixing a fundamental domain for $\exp V / \exp \Lambda_U$, pulling the sum over α outside the integral, and introducing variables $w_j = |\alpha^{(j)}/\alpha^{(n)}|y_j$, $1 \leq j \leq n-1$ (compare (3.4)), we obtain

$$a_\mu(s; p, k, \mathfrak{b}) = \frac{1}{cnR} \sum_{0\neq\alpha\in\mathfrak{b}} N(\alpha)^{-ks/n} p(\alpha) \prod_{j=1}^{n-1} \Big| \frac{\alpha^{(n)}}{\alpha^{(j)}} \Big|^{-2\pi i\mu_j} \times$$

$$\int_{R_\alpha} \frac{\prod_{j=1}^{n-1} w_j^{ks/n-2\pi i\mu_j}}{\left(\sum_{i=1}^{n-1} w_i^k + 1\right)^s} \frac{dw_1}{w_1} \cdots \frac{dw_{n-1}}{w_{n-1}},$$

where R_α is a domain in $\exp V$ depending on α.

Call two nonzero field elements *associate modulo U* if their quotient is in U. Denote this equivalence relation $\sim$. In this sum, we may group the terms corresponding to α that are associate modulo U. Indeed, the term in front is unchanged if we replace α by αu for $u \in U$. This is verified as in Section 3. Collecting terms, the domains R_α for α which are associate modulo U join up without overlap to give the entire region $\mathbb{R}_+^{n-1}$ exactly twice. We arrive at the formula

$$a_\mu(s; p, k, \mathfrak{b}) = \frac{2}{cnR} \sum_{0\neq\alpha\in\mathfrak{b}/\sim} N(\alpha)^{-ks/n} p(\alpha) \prod_{j=1}^{n-1} \Big| \frac{\alpha^{(n)}}{\alpha^{(j)}} \Big|^{-2\pi i\mu_j} I_\mu(s; k)$$

where

$$I_\mu(s; k) = \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{j=1}^{n-1} w_j^{ks/n-2\pi i\mu_j}}{\left(\sum_{i=1}^{n-1} w_i^k + 1\right)^s} \frac{dw_1}{w_1} \cdots \frac{dw_{n-1}}{w_{n-1}}.$$

(Note that the integral $I_\mu(s)$ of Section 3 arises when $k = 2$: $I_\mu(s) = I_\mu(ns/2; 2)$.)

The integral $I_\mu(s; k)$ is evaluated similarly to Section 3: multiply by $\Gamma(s)$ to introduce an additional integration in a variable t and then change variables $w_j \mapsto (w_j/t)^{1/k}$. This gives

$$\Gamma(s)I_\mu(s; k) = k^{-(n-1)}\Gamma\Big(\frac{s}{n} + \frac{2\pi i}{k}\sum_{j=1}^{n-1}\mu_j\Big) \prod_{j=1}^{n-1}\Gamma\Big(\frac{s}{n} - \frac{2\pi i\mu_j}{k}\Big).$$

Suppose now that $p(\alpha u) = \kappa(u)\, p(\alpha)$ for all $\alpha \in \mathfrak{b}$ and $u \in \mathcal{O}_K^\times$, where κ is a character of $\mathcal{O}_K^\times/U$. Then $a_\mu(s; p, k, \mathfrak{b}) = 0$ unless

$$\kappa(u) \prod_{j=1}^{n-1} \Big| \frac{u^{(n)}}{u^{(j)}} \Big|^{-2\pi i\mu_j} = 1 \tag{4.2}$$

for all $u \in \mathcal{O}_K^\times$, and in this case we have

$$a_\mu(s; p, k, \mathfrak{b}) = \frac{2}{nR} \sum_{\mathfrak{b}|(\alpha)\neq 0} N(\alpha)^{-ks/n} p(\alpha) \prod_{j=1}^{n-1} \Big| \frac{\alpha^{(n)}}{\alpha^{(j)}} \Big|^{-2\pi i \mu_j} I_\mu(s; k),$$

where the sum is over nonzero principal integral ideals (α) divisible by $\mathfrak{b}$. For p a combination of signs and powers, the sum over (α) may be expressed in terms of the partial Hecke L-function associated to the integral ideal class of $\mathfrak{b}^{-1}$ in the wide ideal class group; compare (3.1).

In particular, let us analyze the functions we shall use for our estimates of integers of given trace. For each i, $1 \leq i \leq n$, choose $e_i = 0$ or 1, and let $\nu(\alpha) = \prod_{i=1}^n \operatorname{sgn}(\alpha^{(i)})^{e_i}$. (As the vector space V does not enter further, we change p to ν to emphasize the analogy to Hecke [3].) For such a ν, we consider the sum $\Phi(s, \mathbf{1}; \nu, 1, \mathfrak{b})$ where $\mathbf{1} = (1, \dots, 1)$, so all $y_i = 1$. Denote this sum $\Psi(s, \nu, \mathfrak{b})$, that is,

$$\Psi(s, \nu, \mathfrak{b}) = \sum_{0\neq\alpha\in\mathfrak{b}} \frac{\nu(\alpha)}{(|\alpha^{(1)}| + \cdots + |\alpha^{(n)}|)^s}. \tag{4.3}$$

This sum converges for $\Re(s) > n$. This function is 0 unless $\nu(\alpha) = \nu(-\alpha)$. Let U denote the subgroup of units which are either totally positive or totally negative. Let ν_0 correspond to $e_i = 0$ for all i. Then we shall show the following.

