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TOWARDS A PROBLEM IN DEFORMATIONS OF POLARIZED ALGEBRAIC $K3$ SURFACES

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§ 0.

(0.1) Introductory

A nonsingular algebraic surface V is called a $K3$ surface if i) $1 = p_g = \mathfrak{l}(K_V) = \dim. H^0(V, \Omega^2{}_V)$, i.e. a canonical divisor K_V on V is linearly equivalent to zero; and ii) $\mathfrak{h}^1(V) = \dim. H^1(V, \underline{O}_V) = 0$. When the characteristic is zero, condition ii) is equivalent to ii)′ $q =$ dimension of the Albanese variety of $V = 0$, and always ii) implies ii)′ as in fact $\mathfrak{h}^1(V) \geq q \geq 0$, but in non-zero characteristic it can happen that $\mathfrak{h}^1(V) > q$ ([6], [15]). When ii)′ is true, the algebraic and linear equivalences of divisors coincide on V, because of the duality between Picard and Albanese varieties of V, [8]. When i) and ii) are true, the Riemann-Roch theorem for divisors D on V reads

$$\text{(R)} \qquad \mathfrak{l}(D) - \mathfrak{h}^1(D) + \mathfrak{l}(-D) = D^{(2)}/2 + 2$$

because in general $\mathfrak{l}(D) - \mathfrak{h}^1(D) + \mathfrak{l}(K_V - D) = D^{(2)} - I(D, D + K_V)/2 + 1 + p_a(V)$ (cf. [25], ch. 4, app.) and on a $K3$ surface we have that $K_V \sim 0$, $p_g = 1$, $p_a = p_g - \mathfrak{h}^1(V) = 1$. —Here we use a standard notation: $\mathfrak{l}(D) = \dim. H^0(V, \underline{O}_V(D))$, $\mathfrak{h}^1(D) = \dim. H^1(V, \underline{O}_V(D))$, and "$\sim$" denotes linear equivalence of divisors.

Let V be a $K3$ surface. Notice that any self-intersection number $D^{(2)}$ is even, so exceptional divisors X of first kind cannot exist since, for such divisors, the fundamental cycle Z of X satisfies $Z^{(2)} = -1$ by the contractibility criterion of Castelnuovo and M. Artin, [1]; cf. [9]. On the other hand, when a surface W is ruled, $p_g(W) = 0$; so a $K3$ surface is not ruled. Then a $K3$ surface is always a minimal model of

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its birational class, [27]. This shows that birational maps between $K3$ surfaces must be everywhere biregular.

Suppose that V is a polarized $K3$ surface and X is a basic polar divisor on the underlying surface V. This means ([10]) that there is a set $\mathfrak{X}$ of V-divisors with $X \in \mathfrak{X}$; $\mathfrak{X}$ contains a hyperplane section Z of V in some projective embedding; $Y \in \mathfrak{X}$ if and only if Y is numerically equivalent to a multiple of X; and when $Z \in \mathfrak{X}$ is a hyperplane section, Z is numerically equivalent to a positive multiple of X. Since the underlying $K3$ surface V is regular, i.e. $q = 0$, when Y and Z are numerically equivalent divisors on V, there is a positive integer m such that $m \cdot Y$ is linearly equivalent to $m \cdot Z$, [11]. Therefore when X is a basic polar divisor, a positive multiple of X determines a projective embedding. The notation (V, X) will be used for a polarized surface V and basic polar divisor X.

A common example of a polarized $K3$ surface is one where the underlying surface is a general quartic Q in $\boldsymbol{P}^3$ and on which a plane section is a basic polar divisor—denoted (Q, C_Q). Let (V, X) be a polarized $K3$ surface such that the rank $X^{(2)} = 4 =$ the rank of (Q, C_Q). Suppose that (V, X) is an algebraic deformation of (Q, C_Q) as polarized surfaces, i.e. C_Q deforms to X as Q deforms to V. By a theorem of Matsusaka-Mumford, [12], the set of algebraic deformations (V, X) is represented up to isomorphism by members of an algebraic family of nonsingular surfaces in a projective space.

In this paper we try to study a particular specie of polarized $K3$ surfaces (V, X), of rank 4. These "special" surfaces have the characteristic feature that V is $K3$ and that a basic polar divisor X is linearly equivalent to a sum: $X \sim 3E + F$, where E, F are irreducible nonsingular curves on V of respective genera $1, 0$, and $X^{(2)} = (3E + F)^{(2)} = 4$ (whence $I(E, F) = 1$). In § 1 we show that, *if* there exists a deformation (V, X) of (Q, C_Q) with X *reducible*, then V must be one of our special $K3$'s. The rest of the paper is concerned with seeing whether polarized special $K3$ surfaces do exist, and then with setting up a correspondence in which such a surface is associated with a homogeneous form of a certain type. The question, whether a general, special $K3$ $(V, 3E + F)$, is really a specialization of (Q, C_Q) is not considered at all.

The main results and assumptions are as follows. A *variety* V is, by convention here, absolutely irreducible: if V is defined over a field

k and P is a generic point of V over k, then $k(P)$ is by assumption a regular extension of the field k. A *curve*, resp. a *surface*, is a variety of dim. 1, resp. of dim. 2. Let P be a point of an r-dimensional variety V, let V be embedded, locally at P, in an affine space S^n, and let $\{F_i(x) = 0\}$ be a set of local defining equations. P is *simple on* V if the rank of the Jacobian matrix $\|\partial F_i/\partial x_j\|$ is $n-r$. V is *nonsingular* if every point P is simple on V. These are conventions in Weil's book, [22], and by and large we shall adhere to the conventions therein.

It is always assumed that the characteristic $p \neq 2$. There is no other assumption made about p except in § 1 where $p > 3$ is needed for the argument given.

In § 2 the existence of special $K3$ surfaces (not necessarily polarized rank 4) is considered. Referring to the lead paragraph, we show the existence of a nonsingular surface V possessed of properties i) $p_g = 1$, ii)′ $q = 0$ (but ii) $p_a = 1$ is not proved. Thus for V as constructed, the particular Riemann-Roch formula (R) is valid when the characteristic $=$ 0 but might not be valid when the characteristic $\neq 0$. To establish (R) one could for instance produce—without using (R)!—a curve D on V with $\mathfrak{h}^1(D) = 0$ and $\mathfrak{l}(D) = D^{(2)}/2 + 2$.

For the rest we have first to discuss a "branched-cover" representation of special $K3$ surfaces. In § 3 we see—assuming that V is a special $K3$ surface, so that (R) is valid—that the complete linear system Λ containing the divisor $4E + 2F$ determines a morphism λ of degree 2 from V to a rational cone C in $\boldsymbol{P}^5$, so that $\lambda(F)$ is the vertex and $\lambda(E)$ is a generator. (The complete linear system $\langle E\rangle$ containing E is a linear pencil, by (R).) The cone C is the same up to projective equivalence for all special V, and we fix a particular model C in $\boldsymbol{P}^5$. The branch locus of λ is a principal divisor Γ on C, defined there by a form $\Psi = \Psi(X_0, \cdots, X_5)$ of degree 3 (which we might suppose to "represent" V). Conversely, given such a form Ψ which is sufficiently general that Γ is nonsingular, we can reverse the procedure (§ 2), except that the V we obtain, as already said, is such that $p_g = 1$ and $q = 0$ for it but p_a may be less than 1.

In § 4 we show that a special $K3$ surface V represented as a "double cone" branched over Γ on C, as just described, is polarized by $X = 3E + F$ if and only if Γ is nonsingular. In § 5 we see that the correspondence between polarized $K3$ surfaces V and forms $\Psi = \Psi(X_0, \cdots, X_5)$

is 1–1 in a reasonable sense.

Although the branch curve Γ must be nonsingular to correspond to a suitably polarized V, and thus that is the case of main interest (cf. also Prop. 4), there is also some discussion of what occurs when Γ acquires singularities, cf. (2.3), (3.3) for instance.

(0.2)

Here, for purpose of reference, we prove some lemmas which are more or less familiar assertions about $K3$ surfaces (cf. [13], [17]).

First we define:

Special K3 Surfaces. The following particular $K3$ surfaces are discussed in this paper. (V, E, F) is called a *special K3 surface* if V is a $K3$ surface carrying irreducible nonsingular curves E and F such that E has genus one, F is rational, and $I(E, F) = 1$, $E^{(2)} = 0$, $F^{(2)} = -2$.

LEMMA 1. *Let X be an irreducible curve lying on a $K3$ surface V. Then $\mathfrak{h}^1(X) = \mathfrak{h}^1(V, \mathfrak{O}_V(X)) = 0$.*

Proof. There is the exact sequence: $0 \to \mathfrak{O}_V(-X) \to \mathfrak{O}_V \to \mathfrak{O}_X \to 0$ so there is $(\otimes\, \mathfrak{O}_V(X)) : 0 \to \mathfrak{O}_V \to \mathfrak{O}_V(X) \to \mathfrak{O}_X \otimes \mathfrak{O}_V(X) \to 0$, whence: $0 \to k \to L(X) \to H^0(\mathfrak{O}_X \otimes \mathfrak{O}_V(X))$. Since $K_V \sim 0$, $\mathfrak{h}^0(\mathfrak{O}_X \otimes \mathfrak{O}_V(X)) = \mathfrak{h}^1(\mathfrak{O}_X)$, by duality (cf. [15], p. 79), so $\mathfrak{l}(X) - 1 \leq h^1(\mathfrak{O}_X)$. By (R), $\mathfrak{l}(X) - 1 = X^{(2)}/2 + 1 + \mathfrak{h}^1(V, \mathfrak{O}_V(X))$. But $\mathfrak{h}^1(\mathfrak{O}_X) = p_a(X) = X^{(2)}/2 + 1$, (cf. [19]), so $\mathfrak{h}^1(V, \mathfrak{O}_V(X)) = 0$.

COROLLARY. *When X is an irreducible curve on V, $\mathfrak{l}(X) = X^{(2)}/2 + 2 = p_a(X) + 1$.*

LEMMA 2. *Let Λ be a complete linear system without fixed component on a $K3$ surface V, and let λ be the mapping of V determined by Λ. Suppose that $\lambda(V)$ is a curve. Then there is an irreducible linear pencil $\langle E \rangle$ of curves E such that Λ is composite with $\langle E \rangle$. Furthermore $E^{(2)} = 0$.*

Proof. There is an algebraic pencil $\boldsymbol{P} = \boldsymbol{P}(\Delta)$ whose general member Δ is irreducible, with which Λ is composite, by ([27], 1.4.3.b). The complete linear system $\Lambda(\Delta)$ determined by a general member Δ of $\boldsymbol{P}$ is contained in $\boldsymbol{P}$ (as sets of divisors) because Λ is linear, complete and is composite with $\boldsymbol{P}$. On the other hand, since V is a regular surface, all the curves Δ in $\boldsymbol{P}$ are linearly equivalent on V. Therefore $\boldsymbol{P}$ coincides with $\Lambda(\Delta)$ so $\boldsymbol{P}$ is by definition a linear system. Write $\boldsymbol{P} = \langle E \rangle$. $\mathfrak{l}(E) = 2$ as Λ is complete and composite with $\langle E \rangle$, and $\lambda(V)$ is a curve. Consequently $E^{(2)} = 0$ by (R) and the corollary to Lemma 1.

COROLLARY. *If $D^{(2)} > 0$ when D is in Λ then $\lambda(V)$ is a surface.*

Linear systems. Let X be a divisor on a normal variety V. The notation $\Lambda(X)$ means the complete linear system of positive divisors on V determined by X, as in [22]. $\mathfrak{L}(X) = \mathfrak{O}_V(X) =$ the invertible sheaf determined by X; $L(X) = H^0(V, \mathfrak{L}(X))$.

Let Y be an irreducible curve on a complete nonsingular surface V. Let X' be a divisor on V and let $\Lambda = \Lambda(X')$ be the complete determined by X'. Let X be a general member of Λ and assume that X and Y meet properly and that points common to X and Y are simple on Y. Let $D = X \cdot Y$ (considered on Y). There are two linear series on Y containing D which we mention, viz., the "complete" linear series $\Lambda_Y(D)$ determined by D and $Y \subset V$, and the subseries "trace" consisting of the set of divisors $\{X' \cdot Y\}$ when we take for X' all members of Λ meeting Y properly, and delete multiple points of Y from $X' \cdot Y$ (by convention, "divisors" have support at simple subvarieties). The vector spaces of functions associated with the two systems are related as follows. The restriction of $\mathfrak{L}(X)$ to Y determines a sheaf $\mathfrak{L}(X)|_Y$ and there is an exact sequence $0 \to \mathfrak{L}(X - Y) \to \mathfrak{L}(X) \to \mathfrak{L}(X)|_Y \to 0$. When we take for y the functions in $H^0(Y, \mathfrak{L}(X)|_Y)$, $\Lambda_Y(D)$ is the set $\{D + \operatorname{div}(y)\}$. When we take for x all the functions in image $[H^0(V, \mathfrak{L}(X)) \to H^0(Y, \mathfrak{L}(X)|_Y)]$, the trace $\{X' \cdot Y\}$ is the set $\{D + \operatorname{div}(x)\}$. If the trace and $\Lambda_Y(D)$ coincide one says "the trace is complete". In any case there is the long exact sequence $\cdots \to H^0(V, \mathfrak{L}(X)) \to H^0(Y, \mathfrak{L}(X)|_Y) \to H^1(V, \mathfrak{L}(X - Y)) \to \cdots$.

LEMMA 3. *Let E be an irreducible nonsingular curve of genus one on a $K3$ surface V. When n is a positive integer, $\mathfrak{l}(nE) = n + 1$.*

Proof. Consider the trace of $\Lambda(nE)$ on E: there is the exact sequence $0 \to \mathfrak{L}((n-1)E) \to \mathfrak{L}(nE) \to \mathfrak{L}(nE)|_E \to 0$. Since the genus of E is 1 and $E^{(2)} = 0$, $\mathfrak{L}(nE)|_E$ is just $\mathfrak{O}_E$ and we have a surjection $L(nE) \to H^0(E, \mathfrak{O}_E) \to 0$. By Lemma 1 and (R), $\mathfrak{l}(E) = 2$. Then $\mathfrak{l}(nE) = n + 1$ by induction on n.

COROLLARY. *Let $X \in \Lambda(nE)$, $n > 1$. Then X is a sum $X = E_1 + \cdots + E_n$, $E_i \sim E$.*

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Proof. We have $\mathfrak{l}(X) = \mathfrak{l}(nE) = n + 1$. The complete linear system $\Lambda(X)$ contains the linear subsystem consisting of sums $E_1 + \cdots + E_n$, which has arithmetic dimension $n + 1$. Therefore $\Lambda(X)$ coincides with its subsystem.

