III

Symmetries and anomalies

Application of the concept of symmetry leads to some of the most powerful techniques in particle physics. The most familiar example is the use of gauge symmetry to generate the lagrangian of the Standard Model. Symmetry methods are also valuable in extracting and organizing the physical predictions of the Standard Model. Very often when dealing with hadronic physics, perturbation theory is not applicable to the calculation of quantities of physical interest. One turns in these cases to symmetries and approximate symmetries. It is impressive how successful these methods have been. Moreover, even if one could solve the theory exactly, symmetry considerations would still be needed to organize the results and to make them comprehensible. The identification of symmetries and near symmetries has been considered in Chap. I. This chapter is devoted to their further study, both in general and as applied to the Standard Model, with the intent of providing the foundation for later applications.

III–1 Symmetries of the Standard Model

The treatment of symmetry in Sects. I–4, I–6 was carried out primarily in a general context. In practice, however, we are most interested in the symmetries relevant to the Standard Model. Let us briefly list these, reserving for some a much more detailed study in later sections.

Gauge symmetries: As discussed in Chap. II, these are the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge invariances. It is interesting to compare their differing realizations. $SU(3)_c$ is unbroken but evidently confined, whereas $SU(2)_L \times U(1)_Y$ undergoes spontaneous symmetry breaking, induced by the Higgs fields, leaving an unbroken $U(1)_{\rm em}$ gauge invariance.

Fermion-number symmetries: There exist global vector symmetries corresponding to both lepton and quark number. These are of the form

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$$\psi_\alpha \to e^{-iQ_\alpha \theta}\psi_\alpha \tag{1.1}$$

for fields of each chirality. The index α refers to either the set of all leptons or the set of all quarks, and the conserved charges Q_α are just the total number of quarks minus antiquarks and the total number of leptons minus antileptons.[1] Conservation of baryon number B is violated due to an anomaly in the electroweak sector, but $B - L$ remains exact.

Global vectorial symmetries of QCD: If the quarks were all massless, there would be a very high degree of symmetry associated with QCD. Even if $m \neq 0$, symmetries are possible if two or more quark masses are equal. Three of the quarks (c, b, t) are heavy compared to the confinement scale Λ_{QCD} and widely spaced in mass, so they cannot be accommodated into a global symmetry scheme.[2] However, the u, d, and s quarks are light enough that their associated symmetries are useful. The best of these is the *isospin* invariance, which consists of field transformations

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \to \psi' = \exp(-i\boldsymbol{\tau} \cdot \boldsymbol{\theta})\psi \tag{1.2}$$

where $\{\boldsymbol{\tau}^i\}$ $(i = 1, 2, 3)$ are $SU(2)$ Pauli matrices and $\{\theta^i\}$ are the components of an arbitrary constant vector. Associated with the $SU(2)$-flavor invariance are the three Noether currents

$$J_\mu^{(i)} = \bar{\psi}\gamma_\mu \frac{\tau^i}{2}\psi. \tag{1.3}$$

Isospin symmetry is broken by the up–down mass difference,

$$\mathcal{L}_{\text{mass}} = -\frac{m_u + m_d}{2}(\bar{u}u + \bar{d}d) - \frac{m_u - m_d}{2}(\bar{u}u - \bar{d}d) \tag{1.4}$$

and by electromagnetic and weak interactions. Inclusion of the strange quark extends isospin to $SU(3)$-flavor transformations

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \psi' = \exp(-i\boldsymbol{\theta} \cdot \boldsymbol{\lambda})\psi, \tag{1.5}$$

where $\{\lambda^a\}$ $(a = 1, 2, \ldots, 8)$ are the $SU(3)$ Gell-Mann matrices. The $SU(3)$-flavor symmetry is broken significantly by the strange quark mass, and to a lesser extent by other effects. Predictions of isospin symmetry work at the 1% level, whereas $SU(3)$ predictions hold only to about 30%. It is occasionally convenient to employ

[1] In Chap. VI, we return to the study of lepton-number violation through possible Majorana mass terms

[2] See, however, the discussion of the dynamical heavy-quark symmetries in Chap. XII–3

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a particular $SU(2)$ subgroup of $SU(3)$, called *U-spin*, which corresponds to the transformations

$$\begin{pmatrix} d \\ s \end{pmatrix} \to \exp(-i\boldsymbol{\tau}\cdot\boldsymbol{\theta})\begin{pmatrix} d \\ s \end{pmatrix}. \tag{1.6}$$

U spin is also a symmetry of the electromagnetic interaction, since its generators commute with the electric-charge operator. The U-spin symmetry is broken by the large d-quark, s-quark mass difference.

Approximate chiral symmetries of QCD: The vectorial symmetries are valid if quark masses are equal. If the masses vanish, there are additional *chiral* symmetries, because in this limit the left-handed and right-handed components of the fields are decoupled (cf. Sect. I–3),

$$\mathcal{L}_{QCD}\Big|_{m=0} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}_L \not{D}\psi_L + \bar{\psi}_R \not{D}\psi_R, \tag{1.7}$$

i.e., the left-handed and right-handed fields have *separate* invariances. For massless up-and-down chiral quarks, the symmetry operations are

$$\psi_L \to \exp(-i\boldsymbol{\theta}_L\cdot\boldsymbol{\tau})\psi_L \equiv L\psi_L, \quad \psi_R \to \exp(-i\boldsymbol{\theta}_R\cdot\boldsymbol{\tau})\psi_R \equiv R\psi_R \tag{1.8}$$

where $\psi_{L,R}$ are chiral projections of the ψ doublet in Eq. (1.2). These can also be expressed as vector and axial-vector isospin transformations,

$$\psi \to \exp(-i\boldsymbol{\theta}_V\cdot\boldsymbol{\tau})\psi, \qquad \psi \to \exp(-i\boldsymbol{\theta}_A\cdot\boldsymbol{\tau}\gamma_5)\psi \tag{1.9}$$

with $\boldsymbol{\theta}_V = (\boldsymbol{\theta}_L + \boldsymbol{\theta}_R)/2$, and $\boldsymbol{\theta}_A = (\boldsymbol{\theta}_L - \boldsymbol{\theta}_R)/2$. This invariance is variously referred to as chiral-$SU(2)$, $SU(2)_L \times SU(2)_R$, or $SU(2)_V \times SU(2)_A$. In *QCD*, it is broken by quark mass terms,

$$\mathcal{L}_{\text{mass}} = -m_u\bar{u}u - m_d\bar{d}d = -m_u(\bar{u}_Lu_R + \bar{u}_Ru_L) - m_d(\bar{d}_Ld_R + \bar{d}_Rd_L). \tag{1.10}$$

Thus, if $m_u = m_d \neq 0$, separate left-handed and right-handed invariances no longer exist, but rather only the vector isospin symmetry. The generalization to three massless quarks defines chiral $SU(3)$ (or $SU(3)_L \times SU(3)_R$) and is a straightforward extension of the above ideas.