Proposition 4.1 *Each function $\Psi(s, \nu, \mathfrak{b})$ has meromorphic continuation to the right half plane $\Re(s) > 0$. The functions $\Psi(s, \nu, \mathfrak{b})$ are holomorphic in this right half plane for $\nu \neq \nu_0$, while $\Psi(s, \nu_0, \mathfrak{b})$ has a simple pole at $s = n$ of residue*

$$\frac{2^n}{(n-1)!\ \operatorname{disc}(\mathfrak{b})^{1/2}}.$$

Proof By the computation above, for $\Re(s) > n$ we have

$$\Psi(s, \nu, \mathfrak{b}) = \sum_{\mu\in\Lambda_U^*} a_\mu(s; \nu, 1, \mathfrak{b}). \tag{4.4}$$

In this sum only μ satisfying (4.2) (with $\kappa = \nu$) contribute. For such μ, let

$$\lambda_{\mu,\nu}(\alpha) = \prod_{j=1}^{n-1} \left| \frac{\alpha^{(n)}}{\alpha^{(j)}} \right|^{-2\pi i \mu_j} \nu(\alpha).$$

Extend $\lambda_{\mu,\nu}$ to a Hecke character as in Section 3. Let $L(s, \lambda_{\mu,\nu}, A)$ be the partial Hecke L-function associated to the integral ideal class A of $\mathfrak{b}^{-1}$, and let

$$\Gamma(s, \lambda_{\mu,\nu}) = (\pi^{-n} d)^{s/2} \Gamma\Big(\frac{s}{2} + \frac{e_n}{2} + \pi i \sum_{j=1}^{n-1} \mu_j\Big) \prod_{j=1}^{n-1} \Gamma\Big(\frac{s}{2} + \frac{e_j}{2} - \pi i \mu_j\Big)$$

be the standard Gamma factor attached to $L(s,\lambda_{\mu,\nu})$. Let

$$L^*(s,\lambda_{\mu,\nu},A)=\Gamma(s,\lambda_{\mu,\nu})\,L(s,\lambda_{\mu,\nu},A)$$

be the partial L-function for the integral ideal class A with Gamma factors included. Then we may express the coefficients in (4.4) in terms of these Hecke L-functions. Indeed, we see, as in (3.1), that

$$a_\mu(s;\nu,1,\mathfrak{b})=\frac{2}{nR}\mathbb{N}(\mathfrak{b})^{-s/n}\lambda_{\mu,\nu}(\mathfrak{b})L\left(\tfrac{s}{n},\lambda_{\mu,\nu},A\right)\,I_\mu(s;1) \tag{4.5}$$

with

$$I_\mu(s;1)=\Gamma(s)^{-1}\Gamma\Big(\frac{s}{n}+2\pi i\sum_{j=1}^{n-1}\mu_j\Big)\prod_{j=1}^{n-1}\Gamma\Big(\frac{s}{n}-2\pi i\mu_j\Big). \tag{4.6}$$

At this point we could analyze the convergence of the sum (4.4) by the use of Stirling's formula, Phragmen–Lindelöf estimates, and geometric arguments. In fact, we shall do so in Section 8 below. However, to prove the proposition a simpler argument suffices, which we now give. Using the duplication formula

$$\Gamma(z)=\Gamma(z/2)\,\Gamma((z+1)/2)\,2^{z-1}\,\pi^{-1/2},$$

we find that

$$a_\mu(s;\nu,1,\mathfrak{b})=\frac{2}{nR}\mathbb{N}(\mathfrak{b})^{-s/n}\lambda_{\mu,\nu}(\mathfrak{b})L^*(\tfrac{s}{n},\lambda_{\mu,\nu},A)\,J_\mu(s)$$

where

$$J_\mu(s)=\pi^{(s-n)/2}\,2^{s-n}\,d^{-s/2n}\times\\ \Gamma(s)^{-1}\Gamma\Big(\frac{s}{2n}+\frac{1-e_n}{2}+\pi i\sum_{j=1}^{n-1}\mu_j\Big)\prod_{j=1}^{n-1}\Gamma\Big(\frac{s}{2n}+\frac{1-e_j}{2}-\pi i\mu_j\Big).$$

Since each function L^* may be represented as the Mellin transform of a suitable theta function, it is easy to see that the functions $L^*(\frac{s}{n},\lambda_{\mu,\nu},A)$ are bounded on vertical strips independently of μ (except for $\nu=\nu_0$, $\mu=0$, where we must stay away from the pole at $s=n$). By Stirling's formula, one sees that the expansion (4.4) converges uniformly for s in a compact set (in fact, lying in a vertical strip of bounded width) provided one avoids the lines of poles coming from the Gamma factors. In particular, $\Psi(s,\nu,\mathfrak{b})$ continues to the right half plane $\Re(s)>0$, as claimed. Moreover, the Gamma factors have no poles for $\Re(s)>0$, so the functions $\Psi(s,\nu,\mathfrak{b})$ are regular in the right half plane except for $\Psi(s,\nu_0,\mathfrak{b})$.