In the next lemma we need to refer to the theorem of Bertini on reducible linear systems ([22], ch. 9, or [27]). For the purpose of reference we state it now. Let Λ be a complete linear system on a $K3$ surface V. Suppose that

i) Λ has no fixed part, i.e. when G is an irreducible curve on V there is $D \in \Lambda, D \not\succ G$;

ii) Λ is not composed of a pencil, i.e. (using Lemma 2) we do not have an irreducible curve H with $\mathfrak{l}(H) = 2$ and for any $D \in \Lambda$, $D = H_1 + \cdots + H_r, H_i \sim H$;

iii) Λ is separable, i.e. letting p be the characteristic, we do not have a complete linear system Λ' such that Λ is the set of divisors $p \cdot D'$ where $D' \in \Lambda'$.

Then Λ is irreducible, i.e. a general member of Λ is irreducible.

Let (V, E, F) be a special $K3$ surface. Recall: $\mathfrak{l}(E) = 2, F^{(2)} = -2$.

LEMMA 4. a) *When n is a positive integer, the complete linear system $\Lambda(nE + F)$ has a fixed component which is the curve F; furthermore $\mathfrak{h}^1(V, \mathfrak{L}(nE + F)) = 0$.*

Assume now that m, n are integers such that $m \geq 2$ and $n \geq 2m$, and that not both of m, n are divisible by the characteristic. Let $\Lambda = \Lambda(nE + mF)$. Then

b) *Λ is irreducible and $\mathfrak{h}^1(V, \mathfrak{L}(nE + mF)) = 0$.*

c) *Λ has no base points, and when $D \in \Lambda$ then $D^{(2)} > 0$. Consequently:*

d) *The mapping λ of V determined by Λ is a morphism to a projective surface.*

Proof. Consider the complete linear system $\Lambda(nE + F)$. We claim that F is its fixed part. Let $E_1 \sim E, E_1 \neq E$. Then $nE_1 + F$ and $nE + F$ are two members of $\Lambda(nE + F)$ whose sole common component is F. Hence any fixed component of $\Lambda(nE + F)$ must be alone.

F is not a fixed component if and only if $\mathfrak{l}(nE + F) > \mathfrak{l}(nE)$. Evaluate each side: $\mathfrak{l}(nE + F) = n + 1 + \mathfrak{h}^1(nE + F), \mathfrak{l}(nE) = n + 1$. Thus $\mathfrak{l}(nE + F) = \mathfrak{l}(nE) + \mathfrak{h}^1(nE + F)$, so F is not a fixed component if and only if $\mathfrak{h}^1(nE + F) > 0$.

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Now, on the other hand we know by a preceding lemma that $\mathfrak{h}^1(nE+F)=0$ if $\Lambda(nE+F)$ is irreducible. If we assume that $\Lambda(nE+F)$ has no fixed part and prove ii), iii), in Bertini's theorem, then $\Lambda(nE+F)$ would be irreducible, so $\mathfrak{h}^1(nE+F)=0$ and the assumption leads to a contradiction. Then let us prove ii), iii). As in ii), suppose $nE+F\sim kH$ as above, and evaluate $1=I(nE+F,E)=I(kH,E)=kI(H,E)$: thus $k=1$, $nE+F\sim H$. By our assumption $\mathfrak{l}(nE+F)>\mathfrak{l}(nE)\geq 2$; and $\mathfrak{l}(H)=2$, so $nE+F\sim kH$ is impossible. As to iii), F occurs in $nE+F$ with coefficient 1, so $nE+F=pD'$ is impossible. This proves a).

Fix n,m with $m>1$, $n\geq 2m$ and let $\Lambda=\Lambda(nE+mF)$. We want to prove that Λ is irreducible. Let us verify the conditions in Bertini's theorem. First we prove ii): suppose that $nE+mF\sim kH$, then $m=I(nE+mF,E)=I(kH,E)=kI(H,E)$ so $m\geq k$. Now the arithmetic dimension of the linear system in which a general member is $H_1+\cdots+H_k$, is equal to $k+1$; if Λ coincides with that system then $\mathfrak{l}(nE+mF)=k+1\leq m+1$. But here (R) says that $\mathfrak{l}(nE+mF)\geq m(n-m)+2>m+1$. So Λ is not composite with a pencil. iii) follows from $p\nmid n$ or $p\nmid m$. Lastly we prove that Λ is free of fixed components, by induction on $m\geq 2$. We just saw that $\mathfrak{h}^1(nE+F)=0$; suppose that $\mathfrak{h}^1(nE+(m-1)F)=0$. In that case $\mathfrak{l}(nE+mF)>\mathfrak{l}(nE+(m-1)F)$, using (R). As in a), any fixed part of Λ must have F as a component. Thus the inequality just stated says that there is no fixed part when $\mathfrak{h}^1(nE+(m-1)F)=0$. Hence by Bertini, Λ is irreducible, so $\mathfrak{h}^1(nE+mF)=0$ and the induction can proceed. This proves b).

Next we assert that $\Lambda=\Lambda(nE+mF)$ has no base points when $m\geq 2$ and $n\geq 2m$. Clearly a base point must lie on F. However, we have that $I(nE+mF,F)=n-2m\geq 0$, that F is a nonsingular rational curve, and further that the trace of Λ on F is complete because of the exact sequence $0\to\mathfrak{L}(nE+(m-1)F)\to\mathfrak{L}(nE+mF)\to\mathfrak{L}(nE+mF)|_F\to 0$, and the vanishing of $\mathfrak{h}^1$'s according to a), b). It follows that there is no base point on F.

$D^{(2)}=(nE+mF)^{(2)}=2nm-2m^2\geq 2m^2>0$. Thus c) is proved.

d) follows from c) and the corollary of Lemma 2. This proves the lemma.

LEMMA 5. *Let V be a $K3$ surface and let B' be a divisor on V, such that a positive integer multiple $m\cdot B'$ determines a projective*

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embedding of V. Then there is a positive divisor linearly equivalent to B', i.e. $\mathfrak{l}(B') > 0$.

Proof. By assumption there is a positive divisor C with $m \cdot B' \sim C$ and $I(m \cdot B', C) > 0$. As $m > 0, I(B', C) > 0$. We want to show that $\mathfrak{l}(-B') = 0$. If not, there is a positive divisor D with $-B' \sim D$. Since $D \succ 0, D \cdot C \succ 0$. But now $I(D, C) > 0$ so $I(-B', C) > 0$. This and $I(B', C) > 0$ cannot both be true, hence $\mathfrak{l}(-B') = 0$. Now $C^{(2)} > 0$, so $B'^{(2)} > 0$. Then the inequality derived from (R): $\mathfrak{l}(X) \geq X^{(2)}/2 + 2 - \mathfrak{l}(-X)$ gives that $\mathfrak{l}(B') > 0$.

§ **1.** The result in this part implies that, if the polarized quartic surface (Q, C_Q) (cf. (0.1)) deforms to (V, B), a polarized $K3$ surface of rank $B^{(2)} = 4$ with basic polar divisor B a generally reducible divisor, then B is of a certain type. In this section and only here, we suppose that the characteristic >3 (otherwise >2).

PROPOSITION 1. *Let V be a $K3$ surface and let B' be a divisor on V, such that a positive integer multiple $m \cdot B'$ determines a projective embedding. From the complete linear system $\Lambda(B')$ (which exists, by Lemma* 5) *select a general member B. Suppose that $B^{(2)} = 4$ and that B is reducible. Then $B = E_1 + E_2 + E_3 + F$, where E_i, F are nonsingular curves of respective genera* 1, 0 *and* $I(E, F) = 1$.

Proof. By assumption, $m \cdot B \sim C =$ a section of a projective embedding of V by a general hyperplane. C is a connected curve. By the Degeneration principle of Enriques-Zariski, [24], $m \cdot B$ is connected; therefore B is connected. Similarly, $I(C, X) > 0$ for a positive divisor X on V, hence $I(B, X) > 0$ for such X. Also, if $I(B, X) = 1$ then X is irreducible.

By Bertini's theorem (0.2) the assumption that B is reducible implies that $\Lambda(B)$ either is composed of a pencil or has a fixed component. Suppose that there is no fixed component and that $\Lambda(B)$ is composed of an irreducible pencil $\langle G \rangle$, so that $B = G_1 + G_2 + \cdots + G_m$ with $m > 1$. Now $B^{(2)} = 4$ and $G_i{}^{(2)}$ is even: but these facts and the expression for B are incompatible. Hence there must be a fixed component in $\Lambda(B)$.

A fixed component, i.e. an irreducible curve F with $\mathfrak{l}(F) = 1$, must be a nonsingular rational curve as $p_a(F) = \mathfrak{l}(F) - 1$ (Lemma 1, cor.). Let F_1 be one fixed curve of $\Lambda(B)$, let W be the sum of all the others

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so that $F_1 + W$ is the entire fixed part of $\Lambda(B)$, and let Z be such that $B \sim F_1 + W + Z$. $F_1 + W$ is a sum of nonsingular rational curves, and we may assume that no component of Z is a curve of that sort.

Now Z must be reducible. To prove that it is enough to show that $\mathfrak{h}^1(Z) \neq 0$. Since Z is a member of the variable part of $\Lambda(B)$, $\mathfrak{l}(B) = \mathfrak{l}(Z)$; by the Riemann-Roch theorem (R) and the fact that $B^{(2)} = 4$ we have that $\mathfrak{l}(B) \geqq 4$; hence $\mathfrak{l}(Z) \geqq 4$ and by the same theorem it is enough to show that $Z^{(2)} \leqq 2$, to conclude that $s(Z) \neq 0$. We have that $I(B, F_1 + W) > 0$, as observed at the start of the proof, and also that $I(Z, F_1 + W) > 0$ because B is connected, B is the sum $Z + (F_1 + W)$, and there is no component common to Z and to $F_1 + W$. Taking the intersection of the expression for B with $(F_1 + W)$, it follows that $(F_1 + W)^{(2)} + I(Z, F_1 + W) \geqq 1$, hence that $(F_1 + W)^{(2)} + 2I(Z, F_1 + W) \geqq 2$. As $4 = B^{(2)} = (Z + W + F_1)^{(2)}$ we see that $Z^{(2)} \leqq 2$ as required. Therefore Z is reducible.

Z was an arbitrary member of the variable part of $\Lambda(B)$. Consequently the reduced linear system $\Lambda(Z)$ is reducible, hence composed of a linear pencil, by Bertini's theorem. Let E be a general member of that pencil, then for some integer $m > 1$, we have that $Z \sim mE$. Since E moves in a linear system, $E^{(2)} \geqq 0$; since $Z^{(2)} \leqq 2$ by the argument above, and self-intersection numbers on a $K3$ surface are even, $E^{(2)} = 0$. Hence $p_a(E) = 1$. Furthermore if X is an irreducible curve $I(Z, X) \neq 1$.

There remains to prove three facts: that $W = 0$, that $m = 3$, and that E is nonsingular. As to the first, assume that $W \neq 0$; then we must have that $B \sim F_1 + W + Z$, hence that $B^{(2)} = I(B, F_1) + I(B, W) + I(B, Z)$. The first two numbers are each at least one, the last is at least two, and the sum is four, so $I(B, F_1) = I(B, W) = 1$, $I(B, Z) = 2$. B is ample so from $I(B, W) = 1$ it follows that W is irreducible. Next, we can write $1 = I(B, F_1) = F_1{}^{(2)} + I(F_1, W) + I(F_1, Z)$. F_1 is a nonsingular rational curve so $F_1{}^{(2)} = -2$; hence $3 = I(F_1, W) + I(F_1, Z)$. Finally, writing out $4 = B^{(2)} = (F_1 + W + Z)^{(2)} = F_1{}^{(2)} + W^{(2)} + Z^{(2)} + 2((I(F_1, W) + I(F_1, Z)) + I(W, Z)) = (-2) + (-2) + (0) + 2(3 + I(W, Z))$, we see that $I(W, Z) = 1$. But W is irreducible so $I(W, Z) \neq 1$ as we noticed. This is impossible hence $W = 0$.

Hereafter we shall write F for F_1.

Therefore $3 = I(F, Z) = m \cdot I(F, E)$. Since $m > 1$, then $m = 3$ and F meets each member of the pencil $\langle E \rangle$ in a single point with multiplicity one. Thus the intersection-product of E and F must be a simple point of either curve.

Finally we show that a general member of $\langle E\rangle$ is nonsingular. It is here that char. >3 is used.

The pencil $\langle E\rangle$ corresponds to a map $\delta: V\to \boldsymbol{D}$ of V to the projective line $\boldsymbol{D}$. Assume that k is a field of rationality for V, F, and the graph of δ. Let (v) be generic on V over k. Let $(d)=\delta(v)$; then (d) is generic on $\boldsymbol{D}$ over k and $k(v)\supset k(d)$. Let $E_d=\delta^{-1}(d)$; it is a general divisor in $\langle E\rangle$, so it is an irreducible curve defined over $k(d)$, [22]. (d) has dimension one over k so (v) is generic on E_d over $k(d)$, and $k(v)$ is a regular extension of $k(d)$. Let $(u)=E_d\cdot F$. According to intersection theory, (u) is rational over $k(d)$. Then E_d has a simple point rational over a field of definition.

Now we need a lemma. Let k_0 be a field—not necessarily perfect: here, $k(d)$—and let $K=\overline{k_0}$ be an algebraic closure of k_0.

Lemma 6. *Let E be a curve lying on a nonsingular surface S, both E and S being defined over the field k_0. Let $K=\overline{k_0}$ and let*
p_a = arithmetic genus of $E=\frac{1}{2}I(E,E+K_S)+1$
p = genus of the field extension $k_0(E)/k_0$
q = "effective" genus of E = genus of $K(E)/K$.
Then, $p_a\geq p\geq q$; furthermore, $p_a=p$ if and only if E is k_0-normal.

Proof. $p\geq q$ is contained in Theorem 5, [3], ch. 5. For the rest it is only necessary to rearrange a little the results in ch. IV, nos. 6, 7 of Serre, [19].

E is a k_0-variety with a k_0-topology; let $\mathfrak{O}_1$ be the sheaf on E of k_0-rational functions. E is also a K-variety and $\mathfrak{O}_2=\mathfrak{O}_1\otimes_{k_0}K$ is its sheaf of K-rational functions (Weil [22], ch. 9; [16], ch. 2). By [19], $\dim_K H^1(E,\mathfrak{O}_2)=p_a$. By the Künneth formula (quoted in [15] ch. 11), $\dim_K H^1(E,\mathfrak{O}_2)=\dim_{k_0}H^1(E,\mathfrak{O}_1)$.