Discrete symmetries: Since the Standard Model is a hermitian and Lorentz-invariant local quantum field theory, it is invariant under the combined set of transformations *CPT*. Both *QCD* (given the absence of the θ-term) and *QED* conserve P, C, and T separately. By contrast, the electroweak interactions have maximal violation of P and C in the charged-current sector. If a nonzero phase resides in the quark-mixing matrix, there will exist a breaking of *CP*, or equivalently of T, invariance. Otherwise the weak interactions are invariant under the product *CP*.

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In addition to the above exact or approximate symmetries of the Standard Model, there are some important 'non-symmetries' of QCD. By these we mean invariances of the underlying lagrangian, which might naively be expected to appear as symmetries of Nature but which, for a variety of reasons, do not. These include the following.

Axial $U(1)$: The QCD lagrangian would have an axial $U(1)$ invariance of the form

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \psi' = e^{-i\theta\gamma_5}\psi, \tag{1.11}$$

if the u, d, s quarks were massless. However, this turns out not to be even an approximately valid symmetry, as it has an anomaly. We shall return to this point in Sect. III–3.

Scale transformations: If quarks were massless, the QCD lagrangian would contain no dimensional parameters. The lagrangian would therefore be invariant under the scale transformations

$$\psi(x) \rightarrow \lambda^{3/2}\psi(\lambda x), \qquad A_\mu^a(x) \rightarrow \lambda A_\mu^a(\lambda x), \tag{1.12}$$

where ψ and A_μ^a are respectively the quark and gluon fields. This invariance is also destroyed by anomalies (see Sect. III–4).

'Flavor symmetry': Because the gluon couplings are independent of the quark flavor, one often finds reference in the literature to a flavor symmetry of QCD. Unless the specific application is reducible to one of the above true symmetries, one should not be misled into thinking that such a symmetry exists. For example, flavor symmetry is often used in this context to relate properties of the pseudoscalar mesons $\eta(549)$ and $\eta'(960)$ (or analogous particles in other nonets). However, the result is rarely a symmetry prediction. Rather, this approach typically pertains to specific assumptions about the way quarks behave, and is dressed up by incorrectly being called a symmetry. In group theoretic language, this may arise by assuming that QCD has a $U(3)$ symmetry rather than just that of $SU(3)$.

III–2 Path integrals and symmetries

The transition from classical physics to quantum physics is in many ways most transparent in the path-integral formalism. In this chapter we use these techniques to provide a quantum description of symmetries, complementing the treatment at the classical level of Sects. I–4, I–6. A brief pedagogical introduction to those path-integral techniques which are important for the Standard Model is provided in App. A.

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The generating functional

In order to implement a quantum description of currents and current matrix elements, one studies the generating functional, Z, of the theory. For a generic field φ, we have

$$Z[j] = e^{iW[j]} = \int [d\varphi] \exp i \int d^4x \; (\mathcal{L}(\varphi, \partial\varphi) - j\varphi), \tag{2.1}$$

where $j(x)$ is an arbitrary classical source field whose presence allows us to probe the theory by studying its response to the source. The symbol $[d\varphi]$ indicates that at *each* point of spacetime one integrates over all possible values of the field $\varphi(x)$. All the matrix elements needed to describe physical processes in the theory can be obtained from $\ln Z[j]$ by functional derivations, i.e.,

$$\left\langle 0 \left| T \left(\varphi(x_k) \ldots \varphi(x_p)\right) \right| 0 \right\rangle = (i)^n \frac{\delta^n \ln Z[j]}{\delta j(x_k) \ldots \delta j(x_p)}\bigg|_{j=0}, \tag{2.2}$$

where n is the number of fields in the matrix element. If there is more than one field, i.e., the set $\{\varphi_i\}$, a separate source is introduced for each field.

If one wants to study a given current J^μ (not to be confused with the source j) associated with some classical symmetry, one simply adds an extra classical source field v_μ, which is coupled to that current,

$$Z[j, v_\mu] = \int [d\varphi] \exp i \int d^4x \left(\mathcal{L} - j\varphi - v_\mu J^\mu\right). \tag{2.3}$$

In this case all matrix elements involving J^μ can be obtained by functional derivation with respect to v_μ,

$$\bar{J}^\mu(x) = i \frac{\delta \ln Z}{\delta v_\mu(x)}\bigg|_{v_\mu=0}, \tag{2.4}$$

where the bar in $\bar{J}^\mu$ indicates that it is a functional describing matrix elements of the current J^μ. Specific matrix elements are obtained by further derivatives, as in

$$\langle 0 | T \left(J^\mu(x)\varphi(x_1)\varphi(x_2)\right) | 0 \rangle = (i)^2 \frac{\delta^2}{\delta j(x_1) \delta j(x_2)} \bar{J}^\mu(x)\bigg|_{j=0}. \tag{2.5}$$

This device allows one to discuss all possible matrix elements of the current J^μ.

As an example, consider the vector and axial-vector currents of *QED*. We define

$$Z[v_\mu, a_\mu] \equiv \int [d\psi][d\bar{\psi}][dA_\mu] e^{i\int d^4x \left(\mathcal{L}_{QED} - v_\mu \bar{\psi}\gamma^\mu\psi - a_\mu \bar{\psi}\gamma^\mu\gamma_5\psi\right)}. \tag{2.6}$$

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A three-current (connected) matrix element is obtained then as

$$\begin{aligned}T_{\mu\alpha\beta}(x, y, z)_{\rm conn} &\equiv \left\langle 0 \left| T\left(\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)\ \bar{\psi}(y)\gamma_\alpha\psi(y)\ \bar{\psi}(z)\gamma_\beta\psi(z)\right)\right| 0\right\rangle \\ &= (i)^3\left[\frac{\delta^2}{\delta v^\alpha(y)\delta v^\beta(z)}\ \frac{\delta}{\delta a^\mu(x)}\ln Z\right]_{\substack{v^\mu=0\\ a^\mu=0}} \\ &= (i)^2\frac{\delta^2}{\delta v^\alpha(y)\delta v^\beta(z)}\bar{J}_{5\mu}(x)\end{aligned} \tag{2.7}$$

where the axial-vector quantity $\bar{J}_{5\mu}$ is defined in analogy with Eq. (2.4).

Noether's theorem and path integrals

Returning to the general case, let us consider an infinitesimal transformation of a set of fields $\{\varphi_i\}$ (cf. Eq. (I–3.1))

$$\varphi_i \to \varphi_i' = \varphi_i + \epsilon(x) f_i(\varphi) \tag{2.8}$$

such that the current under discussion is

$$J^\mu(x) = \frac{\partial\mathcal{L}'}{\partial(\partial_\mu\epsilon)}. \tag{2.9}$$

If this is a symmetry transformation, one has up to a total derivative,

$$\mathcal{L}' = \mathcal{L}\left(\varphi', \partial\varphi'\right) = \mathcal{L}\left(\varphi, \partial\varphi\right) + J^\mu\partial_\mu\epsilon. \tag{2.10}$$