To analyze the residue at $s=n$ of $\Psi(s,\nu_0,\mathfrak{b})$, observe that all the terms in the Fourier expansion of $\Phi(s,y;1,1,\mathfrak{b})$ are holomorphic in the right half plane except for

$a_0(s; 1, 1, \mathfrak{b})$, which is expressed in terms of a partial Dedekind zeta function evaluated at s/n. This function has its only pole in the right half plane at $s = n$. Since $I_0(n; 1) = 1/\Gamma(n)$, the residue is given by

$$\operatorname{Res}_{s=n} \Psi(s, \nu_0, \mathfrak{b}) = \operatorname{Res}_{s=n} a_0(s; 1, 1, \mathfrak{b}) = \frac{2^n}{(n-1)! \ \operatorname{disc}(\mathfrak{b})^{1/2}}.$$

This completes the proof of the proposition. ■

Remark This proof closely follows Hecke [3] who studied the case $n = 2$. However, the case $n > 2$ is different from this case in a fundamental way. To explain this, let us study the series given by (4.4) in the half plane $\Re(s) \leq 0$. From (4.5)and (4.6), it follows that each function $a_\mu(s, \nu, 1, \mathfrak{b})$ has possible poles at

$$s = -2nk + n(e_j - 1) + 2\pi i n\mu_j, \quad 1 \leq j \leq n-1$$

and at

$$s = -2nk + n(e_n - 1) - 2\pi i n \sum_{j=1}^{n-1} \mu_j$$

for each $k = 0, 1, 2, \dots$. When we sum over $\mu \in \Lambda_U^*$, for $n > 2$, these locations are dense on the vertical lines $\Re(s) = -2nk + n(e_j - 1)$, $1 \leq j \leq n$, $k = 0, 1, 2, \dots$, as we show momentarily. Thus it does not seem possible to use this to continue the functions $\Psi(s, \nu, \mathfrak{b})$ to all complex s.

To see that if $n > 2$ the poles are dense, suppose without loss of generality that the units $u_1, \dots, u_{n-1}$ are totally positive. They are also multiplicatively independent. Define the $(n-1) \times (n-1)$ matrix Λ by $\Lambda_{i,j} = \log u_j^{(i)} - \log u_j^{(n)}$, and let $M = \Lambda^{-1}$. Then the entries of M are the coordinates of the μ's which make up a $\mathbb{Z}$-basis of the lattice Λ_U^*. To prove that the poles are dense on any vertical line with real part as indicated, it suffices to show that at least two of these entries are linearly independent over $\mathbb{Q}$, since if $r_1, r_2 \in \mathbb{R}$ with $r_1/r_2 \notin \mathbb{Q}$, then $\{mr_1 + nr_2 \mid m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Suppose not. Then $M = \theta R$ where $\theta \in \mathbb{R}$ and $R \in \mathrm{GL}(n, \mathbb{Q})$. Let E be a matrix in $\mathrm{GL}(n, \mathbb{Q})$ such that premultiplication by E adds all the other rows to the last row. Since for each j, $\sum_{i=1}^n \log u_j^{(i)} = 0$, we see that $E\Lambda$ has bottom row

$$-n(\log u_1^{(n)}, \dots, \log u_{n-1}^{(n)}).$$

Also, since $(E\Lambda)(RE^{-1}) = \theta^{-1} I$, we see that there is a nonzero integral vector

$$(c_1, \dots, c_{n-1})$$

whose dot product with the bottom row of $(E\Lambda)$ is 0. (A suitable integral multiple of any of the columns of RE^{-1} except the last will do.) Then we have $\sum_i c_i \log u_i^{(n)} = 0$. But this implies that $\prod_i u_i^{c_i} = 1$, which contradicts the multiplicative independence of the units u_i. Hence for $n > 2$, the poles are dense.

5 A Geometric Computation

Let $\mathfrak{a}$ denote a fractional ideal in K. Let $\alpha_j \in \mathfrak{a}$, $j = 1, \ldots, n$, be a $\mathbb{Z}$-basis of $\mathfrak{a}$. The absolute trace from K to $\mathbb{Q}$ defines a linear map Tr from $\mathfrak{a}$ to the free abelian subgroup of $\mathbb{Q}$ generated by $k > 0$. (The parameter k here is not the same as in Section 4 above.) Denote the set of elements of $\mathfrak{a}$ with trace 0 by $\mathfrak{a}_0$. Then we may assume that $\operatorname{Tr}(\alpha_1) = k$ and that $\alpha_j \in \mathfrak{a}$, $j = 2, \ldots, n$, is a $\mathbb{Z}$-basis of $\mathfrak{a}_0$.