Let $\nu: E'\to E$ be the k_0-normalization of E (see [22], app. 1) and let $\mathfrak{O}_3$ be the structure sheaf of k_0-rational functions on E'. Let $\mathfrak{O}_4$ be the sheaf on E whose stalks at k_0-closed points are the normalizations of stalks of $\mathfrak{O}_1$ in $k_0(E)$. Stalks of $\mathfrak{O}_1$ and $\mathfrak{O}_4$ coincide on a nonempty k_0-open set of E, ([22]). When places P of E' lie over a k_0-closed point Q of E, $\mathfrak{O}_{4Q}=\bigcap_{P\to Q}\mathfrak{O}_{3P}$. By Lemma 1 in [19] ch. IV, no. 6 (which does not depend on using an algebraically closed ground field), $H^1(E',\mathfrak{O}_3)=$

$H^1(E, \mathfrak{O}_4)$. By the "Example" in the same reference $p = \dim._{k_0} R/(R(0) + k_0(E)) = \dim._{k0} H^1(E', \mathfrak{O}_3)$ (R being the algebra of repartitions in $k_0(E)$, [3].

Thus p_a depends on E lying on S only, and p is determined by the k_0-normalization.

Now there is the exact sequence of sheaves of k_0-rational functions on the curve $E: 0 \to \mathfrak{O}_1 \to \mathfrak{O}_4 \to \mathfrak{O}_4/\mathfrak{O}_1 \to 0$. The field of constants in $k_0(E)$ is k_0 so $k_0 = H^0(\mathfrak{O}_1) = H^0(\mathfrak{O}_4)$, and $0 \to H^0(\mathfrak{O}_4/\mathfrak{O}_1) \to H^1(\mathfrak{O}_1) \to H^1(\mathfrak{O}_4) \to 0$ is exact. Combining this and previous statements it follows that $p_a = p + H^0(\mathfrak{O}_4/\mathfrak{O}_1) \geq p$ and $p_a = p$ if and only if $H^0(\mathfrak{O}_4/\mathfrak{O}_1) = 0$. The quotient sheaf is supported at finitely many points as we noted so $H^0(\mathfrak{O}_4/\mathfrak{O}_1) = 0$ if and only if $\mathfrak{O}_4/\mathfrak{O}_1 = 0$, i.e. E is k_0-normal.

In our case—$E = E_d, k_0 = k(d)$, etc.—the arithmetic genus $p_a = E_d{}^{(2)}/2 + 1 = 1$ so p and q are bounded between 0 and 1. $p = 0$ is ruled out because in that case V would be a ruled surface (recall that E_d has a rational simple point $E_d \cdot F$). Suppose now that E_d has a multiple point. Then $q < p_a$ by a familiar formula, [19] ch. IV, so $q = 0$ in that case. Thus we must have: $q = 0$, $p = 1$, so when the field $k(d)$ is extended to $\overline{k(d)}$, there is a genus drop of 1 if E_d has a multiple point. By a result of Tate, [21], when such genus change occurs, the change is at least $\frac{1}{2}(r-1)$ (here $r =$ char.). So if $r > 3$, change cannot happen, and therefore E_d has no multiple point.

Remark 1.1. We could also apply Bertini's theorem about variable singularities, [23], to have that E_d is $k(d)$-normal, thus simplifying matters so far as $\mathfrak{O}_1 = \mathfrak{O}_3 = \mathfrak{O}_4$: Serre's "Example" then gives $p = \dim._{k_0} H^1(E_d, \mathfrak{O}_1)$ which $= p_a$ $(=1)$ as before. Then if E_d has a multiple point the field genus is forced to drop by 1 as before.

§ **2.** In this part we shall construct "special" $K3$ surfaces V. It will appear later that whenever such a V exists it must be a branched cover of a certain cone; here, we first present that cone and then prescribe a branch locus and erect a surface V. We fix an algebraically closed ground field k of characteristic $\neq 2$. To start with we discuss the base curve of the cone.

(2.0). Let us call a projective variety U in $\boldsymbol{P}^n$, projectively normal in $\boldsymbol{P}^n$ if each linear system cut by hypersurfaces of a given degree is complete. By a well-known theorem of Zariski, U is thus projectively normal if and only if the homogeneous coordinate ring is absolutely integrally closed; thus a projectively normal curve is nonsingular. When U is projectively normal a reembedding of U determined by a complete linear system of hypersurface sections is also projectively normal, as follows from the definition.

LEMMA 7. *Let B be an irreducible curve spanning a projective space $\boldsymbol{P}^n$. Then* a) *B is a curve of degree n, if and only if* b) *B is a projectively normal rational curve.*

Proof. Suppose a) is true. Let B_0 be a nonsingular model of B. The rational mapping $B_0 \to B$ into a projective space determines a reduced linear system Λ of degree n on B_0; since B spans $\boldsymbol{P}^n$ the geometric dimension of Λ is n. This means that B_0 is rational: if the genus of B_0 is g we have that $n \leq n - g + \mathfrak{l}(K - D)$ whenever $D \in \Lambda$; as $\mathfrak{l}(K - D) \leq \mathfrak{l}(K) = g$ always, then $\mathfrak{l}(K - D) = g$; since the canonical series is without fixed point, $g = 0$. Consequently the degree of the linear system Λ satisfies $n \geq 2g + 1$, so the map determined by Λ is a birational isomorphism. Therefore B is nonsingular.

Let $\boldsymbol{D}$ be the projective line. There is a birational isomorphism $\boldsymbol{D} \to B$ over a field k of definition for $\boldsymbol{D}$, B such that $k(B)/k$ is purely transcendental. This amounts to a rational mapping of $\boldsymbol{D}$ into a projective space. This map is determined by a linear series of degree n, geometric dimension n, on $\boldsymbol{D}$, which is then the complete system of that degree and dimension. Now $\boldsymbol{D}$ is obviously projectively normal in itself. Thus B is a reembedding by hyperplane sections of a projectively normal variety $\boldsymbol{D}$, so, as above, B is projectively normal.

The converse proof is not given as we don't use the result.

Now let t, u be indeterminates over k and let B be the locus of $(t^n, t^{n-1}u, \cdots, u^n)$ over k in $\boldsymbol{P}^n$. B is evidently an irreducible curve of degree n spanning $\boldsymbol{P}^n$. Next consider the ${}_nC_2$ equations (*) obtained by setting equal to zero the 2×2 minors of the matrix: $\begin{vmatrix} \mathfrak{X}_0 & \mathfrak{X}_1 & \cdots & \mathfrak{X}_{n-1} \\ \mathfrak{X}_1 & \mathfrak{X}_2 & \cdots & \mathfrak{X}_n \end{vmatrix}$ where $(\mathfrak{X}_0, \cdots, \mathfrak{X}_n)$ are homogeneous coordinates in $\boldsymbol{P}^n$. Clearly B is included among the zeros of (*). If we take one more equation, for instance $\mathfrak{X}_0 = \mathfrak{X}_n$, there are at most the finite number n solutions to it and (*). This shows that the equations (*) define B in $\boldsymbol{P}^n$.

A curve B of our type is a reembedding of $\boldsymbol{D}$ as above so two such B's are projectively equivalent in $\boldsymbol{P}^n$; thus, whenever B is an irreducible curve of degree n spanning $\boldsymbol{P}^n$, it is defined by equations (*) in a suitable coordinate system.

Let C be the cone in $\boldsymbol{P}^5$ projecting an irreducible curve of degree 4 spanning $\boldsymbol{P}^4$; C is normal and in an appropriately chosen coordinate system is defined by the vanishing of 2×2 minors of the matrix: $\begin{vmatrix} \mathfrak{X}_1 & \mathfrak{X}_2 & \mathfrak{X}_3 & \mathfrak{X}_4 \\ \mathfrak{X}_2 & \mathfrak{X}_3 & \mathfrak{X}_4 & \mathfrak{X}_5 \end{vmatrix}$. Let C_1, C_5 be the open affine subsets of C where respectively $\mathfrak{X}_1 \neq 0$, $\mathfrak{X}_5 \neq 0$. Every point of C except the vertex $(1:0:\ldots:0)$ lies in C_1 or in C_5, because if either $\mathfrak{X}_1$ or $\mathfrak{X}_5$ vanish at a point of C then $\mathfrak{X}_2$, $\mathfrak{X}_3, \mathfrak{X}_4$ do also. The complement $C - C_1$ is the line $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}_3 = \mathfrak{X}_4 = 0$ on C. It is immediate that C_1 is defined by equations $X_3 = X_2^2$, $X_4 = X_2^3$, $X_5 = X_2^4$ in affine coordinates $(X_1, \cdots, X_5) = (\mathfrak{X}_0/\mathfrak{X}_1, \mathfrak{X}_2/\mathfrak{X}_1, \cdots, \mathfrak{X}_5/\mathfrak{X}_1)$ in S^5. Let $x_1 = \mathfrak{x}_0/\mathfrak{x}_1, x_2 = \mathfrak{x}_2/\mathfrak{x}_1$ be regarded as functions induced on C_1. A general point of C_1 has the coordinates $(x_1, x_2, x_2^2, x_2^3, x_2^4)$. Then clearly the projection: $(x_1, x_2, \cdots, x_2^4) \to (x_1, x_2)$ is a biregular isomorphism between C_1 and the affine plane S^2. Thus $k[C_1] = k[x_1, x_2]$. We sometimes designate the inverse by $\iota: (x_1, x_2) \to (x_1, x_2, \cdots, x_2^4)$.

Let $\langle L \rangle$ be the pencil of generators of C and let L' be a general member of $\langle L \rangle$ over k. Let $|B|$ denote support of a cycle B.

LEMMA 7.1. *Let G be an irreducible curve on C, defined over k, such that $|G| \cap |L'|$ contains no simple point of C. Then G is a member of the pencil $\langle L \rangle$.*

Proof. Let $\phi: C \to \boldsymbol{P}^1$ be the rational map corresponding to $\langle L \rangle$. Select a generic point g of G over k. g is simple on C so ϕ is defined at g. Let $L_g = \phi^{-1}\phi(g) \in \langle L \rangle$. L_g is irreducible, defined over k, and passes through g; hence $L_g =$ locus of g over $k = G$.

(2.1). Let $\Psi = \Psi(\mathfrak{X}_0, \cdots, \mathfrak{X}_5)$ be a homogeneous form of degree three with coefficients in k. Assume that the hypersurface defined in $\boldsymbol{P}^5$ by the equation $\Psi = 0$ meets properly with the cone C and does not meet the vertex of C, also that almost every generator of the cone meets the hypersurface in three transversal intersections. Let Γ be the curve cut on C by the hypersurface: $\Psi = 0$.

Let $X_0, \cdots, X_5$ be the homogeneous coordinate functions of a general

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point of C over k. Set $(x_1, \cdots, x_5) = (X_0/X_1, X_2/X_1, \cdots, X_5/X_1)$ and $(y_1, \cdots, y_5) = (X_1/X_0, \cdots, X_5/X_0)$. By (2.0) the vertex of C is at $(y) = (0)$; and $k(C)$—the field of k-rational functions on C, isomorphic to $k(Q)$ for Q a generic point of C over k— $= k(x_1, x_2)$. Write $\psi(x) = \Psi(X)/X_1^3$, $\phi(y) = \Psi(X)/X_0^3$. Then the relation $\psi(x)/x_1^4 = y_1\phi(y)$ is valid.

By (2.0) we may write $\psi(x) = \psi_1(x_1, x_2)$ since the functions x_3, x_4, x_5 are powers of x_2; in other words ψ_1 is the pull-back of ψ by the isomorphism $\iota: (x_1, x_2) \to (x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_2^2, x_2^3, x_2^4)$; that is, $\psi_1 = \psi \circ \iota$.

The isomorphism projection (inverse to ι) of C_1 to S^2 induces an isomorphism from $\Gamma \cap C_1$ to an image $\Gamma_1 \subset S^2$. The equation $\psi_1 = 0$ defines Γ_1 because of the relation $\operatorname{div}(\psi \circ \iota) = \iota^{-1}(\operatorname{div} \psi)$, (cf. [22], ch. 9). As a polynomial in $k[X_1, X_2]$, ψ_1 can have no multiple factors. For, if it did so, Γ_1 would have multiple components, so that Γ would too; since by assumption Γ meets almost every generator of C transversally at each intersection a multiple component must be a generator (Lemma 7.1), hence must go through the vertex—but that is ruled out. Then certainly:

The equation $Z^2 = \psi_1(X_1, X_2)$ defines a field of functions K of degree two over $k(C)$. (k is algebraically closed and of characteristic $\neq 2$.)

Recall that C is normal, (2.0). *Let V' be the normalization of C in K*, ([22]). Then V' is a projective irreducible normal surface and there is a *normalization map* $\lambda': V' \to C$, of degree 2.

We shall see that V' has (isolated) multiple points of two types: one over the vertex and some others over multiple points γ of Γ. We begin by analyzing the first, which in no way depends on Γ.

(2.2). To study V' over the vertex of C we proceed as follows. The vertex has homogeneous coordinates $(1: 0: \ldots : 0)$ according to (2.0). Let C_0 be the affine open subset of C where $\mathfrak{X}_0 \neq 0$. C_0 is defined by equations expressing the vanishing of the 2×2 minors of the matrix: $\begin{vmatrix} Y_1 & Y_2 & Y_3 & Y_4 \\ Y_2 & Y_3 & Y_4 & Y_5 \end{vmatrix}$, as in (2.0).

Next, from the relation $\psi(x)/x_1^4 = y_1 \cdot \phi(y)$, (2.1), it follows that we can write $K = k(C)(y_6)$ where y_6 is a function defined by the equation $y_6^2 = y_1 \cdot \phi(y)$.

Recall that Γ does not pass through the vertex of C, hence $\phi(0) \neq 0$

or $\phi = \phi(y)$ is invertible in the local ring $\mathfrak{o}$ of C at the vertex.

Now consider the Zariski-closed set $\mathfrak{F}$ in S^8 defined by

$$(\sharp) \quad \begin{array}{ll} Y_6^2 = Y_1 \cdot \phi(Y) & Y_7 Y_8 = Y_4 \cdot \phi(Y) \\ Y_6 Y_7 = Y_2 \cdot \phi(Y) & Y_8^2 = Y_5 \cdot \phi(Y) \\ Y_7^2 = Y_3 \cdot \phi(Y) & Y_6 Y_8 = Y_7^2 \end{array}$$

Since there are only six equations a component of $\mathfrak{F}$ has dimension ≥ 2. All components are rational over $k = \bar{k}$. There exists a component W_0 passing through the solution (0) of the equations; as $\phi((0)) \neq 0$, W_0 is not contained in the hypersurface defined by the equation $\phi(Y) = 0$. Let (y') be a point of W_0, other than (0), such that $\phi(y') \neq 0$. Computation shows that the Jacobi matrix formed from the equations $(\sharp)$ has rank 6 at (y'). Therefore dim. $W_0 = 2$ and such points (y') are simple for W_0.

Now when (y') is any point of W_0 such that $\phi(y') \neq 0$, $(y'_1, \cdots, y'_5)$ lies on C_0. Let (y) be a generic point of W_0 over k. Since $k(y_1, \cdots, y_6)$ contains y_7, y_8 and $y_6^2 = y_1 \phi(y_1, \cdots, y_5)$, $(y_1, \cdots, y_5)$ is a generic point of C_0 over k.