If $\epsilon(x)$ is a constant, the lagrangian is invariant under the transformation. This is the statement of the classical symmetry condition. In order to study the consequences of this situation, we rewrite our previous definition of the current matrix elements

$$\bar{J}^\mu(x) = i\frac{\delta}{\delta v_\mu(x)}\ln Z[v_\nu] \tag{2.11}$$

in integral form by noting

$$\delta\ln Z[v_\mu] = \ln Z[v_\mu + \delta v_\mu] - \ln Z[v_\mu] \equiv -i\int d^4x\ \bar{J}^\mu(x)\delta v_\mu(x), \tag{2.12}$$

which is just the inverse of Eq. (2.11). Now choosing the particular form for δv_μ,

$$\delta v_\mu(x) = -\partial_\mu\epsilon(x), \tag{2.13}$$

we have

$$\begin{aligned}\delta_\epsilon\ln Z[v_\mu] &\equiv \ln Z[v_\mu - \partial_\mu\epsilon] - \ln Z[v_\mu] \\ &= i\int d^4x\ \bar{J}^\mu(x)\partial_\mu\epsilon(x) = -i\int d^4x\,\epsilon(x)\partial_\mu\bar{J}^\mu(x).\end{aligned} \tag{2.14}$$

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With this procedure we can isolate a divergence condition for $\bar{J}^\mu$. If $Z[v_\mu - \partial_\mu \epsilon] = Z[v_\mu]$, then $\partial_\mu \bar{J}^\mu(x) = 0$. To check this, consider

$$Z[v_\mu - \partial_\mu \epsilon] = \int [d\varphi_i] \exp i \int d^4x \left(\mathcal{L}(\varphi_i, \partial\varphi_i) - (v_\mu - \partial_\mu \epsilon) J^\mu \right). \tag{2.15}$$

If we can change integration variables so that

$$\int [d\varphi_i] = \int [d\varphi_i'] \tag{2.16}$$

with φ_i' given by Eq. (2.8), then we obtain

$$Z[v_\mu - \partial_\mu \epsilon] = \int [d\varphi_i'] \exp i \int d^4x \left(\mathcal{L}(\varphi_i', \partial\varphi_i') + v_\mu J^\mu \right) = Z[v_\mu], \tag{2.17}$$

and therefore

$$\partial_\mu \bar{J}^\mu(x) = 0. \tag{2.18}$$

This change of variables seems reasonable and in most cases is perfectly legitimate. After all, the symbol $[d\varphi_i(x)]$ means that we integrate over all values of the field φ_i separately at each point in spacetime. Shifting the origin of integration at point x by a constant, $\varphi_i(x) \equiv \varphi_i'(x) - \epsilon(x) f_i$, and then integrating over all values of φ_i' should amount to the original integration. Given this shift, we have obtained in Eq. (2.18) by Noether's theorem a quantum conservation law involving matrix elements. The expression $\partial_\mu \bar{J}^\mu(x) = 0$ means that all matrix elements of J^μ, obtained via further functional derivatives (as in Eq. (2.5)), satisfy a divergenceless condition, i.e., of the current J^μ is conserved in all matrix elements.

It was Fujikawa who first pointed out the consequences if the change of variables, Eq. (2.16), is not a valid operation in a path integral [Fu 79]. Certainly, many procedures involving path integrals need to be examined carefully in order to see if they are well defined. We shall explicitly study some examples in which the change of variable is nontrivial and can be calculated. In such cases one finds $\partial_\mu \bar{J}^\mu(x) \neq 0$, which implies that the classical symmetry is *not* a quantum symmetry. In these situations it is said that there exists an *anomaly*.

III–3 The $U(1)$ axial anomaly

For massless quarks $m_u = m_d = m_s = 0$, the QCD lagrangian contains an invariance $\mathcal{L}_{QCD} \to \mathcal{L}_{QCD}$ under the global $U(1)$ axial transformations

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \psi' = e^{-i\theta\gamma_5}\psi. \tag{3.1}$$

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In this limit, which we shall adopt until near the end of this chapter, Noether's theorem can be applied to identify the classically conserved axial current,

$$J_{5\mu}^{(0)} = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s, \qquad \partial^\mu J_{5\mu}^{(0)} = 0, \tag{3.2}$$

where the superscript on $J_{5\mu}^{(0)}$ denotes an $SU(3)$ *singlet* current. We shall see that this is not an approximate symmetry of the full quantum theory because the current divergence has an anomaly. This can be demonstrated in various ways. For a direct 'hands-on' demonstration, the early discussion [Ad 69, BeJ 67, Ad 70] of Adler and of Bell and Jackiw, which we recount below, has still not been improved upon. However, for a deeper understanding, Fujikawa's path-integral treatment [Fu 79], also described below, seems to us to be the most illuminating. The effect of an anomaly is simply stated, although one must go through some subtle calculations to be convinced that the effect is inescapable. An anomaly is said to occur when a symmetry of the classical action is not a true symmetry of the full quantum theory. The Noether current is no longer divergenceless, but receives a contribution arising from quantum corrections. It is this contribution which is often loosely referred to as the anomaly. The Ward identities which relate matrix elements no longer hold, but rather are replaced by a set of *anomalous* Ward identities, which take into account the correct current divergence.

There are two applications of the axial anomaly which have proved to be of particular importance to the Standard Model. One is in connection with the $SU(3)$ singlet axial current described above. Here the anomaly will end up telling us that the current is not conserved in the chiral limit, but rather that

$$\partial^\mu J_{\mu 5}^{(0)} = \frac{3\alpha_s}{4\pi} F_{\mu\nu}^a \tilde{F}^{a\mu\nu} \qquad \left(\tilde{F}_{\mu\nu}^a \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}^a\right). \tag{3.3}$$

This will serve to keep the ninth pseudoscalar meson, the η', from being a pseudo-Goldstone boson.

The other application is in the decay $\pi^0 \to \gamma\gamma$, which is historically the process wherein the anomaly was discovered. The quantity of interest here is an isovector axial current $J_{5\mu}^{(3)}$ which transforms as the third component of an $SU(3)$-flavor octet,

$$J_{5\mu}^{(3)} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d. \tag{3.4}$$

Without the anomaly, one would expect that the current $J_{5\mu}^{(3)}$ would be conserved in the chiral $SU(2)$ limit even in the presence of electromagnetism. This follows from the apparently correct procedure

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$$\partial^\mu J^{(3)}_{5\mu} = \bar u\left[\left(\overleftarrow{\not\partial} - iQ\not A\right)\gamma_5 - \gamma_5\left(\not\partial + iQ\not A\right)\right]u \\ - \bar d\left[\left(\overleftarrow{\not\partial} - iQ\not A\right)\gamma_5 - \gamma_5\left(\not\partial + iQ\not A\right)\right]d = 0. \tag{3.5}$$

However, explicit calculation shows that the current has an anomaly, such that

$$\partial^\mu J^{(3)}_{5\mu} = 2i\left(m_u\bar u\gamma_5 u - m_d\bar d\gamma_5 d\right) + \frac{\alpha N_c}{6\pi}F_{\mu\nu}\tilde F^{\mu\nu}, \tag{3.6}$$

where $F_{\mu\nu}$ is the electromagnetic field strength. This will be important in predicting the $\pi^0 \to \gamma\gamma$ and $\eta^0 \to \gamma\gamma$ rates and serves as a test for the number of quark colors.

