ORDER AND NORM CONVERGENCE IN BANACH LATTICES

by ANDREW WIRTH†

(Received 27 June, 1972; revised 19 September, 1972)

Let $(V, \leqq, \|\cdot\|)$ be a Banach lattice, and denote $V \backslash \{0\}$ by V'. For the definition of a Banach lattice and other undefined terms used below, see Vulikh [**4**]. Leader [**3**] shows that, if norm convergence is equivalent to order convergence for sequences in V, then the norm is equivalent to an M-norm. By assuming the equivalence for nets in V we can strengthen this result.

THEOREM. *Let $(V, \leqq, \|\cdot\|)$ be a Banach lattice; then the following statements are equivalent:*

(i) *Norm convergence is equivalent to order convergence, for nets in V.*
(ii) *V is finite-dimensional.*

Proof. (i) implies (ii). If $\alpha, \beta \in V'$, write $\alpha \preccurlyeq \beta$ to mean $\|\alpha\| \geqq \|\beta\|$. Then $(V', \preccurlyeq)$ is a preordered set directed to the right. Let $x_\alpha = \alpha$ for all $\alpha \in V'$; then $\|\cdot\| - \lim x_\alpha = 0$, and so 0-lim $x_\alpha = 0$. Hence $(V, \leqq)$ has a strong unit, e say. Define $\|\cdot\|_e$ by $\|x\|_e = \inf\{\lambda : |x| \leqq \lambda e\}$, for $x \in V$. By Birkhoff [**1**], $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent norms. In fact $(V, \leqq, \|\cdot\|_e)$ is a Banach lattice with unity e and so an M-space, Birkhoff [**1**]. So $(V, \leqq, \|\cdot\|_e)$ is isomorphic with $(C(X), \leqq$, sup norm), X compact Hausdorff, by Kelley and Namioka [**2**].

Let $x_0 \in X$ and let g be the characteristic function for the point x_0. Define

$$F = \{f \in C(X) : f \geqq 0 \quad \text{and} \quad f(x_0) = 1\};$$

then $(F, \geqq)$ is directed to the right. Let $f_\alpha = \alpha$ for all $\alpha \in F$. Then, by Urysohn's Lemma, $f_\alpha \downarrow g$ pointwise. If $g \in C(X)$, then 0-lim$f_\alpha = g$; otherwise 0-lim$f_\alpha = 0$. Now $\|\cdot\|_e - \lim f_\alpha = 0$ is impossible; so $g \in C(X)$. Hence $\{x_0\}$ is open; so X is discrete and hence finite.

(ii) implies (i). The proof of this is trivial.

The author is grateful to the referee for suggesting an improvement to the paper.

REFERENCES

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications **25**, 3rd Edition (Providence, R. I., 1967), 366–377.

2. J. L. Kelley and I. Namioka, *Linear topological spaces* (Princeton, 1963), 236–242.

3. S. Leader, Sequential convergence in lattice groups, *Problems in Analysis* (ed. R. C. Gunning), (Princeton, 1970), 273–290.

4. B. Z. Vulikh, *Introduction to the theory of partially ordered spaces* (Groningen, 1967).

MONASH UNIVERSITY
CLAYTON, VICTORIA, AUSTRALIA, 3168

† This research was supported by a Commonwealth Postgraduate Research Award.

https://doi.org/10.1017/S0017089500002032 Published online by Cambridge University Press