If U is another component of $\mathfrak{F}$ not contained in the hypersurface $\phi(Y) = 0$, and (u) is a generic point of U over k, $(u_1, \cdots, u_5)$ is a generic point of C_0 over k for the same reasons. Then there is a generic point (u) of U over k such that $(u_1, \cdots, u_5) = (y_1, \cdots y_5)$. Then $u_6^2 = y_6^2$ so (u), (y) are generic specializations of one another over $k(y_1, \cdots, y_5)$, hence over k.

This shows that there is a unique component W_0 of $\mathfrak{F}$ containing (0) and that the quotient ring of $k[Y]/(\sharp)$ with respect to the complement of (Y) is the local ring $\mathfrak{O}''$ of W_0 at (0). On account of the relation $y_6^2 = y_1 \cdot \phi(y)$, W_0 is birationally equivalent to V'; and we identify $k(W_0)$ with $K = k(V')$.

The functions y_6, y_7, y_8 are integral over the domain $k[y_1, \cdots, y_5]$ and consequently $\mathfrak{O}''$ is integral over $\mathfrak{o} = k[y_1, \cdots, y_5]_{(0)}$. We shall show that $\mathfrak{O}''$ is integrally closed.

For this (and other purposes) we compute the cone tangent to W_0 at (0): we claim that it is isomorphic to the "ordinary cone" defined by $XZ = Y^2$ in S^3. Since the latter is normal (2.0) it will follow that $\mathfrak{O}''$ is also. We need to know the ideal A^* of leading forms of polynomials in the ideal A generated in $k[Y]$ by $(\sharp)$. Certainly A^* contains the set $\{Y_1, \cdots, Y_5, Y_6 Y_8 - Y_7^2\}$ (assuming that $\phi(0) = 1$). Since $(\sharp)$ are local defining equations any element of A^* is of the form $\sum_{i=1}^{5} g_i(Y)Y_i + g_6(Y)(Y_6Y_8 - Y_7^2)$. Therefore $k[Y_1, \cdots, Y_8]/A^* \cong k[X, Y, Z]/(XZ - Y^2)$ as claimed. Now the tangent cone is the spectrum of the associated graded $G(\mathfrak{O}'')$, so by local algebra, ([26], vol. 2), $\mathfrak{O}''$ is normal since $G(\mathfrak{O}'')$ is so.

Now W_0 is an affine surface and is normal in the vicinity of (0). To be quite definite, put $C_{00} = C_0 - (\phi = 0)$, and then let V'_0, W_{00} be respectively the affine open subsets of V', W_0 over C_{00} (cf. [22], app. 1; [16], ch. 3). Both of V'_0, W_{00} are normal, and they are birationally equivalent. Then by the uniqueness of normal covers of a variety (here, C_{00}) in a function field (K), cf. [22], V'_0 and W_{00} are k-isomorphic. This shows that whenever v is a point of V' lying over the vertex of C, and $\mathfrak{O}'$ is the local ring of v on V', then v is the only point over the vertex, and $\mathfrak{O}' = \mathfrak{O}''$. Hereafter we use the letter v for this (multiple) point of V'.

Now apply a quadratic transformation $\rho^* : V^* \to V'$ of V' centered at v. To study the effect of ρ^* we may use the local affine model W_0 in S^8. We shall apply a quadratic transformation $\sigma^* : w^* \to W_0$ centered at (0) and transfer what is learned to V^*. We shall use the same letters F, E^*, etc. to denote certain similar curves to be defined on W^* and on V^*; this should not cause confusion.

Let B be the result of blowing up the origin in S^8—specifically, the closure of the graph of the canonical morphism $p : S^8 - (0) \to \boldsymbol{P}^7$. Let $q : B \to S^8$ be the natural projection. B lies on $S^8 \times \boldsymbol{P}^7$; is nonsingular; and carries a divisor E, isomorphic to $\boldsymbol{P}^7$, projecting to the origin in S^8 (cf. [16], ch. 3). The quadratic transform W^* is then the proper transform of W_0 via q^{-1}, i.e. the k-closure of $q^{-1}((y))$ where (y) is a generic point of W_0 over k. Clearly W^* and E meet properly on B. Consider the intersection product: $(W^* \cdot E)_B$. It is a cycle on W^* and also on E. Now it is shown in [16], p. 319, that $W^* \cdot E$ is defined *on* E (identifying $E = \boldsymbol{P}^7$) by the homogeneous ideal of leading forms of defining equations for W_0 in S^8, viz. $\{Z_1 = \cdots = Z_5 = 0,\ Z_6Z_8 = Z_7^2\}$. These equations define a nonsingular rational curve.

This shows, first, that $W^* \cdot E$ is a nonsingular variety. Therefore W^* and E are transversal at every point of intersection, [22], ch. 6, so W^* is nonsingular at every such point. Since E is locally defined on B by equations $Y_i = 0$ (cf. [16]; the Y_i's are induced by the coordinate functions in S^8), $W^* \cdot E$ is locally defined on W^* by equations $y_i = 0$ (cf. [22], ch. 9; y_i is the restriction of Y_i to W^*). Call $F = W^* \cdot E$: F is a nonsingular rational curve defined over k; every point of F is simple for W^*; and F is locally defined on W^* by equations $y_i = 0$. Thus the singularity at $(0) \in W^*$ is quite eliminated by a single quadratic transformation.

Furthermore: let $\mathfrak{m}''$ be the maximal ideal of the local ring $\mathfrak{O}''$ of W_0 at (0) and let $\lambda(\mathfrak{m}''^n)$ denote the length of the $\mathfrak{O}''$-module $\mathfrak{O}''/\mathfrak{m}''^n$. Then from the explicit form of the associated graded: $G(\mathfrak{O}'') = \oplus_{n \geq 0} \mathfrak{m}''^n/\mathfrak{m}''^{n+1} \cong k[X, Y, Z]/(XZ - Y^2)$, it follows easily that: $\lambda(\mathfrak{m}''^n) = \sum_{i=0}^{n-1} \left[\binom{i+2}{2} - \binom{i}{2}\right] = n^2$ (cf. [9], § 23). Consequently, $\mathfrak{O}''$—and also $\mathfrak{O}'$, therefore—is a two-dimensional ([16], p. 85) normal local ring such that $\lambda(\mathfrak{m}''^n) = n^2$ and whose singularity is eliminated by a quadratic transformation. Then by the criterion: Prop. (23.5) of Lipman, [9], we have that (0) (resp. v) is a rational double point of W_0 (resp. V'): in particular we have that $F^{(2)} = -2$.

Therefore, V^* carries a curve F (using the same letter) which is the exceptional curve of ρ^*; each point of F is simple for V^*; F is a nonsingular rational curve, defined over k; and $F^{(2)} = -2$.

Let $\lambda^* : V^* \to C$ be the composite rational mapping $V^* \xrightarrow{\rho^*} V' \xrightarrow{\lambda'} C$ of V^* to the cone C embedded in $\boldsymbol{P}^5$. We are interested now in the pull-back via λ^* of a section J of C by a hyperplane through the vertex. As to C we shall for now restrict attention to the affine part C_0 in S^5. Then we may continue to use W^*, W_0, etc.

Let $(\mathfrak{O}'', \mathfrak{m}'')$ denote as above the local ring of W_0 at (0). According to the relations (♯), $\mathfrak{m}''$ is generated by y_6, y_7, y_8. Let $h = h(y) = a_1 y_1 + \cdots + a_5 y_5 = 0$ be the defining equation on C_0 for a hyperplane section through the vertex. Call b the unit $1/\phi = b$ in $\mathfrak{O}''$; then we have by (♯) that $h(y) = b \cdot (a_1 y_6^2 + a_2 y_6 y_7 + a_3 y_7^2 + a_4 y_7 y_8 + a_5 y_8^2)$. That is, h is a unit times a homogeneous form of degree 2 in the generators of $\mathfrak{m}''$. Write accordingly: $h(y) = b \cdot H_2(y_6, y_7, y_8)$ (H_2—coefficients in $k(a) = k(a_1, \cdots, a_5)$). Now let v_f denote the one-dimensional valuation of $K = k(V')$ with center F on W^* (or on V^*). The center of v_F on W_0 is the single point (0) and $v_F(y_i) > 0$ for $i = 1, \cdots, 8$. Since y_6, y_7, y_8 generate $\mathfrak{m}''$, min. $v_F(y_i) =$ min. $\{v_F(y_6), v_F(y_7), v_F(y_8)\}$. Set min $\{v_F(y_i)\} = v_F(y_j)$, $j = 6, 7$, or 8. Then generic points of F are at a finite distance with respect to

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the standard affine open subset W^*_j of W^* corresponding to y_j. We have that $v_F(y_j) = 1$ since the equation $y_j = 0$ locally defines F on W^*_j (cf. above). For suitable (a) (for instance: $a_{(2j-11)} = 1$, others $= 0$), $v_F(h(y)) = 2$. As $v_F(y_i) \geq 1$, $i = 6, 7, 8$, and $\deg. H_2 = 2$, $v_F(h) \geq 2$ in any case, hence the value in 2 for general (a).

So far this shows that F appears with multiplicity two in a pull-back of a general section J as above.

Next let J be a section of C by a general hyperplane through the vertex, let $D' = \lambda'^{-1}(J)$ be the V'-divisor lying over J, and let D^* on V^* be the proper transform of D' via ρ^{*-1}: D^* is by definition the $\overline{k(a)}$-closure of the set $\bigcup_{i=1}^{d} \rho^{*-1}(x'_i)$ where points (x'_i) range over generic points of irreducible components of D' over $\overline{k(a)}$. F cannot ever be a component of D^* (as J varies). We want to consider the intersections of F with D^* (recall that V^* is simple at each point of F). Again we use W^*, W_0, etc. As before with $W^* \cdot E$, we have that $(D^* \cdot E)_B$ is defined on $E = \boldsymbol{P}^7$ by homogeneous equations: $\{Z_1 = \cdots = Z_5 = 0,\ Z_6 Z_8 = Z_7^2,\ H_2(Z_6, Z_7, Z_8) = 0\}$. For a general choice of (A)—even $H_2(Z) = Z_6^2 - Z_8^2$ will do—the cycle $D^* \cdot E$ is the sum of four distinct points each counted once. Therefore D^* and E are transversal at each common point; also $D^* \cap F$ consists of simple points of D^* and of F. Since $F = (W^* \cdot E)_B$ we have that: $(D^* \cdot F)_{W^*} = (D^* \cdot (W^* \cdot E)_B)_{W^*} = (D^* \cdot E)_B$ (cf. [22], p. 233). Consequently, $D^* \cdot F$ consists of a sum of four distinct points, each counted once.

Thus for general J, the pull-back $\lambda^{*-1}(J) = 2F + D^*$; D^* and F meet properly and $I(D^*, F) = 4$; and the intersections are generally transversal at four points. A little later (2.4) it will appear that F is never a component of $\lambda^{*-1}(J) - 2F$, even for a special J.

Next we take apart D^*. Let L be a general generator of C and let $E'' = \lambda'^{-1}(L)$ be the V'-divisor lying over L, and let E^* on V^* be the proper transform of E' via ρ^{*-1}. A general section J as above is a sum, $J = \sum L_i$, of four distinct lines L_i so $D' = \sum E'_i$ is a similar sum and likewise D^* is the sum of four E^*_i. Let L_1, L_2 be distinct generic generators (not necessarily independent) over k on C, and let E^*_1, E^*_2 be the corresponding V^*-divisors. By what was just shown, namely, that $\lambda^{*-1}(J) = 2F + D^*$ with F not a component of D^*, for general J, the divisors E^*_1, E^*_2 are linearly equivalent on the normal surface V^*. Applying this to a general $D^* = \sum_{i=1}^{4} E^*_i$, we get that $I(D^*, F) =$

$4 \cdot I(E^*_i, F)$; as $I(D^*, F) = 4$, we get that $I(E^*_i, F) = 1$. Since the intersections of D^* and F are generally transversal, at four points, E^*_1 and E^*_2 meet with F at different points.

Finally we consider the rationality of the divisors E^* and of the points $E^* \cdot F$. Let u be a variable quantity over k. The equations: $Y_2 = u \cdot Y_1$, $Y_3 = u \cdot Y_2$, $Y_4 = u \cdot Y_3$, $Y_5 = u \cdot Y_4$ define a line through the origin of S^5 and lying on C_0, so they define a generator of C_0; which we call L_u. Let E^*_u on V^* correspond to this L_u. E^*_u is rational over $k(u)$. Now V^* and F are also rational over $k(u)$, hence $P_u = E^*_u \cdot F$ is too (cf. [22], ch. 8). Thus, E^*_u has a simple point $E^*_u \cdot F$, rational over a field of definition of E^*_u.

So far we have shown:

LEMMA 8. *Let C, Γ be as above. There exists a surface V^* and a map $\lambda^* : V^* \to C$ satisfying*

(1) *V^* is normal, projective, irreducible.*

(2) *λ^* is a proper morphism of degree 2, finite except over the vertex and ramifying over Γ.*

(3) *There is an irreducible nonsingular rational curve F on V^*, whose support is the set-theoretic inverse image of the vertex; F has coefficient 2 as a component of the pull-back of a general section of C by a hyperplane through its vertex; every point F is simple for V^*; and $F^{(2)} = -2$.*

(4) *When E^* lies over a general generator L of C (as above: E^* is the proper ρ^{*-1}-transform of $\lambda'^{-1}(L)$), then $I(E^*, F) = 1$; and when E^*_1, E^*_2 lie over two different generators L_1, L_2, then $E^*_1 \sim E^*_2$ and $E^*_1 \cdot F \neq E^*_2 \cdot F$.*

(5) *When (u) is a point of F, there is a generator L_u and E^*_u lying over it such that $E^*_u \cdot F = (u)$; and E^*_u is defined over $k(u)$.*

(2.3). The next step is to deal with multiple points of V' lying over simple points of C. (Since $V^* - F$ and $V' - v$ are biregularly isomorphic the following results can apply at once to V^*.)