Let a denote a positive integral multiple of k. In this section, we compute the ratio r_a of two volumes:

(i) the volume of the simplex S_a of totally positive elements of $\mathfrak{a}_0 \otimes \mathbb{R} + \beta$, where $\beta \in \mathfrak{a}$ is a fixed element with trace a;

(ii) the volume of the fundamental cell of the lattice $\mathfrak{a}_0$ in $\mathfrak{a}_0 \otimes \mathbb{R}$.

Since r_a is a ratio, it is independent of the normalization of the volume on $\mathfrak{a}_0 \otimes \mathbb{R}$.

Proposition 5.1 *The ratio r_a is given by*

$$r_a = \frac{ka^{n-1}}{(n-1)!(\operatorname{disc}\mathfrak{a})^{1/2}}.$$

Proof We devote the rest of this section to a proof of this result. The general element $\alpha \in \mathfrak{a}$ can be written as $\alpha = \sum x_j \alpha_j$, and we denote the column vector with coordinates $x_1, \ldots, x_n$ by $\mathbf{x}$. As above, we denote the n embeddings of α into $\mathbb{R}$ by $\alpha^{(i)}$. Let $\underline{\alpha}^{(i)}$ denote the row vector $(\alpha_1^{(i)}, \ldots, \alpha_n^{(i)})$. In $\mathbf{x}$-space, the totally non-negative elements of $\mathfrak{a} \otimes \mathbb{R}$ constitute the cone C^+ which is the intersection of the half-planes $\underline{\alpha}^{(i)}\mathbf{x} \geq 0$, since $\alpha^{(i)} = \sum x_j \alpha_j^{(i)}$. Similarly, since $\operatorname{Tr}\alpha = \sum \alpha^{(i)}$, in $\mathbf{x}$-space the trace becomes the function $t\colon \mathbf{x} \mapsto \sum_{i,j} x_j \alpha_j^{(i)}$.

Let a denote the positive integral multiple of k chosen above. Set Y_a to be the $(n-1)$-dimensional simplex in $\mathbf{x}$-space given by the intersection of the cone C^+ with the hyperplane H_a given by the equation $t(\mathbf{x}) = a$. Let L_a be a fundamental cell for the $(n-1)$-dimensional lattice $\mathbb{Z}^n \cap H_a$ in $\mathbf{x}$-space. Then $r_a = \operatorname{vol}(Y_a)/\operatorname{vol}(L_a)$.

If ξ is a vector in $\mathbf{x}$-space not parallel to H_a and Z is any $(n-1)$-dimensional measurable set in H_a, we denote by $\xi \wedge Z$ the n-dimensional set consisting of the union of $Z + u\xi$, $u \in [0, 1]$. For instance, if Z is a parallelopiped, so is $\xi \wedge Z$. Then $r_a = \operatorname{vol}(\xi \wedge Y_a)/\operatorname{vol}(\xi \wedge L_a)$.

In $\mathbf{x}$-space, the basis α_j becomes the standard basis e_j. We shall choose $\xi = e_1$. Then $\operatorname{vol}(\xi \wedge L_a) = 1$. It remains to compute $\operatorname{vol}(\xi \wedge Y_a)$.

Let Φ denote the $n \times n$ matrix $(\alpha_j^{(i)})$, where i denotes the row and j the column, as usual. Define $\mathbf{y} = \Phi\mathbf{x}$. Then $y_i = \alpha^{(i)}$. In $\mathbf{y}$-space, the cone C^+ becomes the positive "octant", and the trace function becomes $\mathbf{y} \mapsto \sum y_i$. Therefore the simplex Y_a is given in $\mathbf{y}$-space by the hyperplane $\sum y_i = a$ as cut off by the coordinate hyperplanes, so the intercept with the y_j-axis is ae_j. Therefore the simplex in $\mathbf{y}$-space is spanned by the vectors $a(e_2 - e_1), \ldots, a(e_n - e_1)$, and its volume is $1/(n-1)!$ times the volume of the parallelopiped spanned by those $n - 1$ vectors.

Back in $\mathbf{x} = \Phi^{-1}\mathbf{y}$-space, we conclude that

$$r_a = \operatorname{vol}(\xi \wedge Y_a) = \frac{1}{(n-1)!}\det(e_1, z_2, \ldots, z_n)$$

where $z_j = a\Phi^{-1}(e_j - e_1)$, $j = 2, \dots, n$. Thus

$$r_a = \frac{a^{n-1}}{(n-1)!}\det(e_1, \Phi^{-1}(e_2 - e_1), \dots, \Phi^{-1}(e_n - e_1)).$$