Diagrammatic analysis

To review the work of Adler and of Bell and Jackiw, we first consider the Ward identities for the coupling of the $U(1)$ axial current to two gluons. We define

$$T^{ab}_{\mu\alpha\beta}(k,q) \equiv i\int d^4x\, d^4y\, e^{ik\cdot x}e^{iq\cdot y}\left\langle 0\left|T\left(J^{(0)}_{5\mu}(x)J^a_\alpha(y)J^b_\beta(0)\right)\right|0\right\rangle, \tag{3.7}$$

where J^a_α is a flavor-singlet (color-octet) vector current coupled to gluons

$$J^a_\alpha = \sum_{q=u,d,s}\bar q\gamma_\alpha\frac{\lambda^a}{2}q. \tag{3.8}$$

It is important to understand that the $SU(3)$ matrices pertain here to the *color* degree of freedom and should not be confused with analogous matrices which operate in flavor space. The amplitude $T^{ab}_{\mu\alpha\beta}$ is related to the vacuum-to-digluon matrix element by

$$\langle G^a(\lambda_1,q)\; G^b(\lambda_2,-k-q)\big|J^{(0)}_{5\mu}\big|0\rangle = ig_3^2\epsilon_1^{\dagger\alpha}\epsilon_2^{\dagger\beta}T^{ab}_{\mu\alpha\beta}(k,q). \tag{3.9}$$

There are two Ward identities, representing the conservation of axial and vector currents. The vector Ward identity, corresponding to color current conservation, $\partial^\alpha J^a_\alpha = 0$, is

$$q^\alpha T^{ab}_{\mu\alpha\beta}(k,q) = 0. \tag{3.10}$$

The axial Ward identity is derived in a similar fashion using the assumed conservation of the $U(1)$ axial current in the massless limit,

$$\partial^\mu J^{(0)}_{5\mu}(x) = 0, \tag{3.11}$$

to yield

$$k^\mu T^{ab}_{\mu\alpha\beta}(k,q) = 0. \tag{3.12}$$

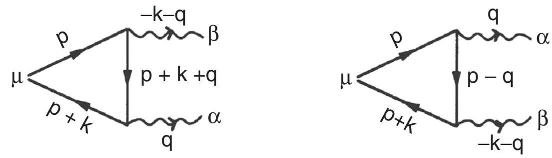

Fig. III–1 Triangle diagrams associated with the axial anomaly.

In order to reveal the anomalous behavior of this coupling, we calculate the vertex in lowest-order perturbation theory via the triangle diagrams of Fig. III–1. With the momenta as labeled in the figures, this produces the amplitude

$$T^{ab}_{\mu\alpha\beta} = -3\int \frac{d^4 p}{(2\pi)^4}\left[\mathrm{Tr}\left(\gamma_\mu\gamma_5\frac{1}{\not p+\not k}\gamma_\beta\frac{\lambda^b}{2}\frac{1}{\not p-\not q}\gamma_\alpha\frac{\lambda^a}{2}\frac{1}{\not p}\right)\right.$$
$$\left. + \,\mathrm{Tr}\left(\gamma_\mu\gamma_5\frac{1}{\not p+\not k}\gamma_\alpha\frac{\lambda^a}{2}\frac{1}{\not p+\not k+\not q}\gamma_\beta\frac{\lambda^b}{2}\frac{1}{\not p}\right)\right], \tag{3.13}$$

where the prefactor of 3 arises from the three massless quarks, each of which contributes equally.

Observe that these integrals are linearly divergent, and so may not be well defined. In particular, there exists an ambiguity corresponding to the different possible ways to label the loop momentum. An example will prove instructive, so we consider the integral

$$I_\gamma = \int d^4 p\left[\frac{p_\gamma}{p^4} - \frac{(p-\ell)_\gamma}{(p-\ell)^4}\right]. \tag{3.14}$$

This is evaluated by transforming to Euclidean space, where $p_0 = ip_4$ and $p^2 = -p_4^2 - \mathbf{p}^2 \equiv -p_E^2$. In order to perform the integration, one may note that for a general function, $F(p)$, whose four-dimensional integral is linearly divergent (i.e., one with $p^3F(p) \neq 0$, but $p^3F'(p) = p^3F''(p) = \ldots = 0$ for $p \to \infty$), one finds by Taylor expanding and using Gauss' theorem that

$$\begin{aligned}\int d^4 p_E\,[F(p) - F(p-\ell)] &= \int d^4 p_E\left[\ell^\mu\partial_\mu F(p) - \frac{1}{2}\ell^\mu\ell^\nu\partial_\mu\partial_\nu F(p) + \cdots\right]\\ &= \ell^\mu\int d^3S_\mu\left[F(p) - \frac{1}{2}\ell^\nu\partial_\nu F(p) + \cdots\right]_{p\to\infty}\\ &= \ell^\mu\int d^3S_\mu\,F(p)\Big|_{p\to\infty}\end{aligned} \tag{3.15}$$

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where $d^3 S_\mu$ indicates integration over a three-dimensional surface at $p \to \infty$.[3] Applying this result to the case at hand, we obtain a surface integral

$$I_\gamma = i \int d^4 p_E \left(\frac{p_\gamma}{p^4} - \frac{(p-\ell)_\gamma}{(p-\ell)^4} \right) = i\ell^\mu \int d^3 S_\mu \frac{p_\gamma}{p^4} = i\ell^\mu \int d^3 S \frac{p_\mu}{p} \frac{p_\gamma}{p^4}. \tag{3.16}$$

Note that from euclidean covariance we can replace $p_\mu p_\gamma$ by $\delta_{\mu\gamma} p^2/4$, to yield

$$I_\gamma = i\frac{\ell_\gamma}{4} \int d^3 S \frac{1}{p^3} = i\frac{\pi^2 \ell_\gamma}{2}, \tag{3.17}$$

where the last step uses the surface area of a three-dimensional surface in four-dimensional euclidean space, $S_4 = 2\pi^2 R^3$.

In the case of $T^{ab}_{\mu\alpha\beta}$, consider the effect of shifting the integration variable of the first term in Eq. (3.13) from p to $p + b_1 q + b_2(-k-q)$. In order to maintain the Bose symmetry of $T^{ab}_{\mu\alpha\beta}$ (i.e., symmetry under the interchange $\alpha \leftrightarrow \beta$ at the same time as $q \leftrightarrow (-k-q)$) we must shift the second integration from p to $p + b_1(-k-q) + b_2 q$. Use of Eqs. (3.14)–(3.17) then yields the change in $T^{ab}_{\mu\alpha\beta}$

$$\begin{aligned}
\Delta T^{ab}_{\mu\alpha\beta} &= \frac{6i\delta^{ab}}{(2\pi)^4} \epsilon_{\mu\alpha\beta\gamma} [I^\gamma (b_1 q + b_2(-q-k)) - I^\gamma (b_1(-q-k) + b_2 q)] \\
&= -\frac{3\delta^{ab}}{16\pi^2} (b_1 - b_2) \epsilon_{\mu\alpha\beta\gamma} (2q+k)^\gamma,
\end{aligned} \tag{3.18}$$

induced by the shift of the original integration variable p^μ. This is an indication that there may be trouble in the calculation of this diagram, but it is not yet proof of any violation of the Ward identities.