Recall that the complement of the vertex on C is covered by two affine surfaces which we called C_1, C_5, which are each set-theoretically the complement of a generator, cf. (2.0). Let V'_1, V'_5 be the open subsets of our normalization V' lying respectively over C_1, C_5. Each V_i is isomorphic to an affine surface by a basic property of normalizations and it will be convenient to have explicit models available. These two models are very similar and are much simpler than W_0 previously examined. Recall that a surface in S^3, defined by an equation $Z^2 = f(X, Y)$ with f a square-free polynomial over a field of characteristic $\neq 2$, is automatically a normal surface ([26], vol. 1). (More generally an irreducible affine hypersurface is normal if it is nonsingular in codimension 1, cf. [16].) In the notations of (2.1), by definition $\psi(x) = \Psi(X)/X_1^3$, and we can write $\psi(x) = \psi_1(x_1, x_2)$ since x_3, x_4, x_5 are powers of x_2. The polynomial $\psi_1(X_1, X_2)$ was shown to be square-free in (2.1). When z is a function defined by the equation $z^2 = \psi_1(x_1, x_2)$ then the field of k-rational functions $K = k(V') = k(x_1, x_2, z)$. Then the domain $k[x_1, x_2, z]$ is automatically the integral closure of $k[x_1, x_2]$ in the field of functions K. Therefore the surface W_1 in S^3 defined by $Z^2 = \psi_1(X_1, X_2)$ is a normal affine model of K. W_1 can be regarded as lying over C_1 via the isomorphism $\iota: S^2 \rightarrow C_1$ in (2.0). By the uniqueness of normal covers up to isomorphisms ([22], app. 1) W_1 and V'_1 are isomorphic.

Similarly there is an explicit model W_5 for V'_5. Incidentally we now have a covering of V' by affine open subsets V'_0, V'_1, V'_5 lying respectively over C_0, C_1, C_5; each V'_i is explicitly represented by an affine model $W_i, i = 0, 1$ and 5.

Let P be a generic point of W_1 over k. We are regarding x_1, x_2, z as k-rational functions on W_1; set $x'_1 = x_1(P)$, etc. Then $K \cong k(x'_1, x'_2, z')$. Now clearly $k(x'_1, x'_2, z')$ is separable over $k(x'_1, x'_2)$, so X_1, X_2 are uniformizing linear forms in S^3 for W_1 at P (cf. [22], ch. 4). More particularly, computing the Jacobian matrix for our equation $Z^2 = \psi_1(X_1, X_2)$ shows that every point of W_1 is simple except for those points lying over multiple points of Γ_1 (cf. (2.1)), which are indeed multiple points on W_1. Repeating this for W_5 we get: V' is nonsingular at points lying over simple points of C, except over (finitely many) points of Γ where V' is singular.

Now we show

LEMMA 9. *Let g be a point of V' lying over a multiple point γ of Γ. Then g is a multiple point of V' and is the only point of V' lying over γ.*

Proof. We must use that deg. $\lambda' = 2$ and that γ is simple on C. One can employ "conservation of number", [22]. Let $\boldsymbol{P}$ be an ambient projective space for V'. Let A be the graph of λ', and let c be a generic point of C over k. Since γ is not the vertex it is simple on C; then every point of $|A| \cap \boldsymbol{P} \times \gamma$ is simple on the ambient space $\boldsymbol{P} \times C$. Therefore $A \cdot V' \times \gamma$ is the unique specialization of $A \cdot V' \times c$ extending the specialization $c \to \gamma$; furthermore if $A \cdot V' \times \gamma$ has 2 components (g_1, γ) and (g_2, γ) then $A \cdot V' \times \gamma = 1 \cdot (g_1, \gamma) + 1 \cdot (g_2, \gamma)$, as deg. $\lambda' = 2$. Consequently A and $V' \times \gamma$ are transversal at each point of intersection on $\boldsymbol{P} \times C$, if $g_1 \neq g_2$. Hence the points of intersection are simple points on $V' \times C$, therefore g_1 and g_2 are simple on V'. But we saw that this could not happen.

At this point we shall digress slightly to consider divisors on V' over generators on C. Let L be a general generator and set $E' = \lambda'^{-1}(L)$. Then E' is an irreducible nonsingular curve of genus one—this is clear from the explicit affine models W_1, W_5. Indeed, $L \cap C_1$ corresponds to a line $X_2 =$ const. $= a$, say. $E' \cap V'_1$ is isomorphic to a plane cubic curve $Z^2 = \psi_1(X_1, a)$, lying on W_1 and defined there with multiplicity one by $X_2 = a$ as X_2 is a uniformizing linear form for W_1 at generic points. Moreover for general $a = X_2, \psi_1(X_1, a)$ has distinct roots. More particularly, since $\Gamma \cdot L = 3$ distinct points counted once, and Γ does not go through the vertex, according to assumption, the polynomial $\psi_1(X_1, X_2)$ always has degree 3 in X_1 for any value of X_2 and, for a general value $X_2 = a$, has distinct roots. Consequently, for any generator $L'', E'' = \lambda'^{-1}(L'')$ is irreducible and, for general $L, E' = \lambda'^{-1}(L)$ is a nonsingular curve of genus one. Clearly every point of a general E' is simple for V' (excepting v of course) as L does not meet Γ at its multiple points, cf. previous lemma.

If we are going to find a $K3$ surface V with field of functions K over k, it will surely be necessary to place some restrictions on allowable types of singularities γ on Γ. Indeed, regarding V, we must have in particular that a canonical divisor K_V on V is linearly equivalent to zero, from which it follows that an irreducible curve G on V which is a component of a divisor collapsing to a multiple point g of V', must be a nonsingular rational curve—$G^{(2)} < 0$ by [14], the arithmetic genus $p_a(G) = (1/2)(G^{(2)} + I(K_V, G)) + 1$ is non-negative, [19], therefore $p_a(G) = 0$ and also $G^{(2)} = -2$. But it is easy to find examples of Γ where, when the singularity at g is resolved, curves G of arithmetic genus

$p_a(G) > 0$ occur (for instance let $\Psi(X) = X_0^3 + X_3^3$ (char $\neq 3$); this induces $x_1^3 + x_2^6 = 0$ as local equation for Γ at $\gamma =$ origin in affine coordinates (X/X_1); then an elliptic curve is seen to occur in a resolution of $z^2 + x_1^3 + x_2^6 = 0$.—Here, we shall give a sufficient condition on the singularities γ of Γ to insure $K_V \sim 0$ is possible.

First of all: let a multiple point g of V' lie over γ, as above. Then

(*): *If γ is restricted to be at most a triple point with at most a double point in its first neighborhood then there exists a sequence $V' \leftarrow V'^{(1)} \leftarrow V'^{(2)} \leftarrow \cdots \leftarrow V'^{(n)}$ of quadratic transformations resolving the singularity at g.*

Proof. We know, by well known work of Zariski and Abhyankar, that a finite sequence of, alternately, quadratic transformations and normalizations will resolve g. The claim is that with the restriction on γ no normalizations are needed.

The given multiple point g lies either on V'_1 or on V'_5, say on the former. The corresponding point on W_1 has Z-coordinate zero, so we can assume that it is at the origin in S^3. Recall that, as noted earlier, a surface in S^3 defined by an equation $Z^2 = f(X, Y)$ with f square-free is automatically normal.

We work with W_1 and shall show that, after a quadratic transformation centered at the origin, the transformed surface is again a double plane $Z^2 = f(X_1, X_2)$ branched over a plane curve f without multiple components and satisfying the condition in (*), that a multiple point of f is at most a triple point with at most a double point in its first neighborhood. Such a branch curve will be called "admissible".

Given: $Z^2 - f(X_1, X_2)$. Let us review the effect of a quadratic transformation centered at the origin O. Suppose that f has an r-fold point at O. Write out $f(X_1, X_2) = \sum_{k\geq 0} L_{r+k}(X_1, X_2)$ as a sum of forms. Suppose that X_2 does not divide L_r, so that the line $X_2 = 0$ is not among the tangents at O. After the transformation $X'_1 = X_1/X_2, X'_2 = X_2$, of the plane, f becomes $f' = f'(X'_1, X'_2) = \sum_{k\geq 0} {X'_2}^k \cdot L_{r+k}(X'_1, 1)$. This as usual shows that there is a 1–1 correspondence, preserving multiplicities, between the set of tangents at O, and the intersections of the transformed curve f' with the fundamental line $X'_2 = 0$: for the former set is equivalent to the set of roots $(a : b)$ of the leading form $L_r(X_1, X_2)$

((1 : 0) is not present just now) and the latter set to the set of roots a/b of $L_r(X'_1, 1)$. Furthermore, under the transformation $X'_1 = X_1/X_2$, $X'_2 = X_2$, $Z' = Z/X_2$, $Z^2 - f(X_1, X_2)$ becomes $Z'^2 - \sum_{k\geq 0} X'_2{}^{(r+k-2)} \cdot L_{r+k}(X'_1, 1)$. This shows that the transformed surface is a double plane branched over a curve in which the fundamental line $X'_2 = 0$ appears with multiplicity exactly $r - 2$ as a component.

Now we show that the transform of a double plane with admissible branch curve is again a double plane with admissible branch curve. With our notation above, $r \leq 3$. If $r = 1$ there is nothing to do. If $r = 2$ the new branch curve after a quadratic transformation of the surface is the transform f' of the old curve f, so is still admissible. Next suppose that $r = 3$. The transform of $Z^2 - f(X_1, X_2)$ is $Z'^2 - \sum_{k\geq 0} X'_2{}^{(k+1)} \cdot L_{k+3}(X'_1, 1)$ and the fundamental line $X'_2 = 0$ appears just once as a component of the new branch curve. f' denoting still the transform of f, the new branch curve is defined by $X'_2 f' = 0$.

We shall proceed to consider cases according to the number of distinct tangential directions at the triple point O on f. If there are three tangents, each is counted once and so, by what we observed above regarding the correspondence between tangents and intersections with the fundamental line, the new branch curve will have three double points arising from O and will so be admissible. If there are two tangential directions at O, f' will intersect the fundamental line at two places, transversally at one and, at the other, with multiplicity two. Then at worst the new branch curve $X'_2 f'$ will have again a triple point with two distinct tangential directions, which is clearly admissible.

There remains to deal with the case in which the branch curve f has a single, 3-fold intersection with the fundamental line. f being admissible, f' cannot have a triple point, so either this intersection consists of a curve with a simple flex meeting with the tangent there, which is an admissible circumstance, or else f' has a double point and the fundamental line is a tangent there. We may assume, in the latter case, that the fundamental line is the tangent at a cusp, as the possibility of two tangents has been considered already. Therefore we have: the new branch curve $X'_2 f'$ has a triple point with one tangential direction, and the number of intersections of the tangent line $X'_2 = 0$ with f' is three. A little more generally, suppose now that a plane curve $g(X_1, X_2) = 0$ has an r-fold point P and that there is just one tangent there, which has $(r+s)$-fold contact with g. Then it is readily seen that, after a quadratic transformation of g centered at P, the transform g' has one point P' corresponding to P, and that P' is at most an s-fold point for g'. In our case, $r=2$ and (the "class") $s=1$ so the new branch curve $X'_2 f'$ is admissible.

It should be remarked that the passage from f to f' does not introduce multiple components as the two curves are in biregular correspondence almost everywhere. Thus it is shown that the transform of a double plane $Z^2 - f(X_1, X_2)$ with admissible branch curve f, is again of the same type, so that a sequence of quadratic transformations is sufficient to resolve the singularities.

This finishes what we have to say about multiple points on V', except for stating that from now on the choice of the form Ψ is to be limited so that multiple points on V' can be eliminated by a sequence of quadratic transformations only. (We already know that this is the case for $v \in V'$ over the vertex of C.) Such multiple points are called "rational double points" sometimes (cf. [9]...)

(2.4). The limitation just stated being now in force, there exists a *unique minimal desingularization* $\mu: V \to V'$ (i.e. every desingularization of V' factors through μ), and μ is a product of quadratic transformations, by Lipman's results ([9], (23.4), (23.5), (4.1)). Order this product so that ρ^* comes first (cf. (2.2)) and μ factors thus:

$$\begin{tikzcd} V \arrow[rr, "\mu"] \arrow[dr, "\mu^*"'] & & V' \\ & V^* \arrow[ur, "\rho^*"'] & \end{tikzcd}$$

Now we may regard F as lying on V^* or on V according to convenience. We have that ρ^* induces a biregular correspondence between $V^* - F$ and $V' - v$, and that μ^* is biregular except wherever V^* has a multiple point g.

Let $\lambda: V \to C$ be the proper morphism $\lambda = \lambda' \circ \mu$.

We wish to discuss divisors on V situated over generators of the cone C. Concerning this subject there is preliminary matter in Lemma 8 and in "digression" in (2.3). Let L be a general generator and, as before, set $E' = \lambda'^{-1}(L)$. We saw in (2.3) that every point of E' is simple for V' except v. Let E on V be the proper transform of E', via μ^{-1}.

Recall that E^* on V^* was defined to be the proper ρ^{*-1}-transform of E', in (2.2). Of course V^* is simple at each point of E^*. Now we may apply Lemma 8 to conclude that, when L_1, L_2 are distinct general generators, the corresponding divisors E_1, E_2 on V are linearly equivalent and meet F transversally and at different points. L_1, L_2 meet only at the vertex so E_1 and E_2 don't meet at all. Incidentally, of the several divisors on V corresponding to multiple points of V', only F has points in common with more than one such E.

Suppose that $E \sim G + F$ with $G > 0$. Then $0 = E^{(2)} = I(E, G) + I(E, F) \geq 1$, impossible. Hence F never appears as a component of a divisor linearly equivalent to E.

Let $\langle E \rangle$ denote the linear pencil of divisors including E which lie over the pencil of generators of C. We have seen that $\langle E \rangle$ is irreducible and base-point-free, and that a general member E is a nonsingular curve of genus one such that $E^{(2)} = 0$, and $I(E, F)$ is always defined for all members of $\langle E \rangle$ and $=1$.

This also applies *a posteriori* to the pencil $\langle E^* \rangle$ on V^* (with obvious notation). $\langle E^* \rangle$ has the additional feature that every member is irreducible (cf. (2.3)).

(*Remark* 2.1. We shall say a word about reducible members of $\langle E \rangle$. V is biregularly equivalent to V^* except wherever the latter has a multiple point g; since the intersection number $I(\Gamma, L)$ of Γ and a generator L is 3, a curve $E^* \in \langle E^* \rangle$ passes through at most one multiple point g. Let g be such a multiple point on V^*, let E^* pass through it, and let $E^\# \in \langle E \rangle$ on V lie over the line $\lambda^*(E^*)$. Write $E^\#$ as a sum $E^\# = E_0 + (r_1 E_1 + \cdots + r_n E_n)$ of irreducible components where $\sum_{i=1}^n r_i E_i$ collapses to g. The multiplicities (r_i) may be determined as follows. Let $Z = \sum_{i=1}^n s_i E_i$ be the "fundamental cycle" of $\cup E_i$ (cf. [2]); then $s_i \leq r_i$, all i, because E^* itself is locally at g a plane section and we have that "$\mathfrak{m} \cdot \mathfrak{O}_V = \mathfrak{I}(Z)$" on account of V^* having rational singularities only, cf. [2], th. 4. Now it is easy to check that, because of the fact that $I(\Gamma, L) = 3, E_0$ meets only components E_i which are in the first neighborhood of g; that is the vector $(I(E_0, E_i))$ has coordinate zero except for such E_i. If that is so the vector is easy to find; assume it is known for our g. The multiplicities (r_i) are then determined uniquely as the solution of the system of n equations: $\sum_{j=1}^n r_j I(E_j, E_i) = -I(E_0, E_i)$,

$i = 1, \cdots, n$—the equations express the fact that $I(E^*, E_i) = 0$ for $i = 1, \cdots, n$ and the solutions are unique because the matrix $\|I(E_i, E_j)\|$ is negative definite, cf. [14]. The solutions are in fact $r_i = s_i$ for all i and they may be found in DuVal [5].)