Note that

$$\begin{aligned}\det(\Phi^{-1}e_1, \Phi^{-1}(e_2 - e_1), \dots, \Phi^{-1}(e_n - e_1)) &= \det(\Phi^{-1}e_1, \Phi^{-1}e_2, \dots, \Phi^{-1}e_n)\\ &= \det\Phi^{-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\det(e_1, \Phi^{-1}(e_2 - e_1), \dots, &\Phi^{-1}(e_n - e_1)) - \det\Phi^{-1}\\ &= \det(e_1 - \Phi^{-1}e_1, \Phi^{-1}(e_2 - e_1), \dots, \Phi^{-1}(e_n - e_1))\\ &= \det\Phi^{-1}\cdot\det(\Phi e_1 - e_1, e_2 - e_1, \dots, e_n - e_1).\end{aligned}$$

The first column of Φ is $(\alpha_1^{(i)})$, so the last determinant mentioned equals

$$\det\begin{pmatrix}\alpha_1^{(1)} - 1 & -1 & -1 & \dots & -1\\ \alpha_1^{(2)} & 1 & 0 & \dots & 0\\ \alpha_1^{(3)} & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \alpha_1^{(n)} & 0 & 0 & \dots & 1\end{pmatrix}$$

which equals $\mathrm{Tr}(\alpha_1) - 1 = k - 1$. Therefore, using the fact that $\det\Phi = (\operatorname{disc}\mathfrak{a})^{1/2}$, we obtain the evaluation of r_a stated in the proposition. ■

6 Comparing the Residues

As in the preceding section, let $\mathfrak{a}$ denote a fractional ideal in K, and let $\mathrm{Tr}(\mathfrak{a})$ be generated by $k > 0$. If a is a positive integral multiple of k, let N_a denote the number of totally positive elements of $\mathfrak{a}$ with trace a. One may obtain an estimate of N_a by standard methods for counting lattice points in homogeneously expanding domains. Indeed, from Lang [5, Ch. VI, Theorem 2], we find that N_a is approximated by exactly the volume ratio r_a, and from that result we conclude that there exists a constant d such that

$$|N_a - r_a| \le d|a|^{n-2} \tag{6.1}$$

for all such a.

Our goal here is to study the difference $E_a = N_a - r_a$ between N_a and its estimate as a volume. If a is not a positive multiple of k, we set $E_a = 0$. To do so, we study the Dirichlet series

$$\varphi(s) = \sum_{a>0}\frac{E_a}{a^s}.$$

By (6.1), this series converges absolutely for $\Re(s) > n-1$. Writing $a = mk$, (6.1) gives a trivial bound: for $s > n-1$ real,

$$|\varphi(s)| \leq \sum_{m>0} \frac{dk^{n-2}m^{n-2}}{k^s m^s} = dk^{n-2-s}\zeta(s-n+2).$$

The right-hand side has a pole at $s = n-1$. However, we should expect considerable cancelation in the series for $\varphi(s)$. In fact, we shall now show

Proposition 6.1 *The function $\varphi(s)$ is holomorphic for $\Re(s) > 0$.*

Proof For $i = 1, \ldots n$, choose $e_i = 0$ or 1. Then for $\alpha \in \mathfrak{a}$ set

$$\nu(\alpha) = \prod_{i=1}^{n} (\operatorname{sgn}(\alpha^{(i)})^{e_i}).$$

There are 2^n possible ν's. Recall that $\Psi(s, \nu, \mathfrak{a})$ is given by (4.3). Summing over all the ν, we get

$$\sum_{\nu} \Psi(s, \nu, \mathfrak{a}) = 2^n \sum_{0 \ll \alpha \in \mathfrak{a}} \frac{1}{\operatorname{Tr}(\alpha)^s} = 2^n \sum_{a>0} \frac{N_a}{a^s}.$$

Substituting $r_a + E_a$ for N_a, and setting $a = mk$ as above, we obtain

$$\sum_{\nu} \Psi(s, \nu, \mathfrak{a}) = 2^n \frac{k^{n-s}}{(n-1)!(\operatorname{disc} \mathfrak{a})^{1/2}} \zeta(s-n+1) + 2^n \sum_{a>0} \frac{E_a}{a^s}. \tag{6.2}$$

On the other hand, by Proposition 4.1, $\sum_{\nu} \Psi(s, \nu, \mathfrak{a})$ is holomorphic for $\Re(s) > 0$ except for a simple pole at $s = n$ with residue

$$\frac{2^n}{(n-1)!(\operatorname{disc} \mathfrak{a})^{1/2}}.$$

Since the same is true of

$$2^n \frac{k^{n-s}}{(n-1)!(\operatorname{disc} \mathfrak{a})^{1/2}} \zeta(s-n+1),$$

we obtain that the Dirichlet series

$$\varphi(s) = \sum_{a>0} \frac{E_a}{a^s}$$

is holomorphic for $\Re(s) > 0$ as claimed. ∎

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7 Some Geometric Preliminaries

Our next task is to analyze the growth of $\varphi(s)$ in vertical strips. To do this, we will combine Stirling's formula with the Fourier expansion (4.4). We require some information concerning the number of points in the intersection of a lattice with a certain family of compact polyhedra. We address this now.

Let us fix a Euclidean metric on $\mathbb{R}^n$ for this whole section. For $\eta \in \mathbb{R}^n$ and $r > 0$, let $B_r(\eta)$ denote the closed ball around η of radius r.