Let us now check the Ward identities. In both cases, use can be made of identities similar to $q^\alpha = p^\alpha - (p^\alpha - q^\alpha)$ in order to change the result into a difference of integrals. We find for the vector Ward identity

$$\begin{aligned}
&q^\alpha T^{ab}_{\mu\alpha\beta}(k,q) \\
&= -\frac{3\delta^{ab}}{2} \int \frac{d^4 p}{(2\pi)^4} \operatorname{Tr} \left[\gamma_\mu \gamma_5 \frac{1}{\not p + \not k} \gamma_\beta \frac{1}{\not p - \not q} - \gamma_\mu \gamma_5 \frac{1}{\not p + \not q + \not k} \gamma_\beta \frac{1}{\not p} \right] \\
&= -6i\delta^{ab} \epsilon_{\mu\beta\rho\sigma} \int \frac{d^4 p}{(2\pi)^4} \left[\frac{(p+k)^\rho (p-q)^\sigma}{(p+k)^2 (p-q)^2} - \frac{(p+k+q)^\rho p^\sigma}{(p+k+q)^2 p^2} \right],
\end{aligned} \tag{3.19}$$

[3] Note that this is just the four-dimensional generalization of the one-dimensional formula

$$\int_{-\infty}^{\infty} dx\, [f(x+y) - f(x)] = \int_{-\infty}^{\infty} dx \left[y f'(x) + \frac{1}{2} y^2 f''(x) + \cdots \right] = y\, [f(\infty) - f(-\infty)],$$

valid for $f(\pm\infty) \neq 0$ but $f'(\pm\infty) = f''(\pm\infty) = \ldots = 0$.

while for the axial-vector case,

$$
\begin{aligned}
k^\mu T^{ab}_{\mu\alpha\beta}(k,q) &= \frac{3\delta^{ab}}{2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_5\gamma_\beta\frac{1}{\not p-\not q}\gamma_\alpha\frac{1}{\not p}+\gamma_5\frac{1}{\not p+\not k}\gamma_\beta\frac{1}{\not p-\not q}\gamma_\alpha\right.\\
&\qquad\left.+\gamma_5\gamma_\alpha\frac{1}{\not p+\not k+\not q}\gamma_\beta\frac{1}{\not p}+\gamma_5\frac{1}{\not p+\not k}\gamma_\alpha\frac{1}{\not p+\not k+\not q}\gamma_\beta\right]\\
&= -6i\delta^{ab}\epsilon_{\alpha\beta\rho\sigma}\int\frac{d^4p}{(2\pi)^4}\left[\frac{(p+k+q)^\rho p^\sigma}{(p+k+q)^2p^2}-\frac{(p+k)^\rho(p-q)^\sigma}{(p+k)^2(p-q)^2}\right.\\
&\qquad\left.+\frac{(p+k)^\rho(p+k+q)^\sigma}{(p+k)^2(p+k+q)^2}-\frac{(p-q)^\rho p^\sigma}{(p-q)^2p^2}\right]. \qquad (3.20)
\end{aligned}
$$

It is easy to see that if one could freely shift the integration variable, each expression would separately vanish. However, direct calculation using Eqs. (3.14)–(3.17) yields

$$
q^\alpha T^{ab}_{\mu\alpha\beta}(k,q) = -\frac{3\delta^{ab}}{16\pi^2}\epsilon_{\mu\beta\rho\sigma}k^\rho q^\sigma \quad \text{and} \quad k^\mu T^{ab}_{\mu\alpha\beta}(k,q) = \frac{3\delta^{ab}}{8\pi^2}\epsilon_{\alpha\beta\rho\sigma}k^\rho q^\sigma. \qquad (3.21)
$$

If, on the other hand, the original integration variable were shifted as in Eq. (3.18) one would obtain

$$
\begin{aligned}
q^\alpha T^{ab}_{\mu\alpha\beta}(k,q) &= -\frac{3\delta^{ab}}{16\pi^2}(1+b_1-b_2)\,\epsilon_{\mu\beta\rho\sigma}k^\rho q^\sigma,\\
k^\mu T^{ab}_{\mu\alpha\beta}(k,q) &= \frac{3\delta^{ab}}{8\pi^2}(1-b_1+b_2)\,\epsilon_{\alpha\beta\rho\sigma}k^\rho q^\sigma.
\end{aligned} \qquad (3.22)
$$

Thus, either *one* of the original Ward identities may be regained by a particular choice of $b_1 - b_2$, but both expressions cannot vanish simultaneously.

Our discussion of the manipulations of Feynman diagrams should not obscure the main physical fact illustrated above, i.e., despite the claim of Noether's theorem that there are *two* sets of conserved currents (vector $SU(3)$ of color and axial-vector $U(1)$), one-loop calculations indicate that only one can in fact be conserved. On physical grounds, we know that in Nature the vector current is conserved, as its charge corresponds to *QCD* color charge. Thus, it must be the axial current which is not conserved. This phenomenon is at first sight quite surprising and it deserves the name 'anomaly' by which it has come to be called. Noether's theorem has misled us, and it is only by direct calculation of the quantum corrections that the true symmetry structure of the theory has been exposed. Note that the situation is not the same as spontaneous symmetry breaking, where the symmetry is hidden by dynamical effects. There the currents remain conserved, as demonstrated in Sect. I–6. Here, current conservation has been violated. In particular, the calculation described above (with $b_1 - b_2 = 1$) is consistent through use of Eq. (3.9) with the operator relation of Eq. (3.3),

$$\partial^\mu J_{5\mu}^{(0)} = \frac{3\alpha_s}{4\pi} F^a_{\mu\nu} \tilde{F}^{a\mu\nu}. \tag{3.23}$$

Both sides of this equation have the same two-gluon matrix elements. It is clear from this that the apparent $U(1)$ symmetry predicted by Noether's theorem is not a symmetry of the quantum theory after all.

Path-integral analysis

In a path-integral treatment [Fu 79], the symmetry of the theory can be tested by considering the generating functional, as described in Sect. III–2. In particular, if we consider a functional of the gluon field A^b_μ and an axial current source a_μ,

$$Z\left[a_\mu, A^c_\lambda\right] = \int [d\psi][d\bar{\psi}] \exp i \int d^4x \left(\mathcal{L}_{QCD}(\psi, \bar{\psi}, A^c_\lambda) - a_\mu J_5^{(0)\mu}\right) \tag{3.24}$$

then the steps leading to Eq. (2.14) produce

$$-i \int d^4x\, \beta(x) \partial^\mu \bar{J}^{(0)}_{5\mu}(x) = \ln Z\left[a_\mu - \partial_\mu \beta, A^b_\mu\right] - \ln Z\left[a_\mu, A^b_\mu\right], \tag{3.25}$$

where $\bar{J}^{(0)}_{5\mu}(x)$ denotes the matrix elements of the current $J_5^{(0)}$,

$$\bar{J}^{(0)}_{5\mu}(x) = i \frac{\delta}{\delta a^\mu(x)} \ln Z\left[a_\nu, A^b_\lambda\right]\Big|_{a_\nu = 0}. \tag{3.26}$$