(2.5). One consequence of our assumption, that multiple points of V', hence of V^*, are rational double points, is that when K^* is a canonical divisor on V^*, then $K = \mu^{*-1}(K^*)$ is defined and is a canonical divisor on V (Artin [1], [2]). Therefore if we can show that $K^* \sim 0$ on V^* we may conclude that $K \sim 0$ on V—to work on V^* will be a convenience as we know that all the curves in $\langle E^* \rangle$ are irreducible. Recall too, that E^* is base-point-free and when E^* is a general member, every point of E^* is simple for V^* and E^* is a nonsingular curve of genus one.

Lemma 10. *Let K^* be a canonical divisor on V^*. Then $K^* \sim 0$.*

Proof. Assume that K^* is k-rational. Suppose we show that $K^* \sim mE^*$, m an integer and $E^* \in \langle E^* \rangle$. We have seen that intersection numbers $I(F, X)$ are defined for divisors X. As F is of genus zero, $I(F, F + K^*) = -2$. But $I(F, E^*) = 1$ and $F^{(2)} = -2$ so $K^* \sim mE^*$ will imply that $K^* \sim 0$.

Now we show $K^* \sim mE^*$. Let E^* be a fixed member of $\langle E^* \rangle$, generic over k. E^* is a nonsingular curve of genus one and each point of E^* is simple for V^*. Then there is a function f on E^* with div $(f) = E^* \cdot (E^* + K^*) = E^* \cdot K^*$. Let $h \in k(V^*)$ induce f on E^*. We have that div $(h) \cdot E^* =$ div (f), [22], ch. 9. Consider now div $(h) - K^*$. It is k-rational so E^* is not a component; let D be an irreducible component such that $I(D, E^*) > 0$. Let $k(t)$ be a field of definition for E^*, t being variable over k. The intersection product $D \cdot E^*$ is $k(t)$-rational ([22], ch. 8); if it were rational over the algebraically closed field k the pencil $\langle E^* \rangle$ would have some base points as $D \cdot E^*$ would belong to every specialization of E^* over k—but $\langle E^* \rangle$ is base-point-free. Therefore when D is an irreducible component of the k-rational divisor div $(h) - K^*$, such that $I(D, E^*) > 0$, then $D \cdot E^*$ is supported at generic points of D over k. Consequently two different such components D will meet E^* at disjoint sets of points; in other words, when P is a point in the support of $\Delta =$ (div $(h) - K^*) \cdot E^*$ it is a generic point over k of a unique

D. So, its coefficient in Δ is the coefficient of D in $\operatorname{div}(h) - K^*$ times $i(D, E^*; P)$. But its coefficient in Δ is zero, as $\operatorname{div}(h) \cdot E^*$ and $K^* \cdot E^*$ are, by choice of h, one and the same E^*-divisor. Therefore when D is an irreducible component of $\operatorname{div}(h) - K^*$ such that $I(D, E^*) > 0$, the coefficient of D is zero; consequently, components D_1 with non-zero coefficient satisfy $I(D_1, E^*) = 0$.

Fix such a D_1. Since all the E^* are irreducible it follows that, if D_1 has a point in common with any $E^*_1 \in \langle E^* \rangle$, then $D_1 = E^*_1$ for otherwise $I(D_1, E^*_1) > 0$. But there must exist E^*_1 meeting D_1. In fact there is a function $f: V^* \to \boldsymbol{P}^1$ on V^* corresponding to the linear pencil $\langle E^* \rangle$. As $\langle E^* \rangle$ is base-point-free, f is actually defined everywhere but in any event it is defined at some point d of D_1; then $f^{-1}f(d)$ will meet D_1. But $f^{-1}f(d)$ is a member of $\langle E^* \rangle$.

Thus $K^* \sim mE^*$ as desired.

So far we have shown:

PROPOSITION 2. *Let Ψ, C and Γ be as defined at the beginning of* (2.1). *Γ is to be restricted as to its multiple points by the condition in* (*) *in* (2.3). *Then there is an algebraic surface V and a map $\lambda: V \to C$ with the following properties:*

a) *V is a nonsingular projective surface and λ is a morphism of degree* 2, *ramifying over simple points of Γ (char. $\neq 2$ as always).*

b) *A canonical divisor on V is linearly equivalent to zero.*

c) *V carries an irreducible linear pencil $\langle E \rangle$ of curves E such that a general member is a nonsingular curve of genus one; $\lambda(E)$ is a generator of C, and the restricted map $\lambda | E$ ramifies over $\Gamma \cdot \lambda(E)$ and the vertex, when E is nonsingular.*

d) *There is a nonsingular rational curve F on V whose support $|F| = \lambda^{-1}(vertex)$ (as sets).*

e) *The pull-back of a section of C by a hyperplane passing through the vertex is linearly equivalent to $4E + 2F, E \in \langle E \rangle$; $F^{(2)} = -2, E^{(2)} = 0$, $I(E, F) = 1$.*

f) *Let (u) be a point of F and let $F_u \in \langle E \rangle$ be the curve through (u) $= E_u \cdot F$. Then E_u is defined over $k(u)$.*

The next proposition shows that the surfaces V constructed here are, in general, $K3$ surfaces—at least when the characteristic is zero.

Proposition 3. *Let A be the Albanese variety of V and let q be the dimension of A. Suppose that every member of the pencil $\langle E\rangle$ is an irreducible curve. Then $q = 0$.*

Proof. Let $\alpha: V \to A$ be the Albanese map. We shall assume that $q > 0$ and show that $q = 1$ in that case. Then we shall see that a multiple of the curve F is a fibre of the map α, which will be impossible on account of $F^{(2)} \neq 0$.

Let K be a field of definition for (V, A, α) which contains k. Let (x) be a generic point of V over K and let E_u be the curve in $\langle E\rangle$ through (x). By the previous proposition, (f), we know that E_u is defined over $K(u)$, where $(u) = E_u \cdot F$. Clearly, (u) is a generic point over K on the k-rational curve F.

E_u has a rational point (u) so we can assume that E_u is an abelian variety defined over $K(u)$ with identity element $(u) = E_u \cdot F$. Since F is a rational curve, $\alpha(F)$ is a point a of A; F and A are both defined over K so we can take a to be the identity element in A. Thus α induces a rational homomorphism $\alpha_u: E_u \to A$, defined over $K(u)$. If α_u is a constant mapping then $\alpha_u(E_u) = \alpha_u(u) = a$. This would mean $A = a$ because (V, α) generates A. Therefore if we assume that $q > 0, \alpha_u$ is not constant and $\alpha_u(E_u)$ is a curve defined over $K(u)$. By a theorem of Chow, ([4]), the K-rational Abelian variety A cannot carry a family of abelian subvarieties, so $\alpha_u(E_u)$ is K-rational. Then $\boldsymbol{E} = \alpha_u(E_u)$ is independent of u, and so $A = \boldsymbol{E}$.

Now we show:

1) α_u is set-theoretically 1–1.

2) α is a morphism and $D^{(2)} = 0$, where $D = \alpha^{-1}(y)$, $(y) \in \boldsymbol{E}$ generic over K.

Indeed, as to 1), this may be shown using Chow's "$K(u)/K$-image" of E_u, (Chow [4]; cf. remarks on p. 255). Let E_0 be the $K(u)/K$-image of E_u: then, E_0 is an abelian variety defined over K and there is a surjective primary homomorphism $\lambda_u: E_u \to E_0$ rational over $K(u)$ and, by the maximal property of E_0, a commuting triangle (1), ϕ being rational over K. Therefore E_0 is an elliptic curve (and not a point) and λ_u is a purely inseparable isogeny over $K(u)$, i.e. λ_u is set-theoretically 1–1. To reverse the direction of ϕ and show that $E_0 = \boldsymbol{E}$, we note that, on the other hand, there is a mapping $\lambda: V \to E_0$ rational over K, induced by λ_u; hence, by the universal character of the Albanese variety $\boldsymbol{E}$ of V, a mapping $\mu: \boldsymbol{E} \to E_0$ such that the triangle (2) commutes.

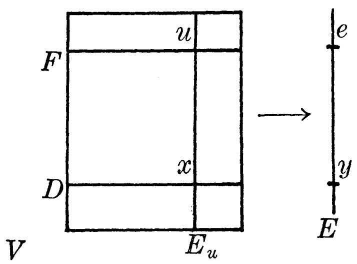
$$\text{(1)} \quad \begin{tikzcd} & E_u \arrow[dl, "\lambda_u"'] \arrow[dr, "\alpha_u"] & \\ E_0 \arrow[rr, "\phi"'] & & \boldsymbol{E} \end{tikzcd} \qquad\qquad \text{(2)} \quad \begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\lambda"] & \\ \boldsymbol{E} \arrow[rr, "\mu"'] & & E_0 \end{tikzcd} .$$

The map μ is rational over K. Now $\mu\alpha = \lambda$, $\phi\lambda = \alpha$, so $(\phi\mu)\alpha = \alpha$ and $\phi\mu$ is the identity mapping of $\alpha_u(E_u)$, that is of $\boldsymbol{E}$. Similarly $\mu\phi$ is the identity on E_0. Hence the curves $\boldsymbol{E}$ and E_0 may be identified. Thus $\alpha_u = \lambda_u$ is a purely inseparable isogeny, which proves (1).

Call E the curve $E = E_0 = \boldsymbol{E} = A$.

2) α is a morphism because a rational map of a nonsingular variety into an abelian variety is a morphism, cf. [8], p. 20. Now we have a morphism of V into a complete nonsingular curve E and we may apply the results of ([22], ch. 9) to conclude that $D^{(2)} = 0$. This proves 1) and 2).

Let e be the identity element of the elliptic curve E/K. $e = \alpha(F)$ as we saw. Let G be the K-rational positive cycle $\alpha^{-1}(e)$ on V. Let $|B|$ denote support of a cycle B. We claim that $|G| = |F|$.



First, F is the sole component of G which meets E_u. In fact, by 1) above, (u) is the only point of E_u lying over (e). The K-locus of (u) is F and components of G are algebraic over K so only F goes through (u). Second, let $v_F(G) = f$ be the coefficient of F and let $G' = G - f\cdot F$. Assume that every member of $\langle E\rangle$ on V is irreducible. Then any component G'' of G' must meet D: in fact, we just saw that no component of G' meets E_u, so $I(G'', E_u) = 0$; this means that G'' is a member of $\langle E\rangle$, as in the proof of Lemma 10. Therefore, $I(G'', D) = I(E_u, D) > 0$.

Since $G \succ 0$ and $I(G, D) = 0$, by 2), there can be no G' except 0. Therefore $|G| = |F|$.

Finally, $F^{(2)} < 0$ implies $G^{(2)} < 0$. But $G^{(2)} = D^{(2)} = 0$. So there can be no such curve $E = A$. Therefore $q = 0$.

Remark 2.2. The argument shows that $q = 0$ in some cases where $\langle E \rangle$ has reducible members. For instance, suppose that every reducible member E' is of the form $E' = E_0 + E_1$, both E_0, E_1 nonsingular rational curves with $E_0^{(2)} = E_1^{(2)} = -2, I(E_0, E_1) = +2$ and $I(E_0, F) = 1, I(E_1, F) = 0$ —corresponding to ordinary double points on Γ. Then no component of $G' = G - f \cdot F$ can be a component of any E'. Indeed, fix an $E' = E_0 + E_1$. One can show that (in any case) D is irreducible. Then if either component of E' is a component of G', E_0 must be so as G is connected and F is a component of G—otherwise G' would have a component G'' with $I(G'', E_u) > 0$, impossible; then $I(D, E') = I(D, E_u) > 0$ and $I(D, G) = 0$ together imply that: $I(D, E_1) > 0, I(D, E_0) = 0$. Then E_1 cannot be a component of G' as $I(D, G) = 0$. Now $(aF + bE_0)^{(2)} < 0$ for integers a, b not both zero. Applying this to the various reducible E', we see that no component of G' is a component of any E'. But in that case a component G'' of G' must satisfy $I(G'', E_u) > 0$, which (again) cannot be. Thus $G' = 0$.

§ **3.** Next we want to check that our constructions account for all of the special $K3$ surfaces. That is, starting with a $K3$ surface V carrying a nonsingular rational curve F and a nonsingular curve E of genus one with $I(E, F) = 1$ we want to produce a homogeneous form Ψ of degree 3 in 6 variables, depending only on V, E, and F, so that the equation $\Psi = 0$ defines the branch curve of the map determined by $\Lambda(4E + 2F)$.

Since V is a $K3$ surface and E is a curve of (arithmetic) genus one we have that $l(E) = 2$. Let $\langle E \rangle$ denote the pencil of curves containing E. By assumption a general curve in $\langle E \rangle$ is irreducible and nonsingular. However, reducible curves may occur in the pencil.

(3.1). Let k be an algebraically closed field of definition for V, F and $\langle E \rangle$, of characteristic $\neq 2$. In order to retrace our steps we start with the complete linear system $\Lambda = \Lambda(4E + 2F)$. First we must get the cone C.

PROPOSITION 4. *$\varDelta$ is a reduced linear system; let λ be the associated rational map of V into a projective space. Then λ is a morphism of degree 2, and the image $\lambda(V)$ is a cone in $\boldsymbol{P}^5$ which is projectively equivalent to C. Moreover, $\lambda(E)$ is a generator line of the cone, and $\lambda(F)$ is the vertex.*

Proof. By Lemma 4, $\varLambda$ is irreducible and base-point free; let λ be the non-degenerate rational morphism associated to $\varLambda$. (Non-degenerate means that the image is not contained in a hyperplane.) Again by Lemma 4, $\lambda(V)$ is a projective surface. Let D be an irreducible curve in $\varLambda$. $\mathfrak{l}(D) = 6$ so the image of V lies in $\boldsymbol{P}^5$, spanning $\boldsymbol{P}^5$. $\lambda(F)$ is a point: it is irreducible since F is so, and it is zero-dimensional since $I(F, D) = 0$. Thus $\lambda(V)$ is a projective surface spanning $\boldsymbol{P}^5$.