Lemma 7.1 *Let $\Lambda \subset \mathbb{R}^n$ be a lattice and $S \subset \mathbb{R}^n$ be a compact set. Fix a fundamental cell Π_0 of Λ. Let ρ be the diameter of Π_0 and d be the diameter of S. Let γ_n be the volume of the unit sphere in $\mathbb{R}^n$. Then*

$$\operatorname{card}(\Lambda \cap S) \leq \gamma_n (d + \rho)^n / \det(\Lambda).$$

Proof If $\operatorname{card}(\Lambda \cap S) = 0$, there is nothing to prove. If not, fix $\lambda_0 \in \Lambda \cap S$. Claim: for any $\lambda \in \Lambda \cap S$, $B_\rho(\lambda) \subset B_{d+\rho}(\lambda_0)$. Indeed, $\operatorname{dist}(\lambda, \lambda_0) \leq d$, and for any $x \in B_\rho(\lambda)$, $\operatorname{dist}(\lambda, x) \leq \rho$. Therefore, $\operatorname{dist}(\lambda_0, x) \leq d + \rho$.

It follows that

$$\bigcup_{\lambda \in \Lambda \cap S} (\lambda + \Pi_0) \subset B_{d+\rho}(\lambda_0).$$

The volume of the union is $(\det \Lambda) \operatorname{card}(\Lambda \cap S)$ and the volume of the ball is $\gamma_n (d+\rho)^n$. ∎

The following lemma is well known as the key to "linear programming".

Lemma 7.2 *If S is a convex compact polyhedron in $\mathbb{R}^n$, then the diameter of S equals* $\sup \operatorname{dist}(v, w)$, *where v, w run over the vertices of S.*

Let H be the hyperplane in $\mathbb{R}^n$ given by the equation $\eta_1 + \cdots + \eta_n = 0$. Let Λ be a lattice in H. (In the application in Section 8 below we will choose Λ to be the lattice consisting of points $(\eta_1, \dots \eta_n) \in H$ such that $(\eta_1, \dots \eta_{n-1}, 0) \in \Lambda_U^*$.) For an integer $M \geq 0$, an integer a such that $0 \leq a < n$, and a real number $T > 0$, let $S_a(T, M)$ be the set of $\eta \in H$ satisfying the inequalities

$$\eta_1 \geq T, \dots, \eta_a \geq T, \eta_{a+1} \leq T, \dots, \eta_n \leq T, -(M+1) \leq 2(aT - (\eta_1 + \cdots + \eta_a)) \leq -M.$$

This last inequality is imposed only if $a > 0$.

Proposition 7.3 *There are positive constants f, g, h such that for all a, T, M,*

$$\operatorname{card}(\Lambda \cap S_a(T, M)) \leq (fT + gM + h)^{n-1}.$$

Proof We will apply Lemmas 7.1 and 7.2. In applying Lemma 7.2, we will use the 1-norm to estimate the 2-norm. Let δ be a constant so that $\|x\|_2 \leq \delta \|x\|_1$ for all $x \in H$.

First let $a = 0$. Then $S = S_0(T, M)$ is independent of M and consists of all $\eta \in H$ with $\eta_i \le T$ for $i = 1, \dots, n$. The vertices of S are the points

$$v_i = (T, \dots, -(n-1)T, \dots, T)$$

with $-(n-1)T$ in the i-th place and T in the remaining places. Then $\|v_i - v_j\|_1 = 2nT$ if $i \neq j$, so the diameter of S_0 is less than or equal to $\delta(2nT)$.

Now suppose $a > 0$. The vertices of $S = S_a(T, M)$ occur at the intersections of $n-1$ bounding hyperplanes L in H. These hyperplanes are of two forms:

(i) $\eta_i = T$ for some j;
(ii) $aT - (\eta_1 + \cdots + \eta_a) = N$ where N is either $-M/2$ or $-(M+1)/2$.

Thus there are two kinds of vertices: those where all bounding hyperplanes L are of type (i) above, and those where one of the bounding hyperplanes L is of the form (ii). The former are exactly the vertices v_i obtained in the $a = 0$ case, with $i = a+1, \dots, n$ (because $T > 0$). As for the latter, let N denote either $-M/2$ or $-(M+1)/2$, so $N < 0$. Then the second kind of vertex occurs when one of the L's is given by $aT - (\eta_1 + \cdots + \eta_a) = N$, and the other L's are of the form $\eta_k = T, k = 1, \dots, n, k \neq i, j$. Taking $i < j$, call this vertex $w_{i,j}$.

Suppose that $1 \le i, j \le a$ (which can only happen if $a > 1$). We then deduce from the equalities that define $w_{i,j}$ that $\eta_i + \eta_j = -(n-2)T < 0$ and also $\eta_i + \eta_j = 2T - N > 0$. This case is therefore impossible. Similarly, the case where $a+1 \le i, j \le n$ is impossible. Thus the vertex $w_{i,j}$ arises exactly when $1 \le i \le a$ and $a+1 \le j \le n$. Solving the equalities we find that

$$w_{i,j} = (T, \dots, T - N, \dots, -(n-1)T + N, \dots, T)$$

with $T - N$ in the i-th place, $-(n-1)T + N$ in the j-th place, and T in the remaining places.