In particular, the two-gluon matrix described above is given by

$$T^{ab}_{\mu\alpha\beta}(x, y, z) = (i)^2 \left[\frac{\delta^2}{\delta A^\alpha_a(y) \delta A^\beta_b(z)} \bar{J}^{(0)}_{5\mu}(x)\right]\Bigg|_{\substack{A^c_\lambda = 0 \\ a_\nu = 0}}. \tag{3.27}$$

In order to solve for $\partial^\mu \bar{J}^{(0)}_{5\mu}$, we note that the $\partial_\mu \beta$ term can be absorbed into a redefinition of the fermion fields. This can be seen from the identity (for infinitesimal β),

$$\bar{\psi} i \partial\!\!\!/ \psi + \partial_\mu \beta\, \bar{\psi} \gamma^\mu \gamma_5 \psi = \bar{\psi}\, (1 - i\beta\gamma_5)\, i\partial\!\!\!/\, (1 - i\beta\gamma_5)\, \psi. \tag{3.28}$$

The following quantities are invariant under this transformation:

$$\begin{aligned} \bar{\psi} A\!\!\!/^a \lambda^a \psi &= \bar{\psi}(1 - i\beta\gamma_5) A\!\!\!/^a \lambda^a (1 - i\beta\gamma_5)\psi, \\ J_\mu = \bar{\psi}\gamma_\mu\psi &= \bar{\psi}(1 - i\beta\gamma_5)\gamma_\mu(1 - i\beta\gamma_5)\psi. \end{aligned} \tag{3.29}$$

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Mass terms would not be invariant, but we are presently working in the massless limit. Therefore, if we define

$$\begin{aligned} \psi' &= (1 - i\beta\gamma_5)\psi = e^{-i\beta\gamma_5}\psi + \mathcal{O}(\beta^2), \\ \bar{\psi}' &= \bar{\psi}(1 - i\beta\gamma_5) = \bar{\psi}e^{-i\beta\gamma_5} + \mathcal{O}(\beta^2), \end{aligned} \tag{3.30}$$

we see that the lagrangian can be written in terms of ψ',

$$\mathcal{L}_{QCD}(\psi, \bar{\psi}, A^a_\mu) + \partial^\mu\beta\, J^{(0)}_{5\mu} = \mathcal{L}_{QCD}(\psi', \bar{\psi}', A^a_\mu). \tag{3.31}$$

Furthermore, we would like to change from ψ to ψ' in the path integration. To be general, we allow for the possibility of a jacobian $\mathcal{J}$ accompanying this change of variables, viz.,

$$\int [d\psi][d\bar{\psi}] \equiv \int [d\psi'][d\bar{\psi}']\mathcal{J}. \tag{3.32}$$

If, as will be shown later, the jacobian $\mathcal{J}$ is independent of ψ and $\bar{\psi}$, it can be taken to the outside of the path integral, resulting in

$$\begin{aligned} Z\left[a_\mu - \partial_\mu\beta, A^a_\mu\right] &= \int [d\psi'][d\bar{\psi}']\mathcal{J} e^{i\int d^4x \left(\mathcal{L}_{QCD}(\psi',\bar{\psi}',A^a_\mu) - a_\mu J^\mu_5\right)} \\ &= \mathcal{J}\, Z\left[a_\mu, A^a_\mu\right]. \end{aligned} \tag{3.33}$$

Thus, the test for the symmetry, Eq. (3.25), depends entirely on $\mathcal{J}$,

$$\ln \mathcal{J} = -i \int d^4x\; \beta(x)\partial^\mu \bar{J}^{(0)}_{5\mu}(x). \tag{3.34}$$

The lesson learned is that if the lagrangian *and* the path-integral measure are invariant under the $U(1)$ transformation, then there exists a $U(1)$ symmetry in the theory, with $\partial^\mu \bar{J}^{(0)}_{5\mu} = 0$. However, if the lagrangian is invariant, as it is in this case, but the path integral is not (i.e. $\mathcal{J} \neq 1$), then the $U(1)$ transformation is not a symmetry of the theory, i.e., $\partial^\mu J^{(0)}_{5\mu} \neq 0$.

We shall show below that the jacobian, when properly regularized, has the form

$$\mathcal{J} = \exp\left(-2i\, \mathrm{tr}\, \beta\gamma_5\right) = \exp\left[-i\int d^4x\; \beta(x)\frac{3\alpha_s}{4\pi}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\right], \tag{3.35}$$

so that the current divergence has the form given in Eq. (3.3),

$$\partial^\mu \bar{J}^{(0)}_{5\mu} = \frac{3\alpha_s}{4\pi}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}.$$

Functional differentiation using Eq. (3.27) yields the same result for $q^\mu T^{ab}_{\mu\alpha\beta}$ as obtained in ordinary perturbation theory. The nontrivial transformation of the path-integral measure has prevented the axial $U(1)$ transformation from being a symmetry of the theory. We now turn to the calculation of the jacobian.

The jacobian in fact diverges, and a regularization is needed in order to make it finite. In Fujikawa's original calculation the regularizer was introduced early into the procedure, allowing each step to be well defined. We will be slightly less rigorous by introducing the regularizer somewhat later. In order to calculate the jacobian we need to review the properties of integration over Grassmann numbers (which are described in more detail in App. A–5). The anticommuting nature of the variables requires that any function constructed from them terminates after linear order in each variable. Thus, a function of two Grassman numbers $z_1,\ z_2$ ($z_1 z_2 = -z_2 z_1,\ z_i^2 = 0$) becomes

$$f(z_1, z_2) = f_0 + f_1 z_1 + f_2 z_2 + f_{12} z_1 z_2, \tag{3.36}$$

where f_0, f_1, f_2, f_{12} are real numbers. The primary property of an integral to be transferred to Grassmann numbers is completeness, i.e.,

$$\int dz\ f(z) = \int dz\ f(z + z'), \tag{3.37}$$

where z' is a constant Grassmann number. Expanding both sides we have

$$\int dz\ (f_0 + f_1 z) = \int dz\ \left(f_0 + f_1 z + f_1 z'\right) \tag{3.38}$$

For this to be true, the condition

$$\int dz = 0 \tag{3.39}$$

is required. Now consider a change of variables

$$z_1 = c_{11} z_1' + c_{12} z_2', \qquad z_2 = c_{21} z_1' + c_{22} z_2', \tag{3.40}$$

involving a matrix of coefficients $\mathbf{C}$. The jacobian is defined by

$$\int dz_1\, dz_2\ f(z) = \mathcal{J} \int dz_1'\, dz_2'\ f(\mathbf{C} z'). \tag{3.41}$$

Application of Eq. (3.36) leads to the consideration of only the f_{12} term,

$$\begin{aligned} f_{12} \int dz_1\, dz_2\ z_1 z_2 &= \mathcal{J} f_{12} \int dz_1'\, dz_2'\ (c_{11} z_1' + c_{12} z_2')(c_{21} z_1' + c_{22} z_2') \\ &= \mathcal{J} f_{12} (c_{11} c_{22} - c_{12} c_{21}) \int dz_1'\, dz_2'\ z_1' z_2', \end{aligned} \tag{3.42}$$

and hence the identification of the jacobian,

$$\mathcal{J} = [\det \mathbf{C}]^{-1}. \tag{3.43}$$

Although derived in the simple 2×2 case, Eq. (3.43) generalizes to arbitrary dimension. Note that, due to the Grassmann nature of the variables, this result is the inverse of what would be expected with normal commuting variables.