We shall show next that the projective degree of $\lambda(V)$ is 4. Since $D^{(2)} = 8$ the degree of $\lambda(V)$ divides 8, in fact $\deg\lambda\cdot\deg\lambda(V) = 8$. Let E be a nonsingular curve of genus one in $\langle E\rangle$. We have $I(D, E) = 2$. The complete linear series on E containing the cycle $D\cdot E$ of degree 2 has geometric dimension one and determines a 2–to–1 rational map. The trace of $\varLambda$ on E must be this complete series: the alternative can only be that $\lambda(E)$ is a point, contradicting $I(D, E) > 0$. Consequently λ induces a 2–to–1 map on E. Since $I(D, E) = 2$ the image has degree 1 in $\boldsymbol{P}^5$. It also follows that $\deg\lambda \geq 2$ so $\deg\lambda(V) \leq 4$.

$\mathfrak{l}(E) = 2$ on V so we may select a function x whose polar divisor is just E. x takes constant values along members of $\langle E\rangle$. Regarding x as a mapping to the projective line, call $x(E)$ the constant value along E; then $x(E) \neq x(E')$ when $E \neq E'$ are in $\langle E\rangle$. Since $L(4E + 2F) \supset L(E)$, $\lambda(E) \neq \lambda(E')$ as lines. Now $\lambda(E)$ and $\lambda(E')$ are different lines in $\boldsymbol{P}^5$ which meet at $\lambda(F)$.

Let P, P' be points of V generic over k. Let E and E' in $\langle E\rangle$ contain respectively P, P'. Then E, E' are generic curves in $\langle E\rangle$ over k and are irreducible. $\lambda(F)$ is k-rational and $\lambda(P), \lambda(P')$ are generic over k on the respective curves, so $\lambda(P) \neq \lambda(F) \neq \lambda(P')$. Suppose $E \neq E'$. Then it follows that $\lambda(P) \neq \lambda(P')$ since the two lines $\lambda(E), \lambda(E')$ cannot meet except at $\lambda(F)$.

Therefore $\deg\lambda \leq 2$ so $\deg\lambda = 2$ and $\deg\lambda(V) = 4$.

Call $C^* = \lambda(V)$. We have also shown that C^* is the union of the lines $\lambda(E)$, all of which pass through the point $\lambda(F)$. In consequence of this and of the property that $\lambda(V)$ spans $\boldsymbol{P}^5$, a general section of C^* by a hyperplane not through $\lambda(F)$ will be an irreducible curve B of degree 4, spanning a $\boldsymbol{P}^4$. We saw in (2.0) that B is then rational and projectively normal. C^* is the cone projecting B from the vertex $\lambda(F)$.

To see finally that $C^* = \lambda(V)$ is projectively equivalent to C, first move the cone C^* by a projective transformation to a cone C' whose vertex coincides with the vertex of C. Then cut $C + C'$ with a hyperplane not through the common vertex. Let B, B' be the base curves of C, C' cut that way. They are 4-fold images of the projective line (2.0), hence are projectively equivalent. This equivalence extends to move C' to C. Thus C and C^* are projectively equivalent. This ends the proof.

(3.2). Next we must produce a form Ψ of degree three such that the hypersurface defined in $\boldsymbol{P}^5$ by the equation: $\Psi = 0$ cuts out on C the branch curve Γ of the mapping λ. We shall see that Γ does not pass through the vertex of C. First, let us prove:

LEMMA 11. *Let A be an irreducible curve on C, defined over k, not passing through the vertex. Then there is a hypersurface $\mathfrak{H}$ in $\boldsymbol{P}^5$ such that $A = C \cdot \mathfrak{H}$.*

Proof. It is enough to show that A is linearly equivalent to a hypersurface section of C. For, we saw that a base curve B of C is projectively normal, (2.0). Let b be a homogeneous generic point of B over k, and let t be variable over $k(b)$. Then (t, tb) is a homogeneous generic point of C over k. Since b is a generic point over k of an affine cone (C_0), $\dim_k(b) = \dim_k(tb)$; hence $\dim_k(t, tb) = 1 + \dim_k(b) = 1 + \dim_k(tb)$, so t is variable over $k(tb)$. tb is a homogeneous generic point over k of the projectively normal curve B, hence the ring $k[tb]$ is integrally closed. It is an exercise to show that, when a domain R is integrally closed in its quotient field K and t is variable over K, then $R[t]$ is integrally closed. From this it follows that $k[t, tb]$ is integrally closed. Therefore, as $k = \bar{k}$, C is projectively normal, so that each system of hyperplane sections is complete.

Let $\langle L \rangle$ be the pencil of generators of C and let $\phi: C \to \boldsymbol{P}^1$ be the corresponding rational map. Select a generic point u of $\boldsymbol{P}^1$ over k and let $L_u = \phi^{-1}(u)$ be a generic member of $\langle L \rangle$ over k. Take H to be a section of C by a k-rational hyperplane not passing through the vertex; then L_u and H meet properly; set $P = L_u \cdot H$. L_u and A meet properly; set $\boldsymbol{a} = L_u \cdot A$. There is an integer m and a function f_u rational over

$k(u)$ on L_u, such that: $m\cdot P - \boldsymbol{a} = \operatorname{div}(f_u)$, since L_u is a straight line. Let f be a function rational over k on C inducing f_u on L_u. The relation: $\operatorname{div}(f)\cdot L_u = \operatorname{div}(f_u)$ holds ([22], ch. 9). Let $Y = m\cdot H - A$. Consider $\operatorname{div}(f) - Y$ and let Z be an irreducible component of it with non-zero coefficient—Z is necessarily k-rational. We shall show, using Lemma 7.1, that $Z \in \langle L\rangle$. Indeed, suppose that Q is a simple point of C common to Z and to L_u. If Q were algebraic over k it would be common to every generator in $\langle L\rangle$; distinct generators meet only at the vertex so Q must be a generic point of Z over k. This shows that when Z' is another component of $\operatorname{div}(f) - Y$ distinct from Z, Q is not a point of Z'. Now $L_u\cdot(\operatorname{div}(f) - Y) = \operatorname{div}(f_u) - L_u\cdot Y = \operatorname{div}(f_u) - L_u\cdot(m\cdot H - A)$. By choice of f_u, m, every component of this 0-cycle on L_u has zero coefficient. Therefore the point Q above cannot exist. Consequently $Z \in \langle L\rangle$.

Thus we may write: $\operatorname{div}(f) - Y = \sum a_i L_i, L_i \in \langle L\rangle$. Let H^* be a generic hyperplane section of C over k. By choice of f, m, $\deg(H^*\cdot Y) = 0$. Also, $\deg(H^*\cdot\operatorname{div}(f)) = 0$. Therefore $\sum a_i = 0$. But this means that $\sum a_i L_i \sim 0$ on the rational cone C. Hence $A \sim m\cdot H$.

According to this lemma, applied to the components of Γ, in order to produce the form Ψ above we need to show that the branch curve Γ has degree 12 in $\boldsymbol{P}^5$ (deg. $C = 4$) and that no component of it passes through the vertex. For this we shall consider the maps induced by λ on two kinds of curves on V where knowledge of branching properties is available, the curves being namely the general members E and D of the linear systems $\langle E\rangle$ and $\Lambda = \Lambda(4E + 2F)$ respectively. By basic assumptions E is nonsingular. We shall have to show that D is too.

First we introduce the normalization V' of C in the field of functions $K = k(V)$ over the algebraically closed field k.

By (3.1) we have that K contains a rational subfield isomorphic to $k(C)$ over which K is a quadratic extension; it is a separable extension as the characteristic is not 2. Identify $k(C)$ with its isomorphic image in K. Let $X_0, \cdots, X_5$ be the induced homogeneous coordinates on C when it is defined by the equations in (2.0) and set $x_1 = X_0/X_1$, $x_2 = X_2/X_1$. Then $k(C) = k(x_1, x_2)$ and $k[x_1, x_2]$ is integrally closed in $k(C)$. By "completing the square" we can find $z \in K$ and $\psi_1 \in k[x_1, x_2]$ such that $z^2 = \psi_1(x_1 x_2)$, ψ_1 is a square-free polynomial and $K = k(x_1, x_2, z)$. Then $\{1, z\}$ is an integral basis in the field K with respect to the domain $k[x_1, x_2]$

and the principal ideal $(\psi_1) \subset k[x_1, x_2]$ is the discriminant ideal of our quadratic extension (cf. [26], vol. 1). Thus ψ_1 is determined up to a scalar factor.

Now let V' be the normalization of C in the field K. Let C_1 be the affine open subset of C where $X_1 \neq 0$. Let V'_1 be the affine open subset of V' lying over C_1. Let W_1 be the normal affine surface in S^3 defined by the equation $Z^2 = \psi_1(x_1, x_2)$. W_1 can be regarded as lying over C_1 via composition of $W_1 \to S^2$ with the biregular isomorphism $\iota: S^2 \to C_1$, cf. (2.0). By uniqueness of normal covers ([22], app. 1), W_1 and V'_1 are isomorphic.

There is a commuting triangle:

$$\begin{tikzcd} V \arrow[d, "\lambda"'] \arrow[dr, "\mu"] & \\ C & V' \arrow[l] \end{tikzcd} .$$

Since V is a $K3$ surface, V carries no exceptional curves of 1st kind, (0.1). Hence μ is biregular at all simple points of V'.

Let Γ_1 be the divisor on C_1 isomorphic to the curve in S^2 defined by setting $\psi_1(x_1, x_2) = 0$. Computing the Jacobi matrix for the equation $Z^2 - \psi_1(x_1, x_2)$ shows that V'_1 is nonsingular over any point of C_1 which is not a multiple point of the curve Γ_1. Using this and the remark just above, it then follows that λ, over simple points of Γ_1 on C_1, is finite and ramifies.

When we make a similar construction using another affine part of C where $X_5 \neq 0$, for instance, we obtain a curve whose intersection with C_1 is Γ_1 because λ ramifies over it too. Thus we obtain a complete curve Γ lying on C. —It is not yet proved that Γ does not pass through the vertex.

Since ψ_1 is square-free Γ has no multiple components, so any multiple points of Γ must be isolated.

Let G be the graph of the mapping $\lambda: V \to C$, and let Q be a simple point of C. When $Q \notin \Gamma$, then $G \cdot (V \times Q) = P \times Q + P' \times Q$, $P \neq P'$; when $Q \in \Gamma$ and not one of the multiple points of Γ, then $G \cdot (V \times Q) = 2 \cdot P \times Q$.

To show that $\Lambda = \Lambda(4E + 2F)$ contains nonsingular members we shall need

LEMMA 12. *Suppose that H is a hyperplane section of C which does* *not pass through the vertex and which meets Γ transversally at each intersection point. Let D be the divisor in Λ corresponding to H. Then D is a nonsingular irreducible curve of genus* 5.

Proof. The hyperplane cutting H on C is transversal in $\boldsymbol{P}^5$ to each generator because H does not pass through the vertex. Moreover the curve H meets a generator at simple points of C. Thus H is transversal on C to each generator, therefore H is a nonsingular curve. We know that Λ is an irreducible system, hence every curve in Λ is connected ([24]); thus, if D is nonsingular it is irreducible. To show that D is nonsingular we use the "calculus of cycles", [22], ch. 8.

Let $P \in D$ with $Q = \lambda(P) \notin \Gamma \cdot H$. Then $Q \notin \Gamma$ so, with G as above, $G \cdot (V \times Q) = P \times Q + P' \times Q$, $P \neq P'$. This intersection is taken on $V \times C$. Let us restrict the ambient space, first from $V \times C$ to $V \times H$, then to $D \times H$. Clearly G intersects both $V \times H$ and $V \times Q$ properly on $V \times C$. Set $G_D = (G \cdot V \times H)_{V \times C}$. Then we have (cf. [22], ch. 8, Th. 10):

$$
\begin{aligned}
(G \cdot V \times Q)_{V \times C} &= ((G \cdot V \times H)_{V \times C} \cdot V \times Q)_{V \times H} = (G_D \cdot V \times Q)_{V \times H} \\
&= (G_D \cdot (V \times Q \cdot D \times H)_{V \times H})_{D \times H} = (G_D \cdot D \times Q)_{D \times H} .
\end{aligned}
$$

Also, $(G \cdot V \times Q)_{V \times C} = P \times Q + P' \times Q$. Therefore G_D and $D \times Q$ are transversal at $P \times Q$ on $D \times H$ so P is a simple point of D.

Let $P \in D$ with $Q = \lambda(P) \in \Gamma \cdot H$, so that $G \cdot (V \times Q) = 2 \cdot P \times Q$. Let Γ' be the component of Γ containing Q (Q is simple on Γ) and let Γ'' be the curve on V determined by

$$2 \cdot \Gamma'' = \mathrm{pr}_V\,(G \cdot (V \times \Gamma')) .$$

Then $\lambda(\Gamma'') = \Gamma'$. λ is finite over all the points of $\Gamma' \cdot H$ so $\Gamma'' \times H$ and G intersect properly on $V \times C$. Put $T = G \cdot \Gamma'' \times H$. By Th. 9 in ch. 8, [22], $\mathrm{pr}_C\, T = \lambda(\Gamma'') \cdot H = \Gamma' \cdot H$; this is by hypothesis a 0-cycle on C in which each component has coefficient one; then the same is true of T on $V \times C$. On the other hand the theorem says that $\mathrm{pr}_V\, T = \Gamma'' \cdot \lambda^{-1}(H)$ $= \Gamma'' \cdot D$. Consequently Γ'' and D are transversal at P on V so P is simple on D.

Thus D is nonsingular and irreducible. D has self-intersection number $D^{(2)} = 8$ so the genus of D is 5 by the formula in (0.2).

The existence of such a nonsingular irreducible D depends on finding H satisfying the conditions of the lemma. There is such an H because

Γ has no multiple components and, when Γ' is a component of Γ the linear series of hyperplane sections of Γ' is a separable linear series ([22]), so that a general member of that series is a sum of distinct points each with multiplicity one. (This argument would not guarantee that the generators of C cut a separable linear series on Γ'.)

Now let D be a general member of $\Lambda = \Lambda(4E + 2F)$ and let E be a general member of $\langle E \rangle$ over k. E and D are irreducible nonsingular curves of genera 1 and 5 respectively. The trace of Λ on E is the linear series of degree 2, dimension 1 containing $2E \cdot F$ (proof of Prop. 4). The trace of Λ on D is the complete canonical series, by the trace exact sequence.

The complete linear series on D corresponds to the map induced by λ from D to H; obviously this map is not an isomorphism (H is rational), hence D is hyperelliptic and, taking k_0 to be an algebraically closed field of definition containing k over which D and H are defined, $k_0(H)$ is the rational subfield of degree two in $k_0(D)$ generated by the ratios of differentials of first kind ([3], ch. 4). By the Riemann-Hurwitz relation $2G - 2 = n(2g - 2) + \deg. \mathfrak{D}$ ([3], ch. 6, § 2) the degree of the different $\mathfrak{D}$ of $k_0(D)$ with respect to $k_0(H)$ is 12. Since the characteristic is not two it follows ([3], ch. 4, § 8) that the different is a product of distinct places with exponent one. This shows that λ, restricted to D, ramifies at 12 distinct places.