We estimate the diameter by computing the 1-norm of the difference of pairs of vertices. We have $\|v_k - w_{i,j}\|_1 = 2nT - 2N$ if $k \neq j$ and $-2N$ if $k = j$. Similarly, there are four possibilities for $\|w_{i,j} - w_{k.m}\|_1$; these give $(0 \text{ or } -2N) + (0 \text{ or } 2nT - 2N)$. Thus the diameter of S is less than or equal to $\delta(2nT + 2M + 2)$ in all cases, including where $a = 0$.

We now apply Lemma 7.1. If we let ρ denote the diameter of our fundamental cell in Λ, then

$$\operatorname{card}(\Lambda \cap S) \le \gamma_n(\delta(2nT + 2M + 2) + \rho)^{n-1} / \det \Lambda.$$

This proves the proposition. ■

8 The Distribution of Totally Positive Integers of Given Trace

In this section we further study the function φ and use this to obtain our main result.

Proposition 8.1 *Given $\epsilon, \delta > 0$, then $\varphi(s) = \mathrm{O}(|t|^{n-1/2-\sigma/2+\epsilon})$ as $t \to \infty$ uniformly for $s = \sigma + it$ with $\delta \le \sigma \le n$, with the implied constant depending on ϵ and $\mathfrak{a}$.*

Proof An estimate of this form for $\zeta(s-n+1)$ follows immediately from the functional equation and convexity. In view of (6.2), it suffices to establish such an estimate for each function $\Psi(s,\nu,\mathfrak{a})$. Denote the Fourier coefficients of such a function $a_\mu(s)$. For convenience, let $\mu_n = -\sum_{j=1}^{n-1}\mu_j$. We use the expressions (4.5) and (4.6). By the convexity bound for the L-function obtained from Phragmen–Lindelöf we have

$$L(\tfrac{s}{n},\lambda_{\mu,\nu},A) \ll \prod_{j=1}^{n}\left|\tfrac{t}{n}-2\pi\mu_j\right|^{(1-\sigma/n)/2+\epsilon}$$

as $|t|\to\infty$, uniformly in the strip. Combining with Stirling's formula, one sees that

$$a_\mu(s) \ll |t|^{1/2-\sigma}e^{\pi|t|/2}\prod_{j=1}^{n}\left|\tfrac{t}{n}-2\pi\mu_j\right|^{(\sigma/2n)+\epsilon}e^{-\frac{\pi}{2}|\frac{t}{n}-2\pi\mu_j|}.$$

Let

$$F(s,\mu)=\prod_{j=1}^{n}\left|\tfrac{t}{n}-2\pi\mu_j\right|^{(\sigma/2n)+\epsilon}e^{-\frac{\pi}{2}|\frac{t}{n}-2\pi\mu_j|}.$$

Then

$$\sum_{\mu\in\Lambda_U^*}|a_\mu(s)| \ll |t|^{\frac{1}{2}-\sigma}e^{\frac{\pi}{2}|t|}\sum_{\mu}F(s,\mu).$$

We must estimate this sum.

To do so, suppose without loss of generality that $t>0$. We change notation slightly for convenience. Introduce $T=\frac{\pi t}{2n}$, $\beta=\frac{\sigma}{2n}+\epsilon$, and let $\eta_j=\pi^2\mu_j$ for $1\le j\le n$. Note that as μ ranges over Λ_U^*, η ranges over a lattice Λ in the hyperplane H in $\mathbb{R}^n$ studied in Section 7 above. Let

$$F_1(s,\eta)=\prod_{j=1}^{n}|T-\eta_j|^{\beta}e^{-|T-\eta_j|}.$$

We wish to estimate $\sum_{\eta\in\Lambda}F_1(s,\eta)$. Let us break the sum over η up into subsums corresponding to $|T-\eta_j|=\pm(T-\eta_j)$. That is, given T and given a with $0\le a<n$, let $B_{T,a}$ be the subset of $\eta\in H$ satisfying $\eta_j\ge T$ for $1\le j\le a$, $\eta_j\le T$ for $a<j\le n$. Up to reordering of the η_j, every subsum is over one of these regions. For $\eta\in B_{T,a}$, we have (using $\eta\in H$)

$$\sum_{j=1}^{n}|T-\eta_j|=\sum_{j=1}^{a}(\eta_j-T)+\sum_{j=a+1}^{n}(T-\eta_j)=(2n-a)T+2(\eta_1+\cdots+\eta_a).$$

(Here the last sum of η's is 0 if $a=0$.) Moreover, since the geometric mean is less than or equal to the arithmetic mean,

$$\prod_{j=1}^{n}|T-\eta_j|^{\beta}\le\left(\frac{(n-2a)T+2(\eta_1+\cdots+\eta_a)}{n}\right)^{n\beta}.$$