Turning now to the path integral, we temporarily consider $\psi(x)$ as a finite number of Grassmann variables corresponding to four Dirac indices at each point of spacetime (i.e., imagine that the spacetime label is discrete and finite). At each point, the transformation is from $\psi \to \psi'$

$$\psi(x) = e^{i\beta(x)\gamma_5}\psi'(x), \qquad \bar{\psi}(x) = \bar{\psi}'(x)\, e^{i\beta(x)\gamma_5}, \tag{3.44}$$

so that the overall jacobian has the form

$$\mathcal{J} = \left[\det\left(e^{i\beta\gamma_5}\right)\right]^{-1}\left[\det\left(e^{i\beta\gamma_5}\right)\right]^{-1} \tag{3.45}$$

with one factor from each of the ψ and $\bar{\psi}$ variables. The determinant runs over the 4×4 Dirac indices, the three flavors, colors, and also the spacetime indices. This is a rather formal object, but can be made more explicit by using

$$\det \mathbf{C} = e^{\operatorname{tr}\, \ln \mathbf{C}}, \tag{3.46}$$

valid for finite matrices, to write

$$\mathcal{J} = e^{-2i\, \operatorname{tr}\, \beta\gamma_5}. \tag{3.47}$$

The symbol tr denotes a trace acting over spacetime indices plus Dirac indices, flavors, and colors,

$$\operatorname{tr} \beta\gamma_5 = \operatorname{Tr}' \int d^4x \, \langle x|\beta\gamma_5|x\rangle \,, \tag{3.48}$$

with Tr' indicating the Dirac, color, and flavor trace. This will become clearer through direct calculation below.

The jacobian still is not regulated. Fujikawa suggested the removal of high-energy eigenmodes of the Dirac field in a gauge-invariant way. Consider, for example, the simple extension

$$\mathcal{J} = \lim_{M\to\infty} \exp\left[-2i\, \operatorname{tr}\left(\beta\gamma_5\, e^{-(\not{D}/M)^2}\right)\right], \tag{3.49}$$

where $\not{D}$ is the *QCD* covariant derivative. The insertion of a complete set of eigenfunctions of $\not{D}$ exponentially removes those with large eigenvalues. There has been an extensive literature demonstrating that other regularization methods produce the same results as Fujikawa's, provided that the regulator preserves the vector gauge invariance.

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In order to complete the calculation we employ the following identity:

$$\begin{aligned}
\not{D}\not{D} &= \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} D^\mu D^\nu + \frac{1}{2}[\gamma_\mu, \gamma_\nu] D^\mu D^\nu \\
&= D_\mu D^\mu + \frac{1}{4}[\gamma_\mu, \gamma_\nu]\left[D^\mu, D^\nu\right] \\
&= D_\mu D^\mu + \frac{g_3 \lambda^a}{4} \sigma^{\mu\nu} F^a_{\mu\nu}.
\end{aligned} \tag{3.50}$$

In this case the expression

$$\langle x | \exp -(\not{D}/M)^2 | x \rangle \tag{3.51}$$

has the same form as given in Eqs. (B–1.1), (B–1.9), (B–1.17–18) with the identifications

$$d_\mu = D_\mu, \quad \sigma = \frac{g_3}{4}\sigma^{\mu\nu}\lambda^a F^a_{\mu\nu}, \quad \tau = \frac{1}{M^2}. \tag{3.52}$$

Applying the calculation done there to our present situation yields

$$\begin{aligned}
\mathcal{J} &= \lim_{M\to\infty} e^{-2i \int d^4x \ \mathrm{Tr}\,(\beta(x)\gamma_5 H(x, M^{-2}))} \\
&= \lim_{M\to\infty} e^{\frac{1}{8\pi^2}\int d^4x \ \mathrm{Tr}\,(\beta(x)\gamma_5[M^4 a_0 + M^2 a_1 + a_2 + \mathcal{O}(M^{-2})])}.
\end{aligned} \tag{3.53}$$

The notation is defined in App. B–1. The first two traces vanish, leaving only the factor with two $\sigma^{\mu\nu}$ matrices in a_2. From the result

$$\mathrm{Tr}\,(\gamma_5 \sigma^{\mu\nu}\sigma^{\alpha\beta}) = -\,\mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = -4i\epsilon^{\mu\nu\alpha\beta}, \tag{3.54}$$

it is easy to calculate

$$\begin{aligned}
\mathcal{J} &= \exp\left(\frac{1}{16\pi^2}\int d^4x\, \beta(x)\mathrm{Tr}'\left(\gamma_5 \frac{g_3^2\lambda^a\lambda^b}{16}\sigma^{\mu\nu}F^a_{\mu\nu}\sigma^{\alpha\beta}F^b_{\alpha\beta}\right)\right) \\
&= \exp\left(\frac{-1}{16\pi^2}\int d^4x\, \beta(x)\ 3\cdot 2\delta^{ab}\cdot 4\, i\epsilon^{\mu\nu\alpha\beta}\frac{g_3^2}{16}F^a_{\mu\nu}F^b_{\alpha\beta}\right) \\
&= \exp\left(-i\int d^4x\, \beta(x)\frac{3\alpha_s}{4\pi}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\right)
\end{aligned} \tag{3.55}$$

where the trace Tr' has produced factors for three flavors, color, and the Dirac trace.

Although the calculation of the jacobian has been somewhat involved, we have succeeded in making sense out of what seemed to be a rather abstract object. The fact that it is not unity is an indication that the $U(1)$ transformation is not a symmetry of the theory. Applying Eq. (3.34) we see that

$$\ln\mathcal{J} = -i\int d^4x\, \beta(x)\partial^\mu \bar{J}^{(0)}_{5\mu}(x) = -i\int d^4x\, \beta(x)\frac{3\alpha_s}{4\pi}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}, \tag{3.56}$$

or once again

$$\partial^\mu \bar{J}^{(0)}_{5\mu} = \frac{3\alpha_s}{4\pi} F^a_{\mu\nu} \tilde{F}^{a\mu\nu}. \tag{3.57}$$

The choice of a regulator which preserves the vector $SU(3)$ gauge symmetry is important. Whereas in the Feynman diagram approach, we had the apparent freedom to shift the integration variable to preserve either the vector or axial-vector symmetries, the corresponding freedom in the path-integral case is in the choice of regularization.

If quark masses are included, the operator relation becomes

$$\partial_\mu J^{(0)}_{5\mu}(x) = 2i(m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d + m_s \bar{s}\gamma_5 s) + \frac{3\alpha_s}{4\pi} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}. \tag{3.58}$$

Masses do not modify the coefficient of the anomaly, basically because it arises from the ultraviolet divergent parts of the theory, which are insensitive to masses.