There must be ramification of λ/D over any point of H where Γ meets H. Indeed, as in the proof of Lemma 12, let G be the graph of λ and let $G_D = (G \cdot V \times H)_{V \times C}$ be the graph of λ/D. D was chosen to be general in Λ, so H and Γ meet at simple points of Γ. Therefore when Q is a point common to H and Γ, G intersects $V \times Q$ properly on $V \times C$. Consequently,

$$2 \cdot P \times Q = (G \cdot V \times Q)_{V \times C} = (G_D \cdot D \times Q)_{D \times H} \,.$$

Therefore λ/D ramifies as claimed. Clearly $I(\Gamma, H)$ is defined on C as H does not pass through the vertex. Then certainly $I(\Gamma, H) \leq 12$. Thus the degree of the 1-cycle Γ in $\boldsymbol{P}^5$ is ≤ 12.

The same arguments show that the restriction of λ to E ramifies at four distinct points, among which are all the points over $\Gamma \cdot L$, L being the generator such that $E = \lambda^{-1}(L)$. There is also the ramification of λ/E over the vertex arising from the presence of $2F \cdot E$ in the trace of Λ on E.

Now a section of C by a general hyperplane through the vertex consists of four lines, which correspond to four curves E_i in $\langle E \rangle$. On each line there are four distinct ramifiication points, one of which is the vertex. Consequently if Γ goes through the vertex it must meet the hyperplane at more than 12 points. We saw this cannot happen. Therefore, no component of Γ goes through the vertex and deg. $\Gamma = 12$ in $\boldsymbol{P}^5$.

Thus we have shown

PROPOSITION 5. *There is a homogeneous form Ψ of degree 3 such that the hypersurface defined in $\boldsymbol{P}^5$ by the equation $\Psi = 0$ cuts out on C the branch curve Γ of the map $\lambda: V \to C$.*

(3.3). Using $\Lambda = \Lambda(4E + 2F)$ we can check that when a curve in $\langle E \rangle$ is reducible, the possibilities are the same as in the constructions in § 2.

PROPOSITION 6. *Let E' be a reducible member of the pencil $\langle E \rangle$. Then $E' = E_0 + Z$, where E_0 is a nonsingular rational curve with $I(E_0, F) = 1$, and Z is contractible to a rational double point of the normalization V'.*

Proof. F is not a component of E' because, when E is a general member of $\langle E \rangle$, $I(E', E) = 0$ and $I(E, F) = 1$, and E is irreducible. We have that $I(E', F) = 1$. Let E_0 be the component of E' meeting F and take $Z = E' - E_0$. Then $I(Z, F) = 0$ and $I(E_0, F) = 1$. E is irreducible and $I(E, E') = 0$, so $I(E, E_0) = 0$; therefore $I(Z, E_0) + E_0^{(2)} = 0$. Since E' is connected and E_0 is not a component of Z, $E_0^{(2)} < 0$. Hence $E_0^{(2)} = -2$ on the $K3$ surface. This means that E_0 is an irreducible curve with $p_a(E_0) = 0$; thus E_0 is a nonsingular rational curve.

Let D be an irreducible curve in Λ. We have that $I(D, E') = 2$ and that $I(D, E_0) = 2 \cdot I(F, E_0) = 2$ also, hence that $I(D, Z) = 0$. Then the geometric image $\lambda(Z)$ is a bunch of points of C. Therefore $\mu(Z)$ is a bunch of points of the normalization V' since V' is finite over C. This bunch $\mu(Z)$ is supported at multiple points of V' so $\lambda(Z)$ is supported at points of C which are multiple points of Γ, cf. (3.2). Points of $\lambda(Z)$ are all points of the single generator $\lambda(E')$. Now $I(\Gamma, \lambda(E')) = 3$, so there is at most one multiple point of Γ on $\lambda(E')$. Therefore $\lambda(Z)$ consists of one point $\gamma \in \Gamma$. Lemma 9 is effective here to show that there is a single point g on V' lying over γ. Therefore Z is connected, [24]. μ is biregular near $\mu(Z) = g$ except over g, (3.2). Hence Z is "contractible": especially, the intersection matrix of its components is negative definite, [14]. Therefore, for every component $X \prec Z$, $X^{(2)} < 0$, so $X^{(2)} = -2$ on the $K3$ surface. Therefore $\mu(Z)$ is a rational double point, [1].

Using Lemma 12 we show:

LEMMA 13. *Suppose that Γ has no multiple points. Let P be a point of V not lying on F. Then there is an irreducible nonsingular curve D in $\Lambda(4E + 2F)$ passing through P.*

Proof. Let $Q = \lambda(P)$, λ being determined by $\Lambda(4E + 2F)$. According to Lemma 12 we have to find a hyperplane section H of C which satisfies the conditions of the lemma and also passes through Q. Since P is not on F, Q is not the vertex.

We must be careful of the fact that, when the char. $\neq 0$, a projective curve can have the "strange" property that, although it is not a line, all its tangents have a point in common (cf. [18]). However, Samuel showed that a nonsingular curve is not strange when the characteristic is not 2; thus Γ is not strange.

Components of Γ are principal on C (Lemma 11), so any two have to meet. Therefore Γ is irreducible.

Let k be a field of definition for Γ and Q. Since Γ is not strange the tangent t to Γ at a generic point R over k does not go through the k-rational point Q. Then there is a hyperplane H through both Q and R not containing t; as Γ is nonsingular H must be transversal to Γ at R.

Let Λ be the reduced linear series of section of Γ cut by hyperplanes passing through Q. If Λ is not separable, that is, if a general divisor $\mathfrak{s}$ of Λ is not a sum of distinct points each counted once, then the coefficient of every point in every divisor $\mathfrak{s}' \in \Lambda$ is divisible by the characteristic (Bertini's theorem). But the divisor cut by H is not of this type: the component R is counted once. Therefore Λ must be separable so there are hyperplanes through Q transversal to Γ at each intersection.

§ **4.** Here we see that a special $K3$ surface V represented as a "double cone" branched over Γ on C, is polarized by the divisor $3E + F$ if and only if Γ is nonsingular.

PROPOSITION 7. *Let (V, E, F) be a special $K3$ surface. The following assertions about V are equivalent*:

(1) *The complete linear system $\Lambda(7E + 3F)$ determines a projective embedding of V.*

(2) *Every member of the pencil $\langle E \rangle$ is an irreducible curve.*

(3) *The branch curve Γ of the mapping $\lambda: V \to C$ determined by $\Lambda(4E + 2F)$ is an irreducible nonsingular curve.*

Proof. (1) $\Rightarrow$ (2). When $E' \in \langle E \rangle$ is reducible, there is an irreducible component E'' of E' such that $I(E'', E) = I(E'', F) = 0$, by Proposition 6. But if (1) is true, then either $I(G, E) > 0$ or $I(G, F) > 0$ for any irreducible curve G, since the image of G has positive intersection with a hyperplane section.

(2) $\Rightarrow$ (3). We saw earlier that when Γ has a multiple point γ the normalization V' of C in $k(V)$ has a multiple point P lying over γ (γ is simple on C). Some divisor G on V collapses to this P. There is a curve E' through P on V' whose proper transform on V is a divisor E in the pencil $\langle E \rangle$. When $x \in k(V)$ is a function whose zero locus on V' is E', x has positive order along components G_i of G, since $x(P) = 0$ and P is the center on V' of the several valuations v_{G_i}. Hence E is reducible. Therefore Γ has no multiple point γ.

Components of Γ are principal on C (Lemma 11) so any two have to meet. Therefore Γ is irreducible if it is nonsingular.

(3) $\Rightarrow$ (1). By Lemma 4, the map λ determined by $\Lambda(7E + 3F)$ is a morphism to a projective surface. Under the hypothesis it follows from Lemma 13 that, when $P \in V - F$, there is an irreducible nonsingular curve D in $\Lambda(4E + 2F)$ passing through P. By Lemma 4 and the trace exact sequence, the trace of $\Lambda(7E + 3F)$ on F and D respectively, is a complete linear series of degree 1 and 14 respectively. Therefore λ induces isomorphisms both on F and on D, since in each case the degree of the induced linear series is at least $2\cdot(\text{genus}) + 1$, cf. [19].

λ separates points: When P and Q are points of V, we have to find $G \in \Lambda(7E + 3F)$ containing one point and not the other. There are three cases to consider: (i) $P \notin F$, $Q \notin F$. Take $D \in \Lambda(4E + 2F)$ as above through P. If $Q \notin D$, take $G = D + 3E + F$ where $Q \notin E \in \langle E \rangle$. If $Q \in D$ there is $G \in \Lambda(7E + 3F)$ containing P and not Q, by what was noted above. (ii) $P \in F$, $Q \in F$. Again there is $G \ni P$, $G \not\ni Q$ by the above. (iii) $P \in F$, $Q \notin F$. If $Q \in E_Q \in \langle E \rangle$, then take $E_Q \neq E' \in \langle E \rangle$ and let $G = 7E' + 3F$.

λ separates tangential directions: First, the morphism λ is birational because we have: $k(V) \supset k(\lambda(V)) \supset k(C)$; $[k(V) : k(C)] = 2$ and characteristic $\neq 2$; and λ is set-theoretically 1–1. Call W the image $\lambda(V)$. There is the birational morphism $\lambda : V \to W$; this does not guarantee even that W is normal, but if we show that W is nonsingular then λ will be biregular everywhere (since the local ring $\mathfrak{O}$ of a point P of V will be integral over the integrally closed local ring of the corresponding $\lambda(P)$ on W).

Let $P \in V$ and put $Q = \lambda(P)$. When G is a divisor in $\Lambda(7E + 3F)$, G corresponds to a hyperplane section of W, say H_G. If we find a divisor G corresponding to a section H_G which, locally at Q, is a nonsingular curve—i.e. there is just one component of H_G through Q with coefficient 1 and Q is simple on it—then Q will be a simple point of the surface W since H_G is principal on W, (cf. (2.2) or [16], p. 384).

Suppose, first, that $P \in F$. The trace of $\Lambda(7E + 3F)$ on F is a complete linear series of degree 1, as we mentioned, so the image of F is a line. By Lemma 4 and the trace exact sequence again, there is an irreducible curve $G' \in \Lambda(7E + 2F)$ such that $G' \not\ni P$. Then for our G we can take $F + G'$. On the other hand, if $P \in V - F$, there is a nonsingular curve $D \in \Lambda(4E + 2F)$ through P which is mapped isomorphically by λ. Then take $G = D + F + 3E'$, where $E' \not\ni P$.

Remark 4.1. By the Nakai-Moisezon test, [7], it is easy to see that a multiple of $3E + F$ determines a projective embedding of V whenever (2) is true, and only in that case.

Remark 4.2. We would like to mention here how the necessity of the condítition (*) in (2.3) on types of multiple points γ on Γ follows from an assertion of Enriques and Campedella, [5], p. 458. By Propositions 6 and 7 we see that, when γ is a multiple point of Γ and L is the generator through γ, the divisor E' in $\langle E \rangle$ over L on V is reducible and has a part Z which collapses to a rational double point g of the normal V'. Therefore the pull-back of a canonical divisor through g on V' is a canonical divisor on V, [1], or "the conditions of adjunction are not affected by the isolated singularity", [5]. Now according to Enriques and Campedella, if the condition (*) is violated by γ then "conditions are imposed on the adjoint curves", i.e. g cannot be a rational double point...

§ **5.** Here we verify that isomorphism classes of special polarized $K3$ surfaces correspond to projective equivalence classes of branch curves Γ on C.

LEMMA 14. *If $f: V \to V'$ is an isomorphism of special polarized $K3$ surfaces then $f(E) \in \langle E' \rangle$, $f(F) = F'$ (where E, E' etc. have the usual meanings on V, V').*

Proof. Let $E \in \langle E \rangle$ be a general member and let $E' \in \langle E' \rangle$ be irreducible. There is an isomorphism $f: V \to V'$ of underlying varieties such that $f(F + 3E)$ is numerically equivalent to $F' + 3E'$:

$$f(F) + 3 \cdot f(E) \equiv f' + 3 \cdot E' \qquad (\text{mod. } \mathfrak{G}_n) .$$

Then taking the intersection product with $f(E)$,

$$1 = I(F', f(E)) + 3 \cdot I(E', f(E)) .$$

As F', $f(E)$ are distinct irreducible curves, neither intersection number is negative; therefore in particular $I(E', f(E)) = 0$. Now f induces an isomorphism from the nonsingular curve E of genus one to $f(E)$. On the other hand the only irreducible solutions X to $I(E', X) = 0$ are nonsingular rational curves, according to Proposition 6, or else members of the pencil $\langle E' \rangle$. Therefore

$$f(E) \in \langle E' \rangle .$$

Consequently $f(F) \equiv F'$ (mod. $\mathfrak{G}_n$). This means that $r \cdot f(F) \sim r \cdot F'$ for a non-zero integer r ([11]). If $f(F) \neq F'$ then $I(F, F') \geq 0$, therefore $r^2 \cdot F'^{(2)} \geq 0$ which cannot be as $F'^{(2)} = -2$. Therefore $f(F) = F'$.

This lemma allows the next assertion to apply to polarized surfaces.

PROPOSITION 8. *Let V, V' be special $K3$ surfaces and let Γ, Γ' be the branch curves on C of rational maps determined respectively by $\Lambda(4E + 2F)$, $\Lambda(4E' + 2F')$. Then there is an isomorphism $f: V \to V'$ with $f(E) \in \langle E' \rangle$, $f(F) = F'$ if and only if Γ and Γ' are projectively equivalent in $\boldsymbol{P}^5$.*

Proof. Suppose $f: V \to V'$ is such an isomorphism. Γ and Γ' are projectively equivalent if, after a change of basis of $L(4E + 2F)$ we have a commuting triangle

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$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "\lambda"'] & & V' \arrow[dl, "\lambda'"] \\ & C & \end{tikzcd}$$

where λ, λ' are respectively determined by $\Lambda = \Lambda(4E + 2F)$, and by $\Lambda' = \Lambda(4E' + 2F')$. The assumptions imply that $f(\Lambda) \subset \Lambda'$; as $\dim. \Lambda = \dim. \Lambda'$, we have $\Lambda = \Lambda'$. Now a basis of $L(4E' + 2F')$ on V' pulls back to a basis of $L(4E + 2F)$, [22], so there is such a triangle.

Conversely, if Γ and Γ' are projectively equivalent we may suppose that $\Gamma = \Gamma'$ on C; then the normalizations of C in $k(V)$, $k(V')$ are isomorphic since as we say they depend only on Γ; then V, V' are birational, hence isomorphic since they are $K3$ surfaces. The isomorphism fits into a triangle as above because it extends the isomorphism of normalizations, so $f(E) \in \langle E' \rangle$, $f(F) = F'$.

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