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Thus we see that

$$\sum_{\eta\in B_{T,a}} F_1(s,\eta) \leq n^{-n\beta}e^{-nT}\sum_{\eta\in B_{T,a}}((n-2a)T+2(\eta_1+\cdots+\eta_a))^{n\beta}e^{2aT-2(\eta_1+\cdots+\eta_a)}.$$

Let M be a non-negative integer, and

$$H_{T,M,a}=\{\eta\in H \mid -(M+1)<2aT-2(\eta_1+\cdots+\eta_a)\leq -M\}.$$

Then

$$\sum_{\eta\in B_{T,a}} F_1(s,\eta) \leq n^{-n\beta}e^{-nT}\sum_{M=0}^{\infty}\operatorname{card}(\Lambda\cap B_{T,a}\cap H_{T,M,a})(nT+M+1)^{n\beta}e^{-M}.$$

Now the closure of each region $H_{T,M,a}\cap B_{T,a}$ is of the form $S_a(T,M)$. By Proposition 7.3 we conclude that

$$\sum_{M=0}^{\infty}\operatorname{card}(\Lambda\cap B_{T,a}\cap H_{T,M,a})(nT+M+1)^{n\beta}e^{-M} \leq (fT+gM+h)^{n-1+n\beta}e^{-M}$$

(increasing the constants f,g,h if necessary). Also, there are only n possibilities for a, and (as noted above) up to reordering every $\eta\in\Lambda$ is in such a region. Therefore,

$$\sum_{\mu\in\Lambda_U^*} F(s,\mu) \ll n^{-n\beta}e^{-nT}\int_{M=0}^{\infty}(fT+gM+h)^{n-1+n\beta}e^{-M}\,dM. \tag{8.1}$$

But we have the following lemma.

Lemma 8.2 *For $x>0$, $r>1$, set $G(x)=\int_0^\infty (t+x)^r e^{-t}\,dt$. Then $G(x)=O(x^r)$.*

Proof Note that $t+x<2t$ if and only $x<t$. Breaking up the integral into two, we get

$$G(x)\leq \int_0^x (t+x)^r e^{-t}\,dt+\int_x^\infty (2t)^r e^{-t}\,dt.$$

The first integral on the right equals $x^r\int_0^x(1+t/x)^r e^{-t}\,dt$. Since $1+t/x\leq 2$ for $t\in[0,x]$, it is bounded above by $2^r x^r$. The second integral is bounded above by $2^r\Gamma(r+1)$. The lemma follows. ∎

Applying Lemma 8.2 to the expression (8.1) we find that

$$\sum_{\mu} F(s,\mu)\leq cn^{-n\beta}e^{-nT}((f/g)T+1+h/g)^{n-1+n\beta}$$

where c is a constant. The right-hand side equals

$$e^{-nT}\,O(T^{n-1+n\beta}) = e^{-\frac{\pi}{2}|t|}\,O(|t|^{n-1+\frac{\sigma}{2}+n\epsilon})$$

since $\frac{\pi}{2n}<1$. Therefore, $\sum_\mu |a_\mu(s)| \ll |t|^{\frac12-\sigma}e^{\frac{\pi}{2}|t|}e^{-\frac{\pi}{2}|t|}|t|^{n-1+\frac{\sigma}{2}+n\epsilon} = |t|^{n-\frac12-\frac{\sigma}{2}+n\epsilon}$. The implied constant depends on ϵ and $\mathfrak{a}$ (and K and n, which may be obtained from $\mathfrak{a}$). This completes the proof of Proposition 8.1. ∎

Finally, to complete our study of $\varphi(s)$, we appeal to the Schnee–Landau theorem. Since this theorem seems not to be well known, we recall its statement. See Landau [4, Satz 54] or Schnee [7] for a somewhat weaker version.

Theorem 8.3 (Schnee-Landau) *Suppose that for each $\delta > 0$, $|a_n| < n^{\ell-1+\delta}$, so that the series $f(s) = \sum a(n)n^{-s}$ converges absolutely for $\Re(s) > \ell$. Suppose that the function $f(s)$ is regular for $\Re(s) \geq m$, and that there exists a number k such that for all $\sigma > m$, $|t| \geq 1$, we have $f(\sigma + it) \leq A|t|^k$ for some fixed constant A. Then the series for $f(s)$ converges for $Re(s) > (m + k\ell)/(1 + k)$.*

Combining this result with Propositions 6.1 and 8.1, we obtain (taking $\ell = n - 1$, $m = 2\epsilon > 0$, $k = n - 1/2$) the following.

Theorem 8.4 *The series $\sum E_a a^{-s}$ converges for $\Re(s) > n - 1 - (2n - 2)/(2n + 1)$.*

Then, by partial summation, we get our result concerning the distribution of the totally positive elements of the fractional ideal $\mathfrak{a}$ of trace a.

Main Theorem *For $\epsilon > 0$,*

$$\sum_{a<X} E_a = \mathrm{O}\left(X^{n-1-\frac{2n-2}{2n+1}+\epsilon}\right),$$

with the implied constant depending on ϵ and $\mathfrak{a}$.

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