One does not have to go through these lengthy calculations for each new application of the anomaly. The anomalous coupling for currents

$$V^{(b)}_\mu = \bar{\psi}\gamma_\mu T^{(b)}_{\rm v}\psi, \qquad A^{(b)}_\mu = \bar{\psi}\gamma_\mu\gamma_5 T^{(b)}_{\rm a}\psi, \tag{3.59}$$

where $T^{(b)}_{\rm v}$, $T^{(b)}_{\rm a}$ are matrices in the space of quark flavors, is of the form

$$\partial^\mu A^{(b)}_\mu = \frac{D^{bcd}}{16\pi^2}\,\epsilon^{\mu\nu\alpha\beta} F^c_{\mu\nu} F^d_{\alpha\beta} + \text{mass terms}, \tag{3.60a}$$

$$D^{bcd} \equiv \frac{N_c}{2}\,\text{Tr}\left(T^{(b)}_{\rm a}\left\{T^{(c)}_{\rm v}, T^{(d)}_{\rm v}\right\}\right), \tag{3.60b}$$

where N_c is the number of colors. In particular, for the electromagnetic coupling to the isovector axial current we have

$$\begin{aligned} J^{(3)}_{5\mu} &= \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d, \\ D^{bcd} &= e^2 N_c \,\text{Tr}\, \tau_3 Q^2 = \frac{N_c}{3} e^2, \end{aligned} \tag{3.61}$$

leading to the result already quoted in Eq. (3.6).

The full content of the anomaly was given by Bardeen [Ba 69]. Consider a fermion with η internal degrees of freedom (flavor or color) coupled to vector and axial-vector currents v_μ, a_μ,

$$\mathcal{L} = \overline{\psi}\,(i\not{\partial} - \not{v} - \not{a}\gamma_5)\,\psi. \tag{3.62}$$

These currents are in an $\eta \times \eta$ representation

$$v_\mu = v^0_\mu I + v^k_\mu \lambda^k, \qquad a_\mu = a^0_\mu I + a^k_\mu \lambda^k. \tag{3.63}$$

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Thus, the axial current is $J^{(k)}_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5\lambda^k\psi$, and the anomaly equation becomes

$$\begin{aligned}
\partial^\mu J^{(k)}_{5\mu} &= \frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left[\lambda^k\left(\frac{1}{4}v_{\mu\nu}v_{\alpha\beta} + \frac{1}{12}a_{\mu\nu}a_{\alpha\beta}\right.\right.\\
&\qquad \left.\left. -\frac{2i}{3}a_\mu a_\nu v_{\alpha\beta} - \frac{2i}{3}v_{\mu\nu}a_\alpha a_\beta - \frac{2i}{3}a_\mu v_{\nu\alpha}a_\beta - \frac{8}{3}a_\mu a_\nu a_\alpha a_\beta\right)\right],\\
v_{\mu\nu} &= \partial_\mu v_\nu - \partial_\nu v_\mu + i[v_\mu, v_\nu] + i[a_\mu, a_\nu],\\
a_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu + i[v_\mu, a_\nu] - i[v_\nu, a_\mu].
\end{aligned} \tag{3.64}$$

This may also be expressed in terms of the left-handed and right-handed field tensors $\ell_{\mu\nu}$ and $r_{\mu\nu}$ by using the identities,

$$\begin{aligned}
&\ell_{\mu\nu} \equiv \partial_\mu\ell_\nu - \partial_\nu\ell_\mu + i[\ell_\mu, \ell_\nu] = v_{\mu\nu} + a_{\mu\nu},\\
&r_{\mu\nu} \equiv \partial_\mu r_\nu - \partial_\nu r_\mu + i[r_\mu, r_\nu] = v_{\mu\nu} - a_{\mu\nu},\\
&\frac{1}{4}v_{\mu\nu}v_{\alpha\beta} + \frac{1}{12}a_{\mu\nu}a_{\alpha\beta} = \frac{1}{12}\left(\ell_{\mu\nu}\ell^{\mu\nu} + r_{\mu\nu}r^{\mu\nu}\right) + \frac{1}{24}\left(\ell_{\mu\nu}r^{\mu\nu} + r_{\mu\nu}\ell^{\mu\nu}\right).
\end{aligned} \tag{3.65}$$

In the language of Feynman diagrams, one encounters the anomaly contributions not only in the triangle diagram, but also in square and pentagon diagrams (e.g. from the $a_\mu a_\nu a_\alpha a_\beta$ term). Our previous result, Eq. (3.57), is obtained for $a_\mu = 0$, $v_\mu = g_3 A^k_\mu \lambda^k/2$, with three flavors and three colors of quarks.

We have seen that symmetries of the classical lagrangian are not always symmetries of the full quantum theory. This is the general situation when there are anomalies. These appear in perturbation theory and are associated with divergent Feynman diagrams. This sometimes gives the mistaken impression that the dynamics has 'broken' the symmetry, and hence one might expect a massless particle through the application of Goldstone's theorem. In the path-integral framework the impression is different. There the symmetry never exists in the first place, as the calculation performed above is simply the path-integral test for a symmetry, generalizing Noether's theorem. Hence there is in general no expectation for a Goldstone boson.

Can anomalies cause problems? When the anomaly occurs in a global symmetry, such as the above $U(1)$ example, the answer is, 'no'. They just need to be taken properly into account, e.g., as in Eq. (3.61). Given the specific form of the anomaly operator relation, there exist 'anomalous Ward identities' which contain terms attributable to the anomaly [Cr 78]. These anomalies can even be associated with a variety of specific phenomena. For example, in Sect. VII–6 we shall see how the decay $\pi^0 \to \gamma\gamma$ is attributed to the axial anomaly.

The presence of anomalies in gauge theories is far more serious because they destroy the gauge invariance of the theory and wreak havoc with renormalizability. Thus, one attempts to employ only those gauge theories which have no anomalies.

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In some cases this can be arranged by ensuring, through the group or particle content of the theory, that the coefficient D^{bcd} of Eq. (3.60b) vanishes. For example, in the Standard Model it must be checked that this occurs for all combinations of the $SU(3)_c \times SU(2)_L \times U(1)_Y$ generators. These were already compiled in Eqs. (II–3.5a–c) and were seen to lead to the quantized fermion charge values observed in Nature.

III–4 Classical scale invariance and the trace anomaly

If the fermion masses were zero in either *QED* or *QCD*, these theories would contain no dimensional parameters in the lagrangian, and they would exhibit a classical scale invariance. The associated quark and gluon scale transformations would be $\psi(x) \to \lambda^{3/2}\psi(\lambda x)$ and $A^a_\mu(x) \to \lambda A^a_\mu(\lambda x)$ for arbitrary λ. We saw in Sect. I–4 that this leads to a traceless energy-momentum tensor, with conserved dilation current $J^\mu_{\rm scale}$,

$$J^\mu_{\rm scale} = x_\nu \theta^{\mu\nu}, \qquad \partial_\mu J^\mu_{\rm scale} = \theta^\nu{}_\nu = 0, \tag{4.1}$$

where $\theta^{\mu\nu}$ is the energy-momentum tensor. Such a situation would have drastic consequences on the theory, since all single particle states would be massless. This can be seen as follows. For any hadron H, the matrix element of the energy-momentum tensor at zero-momentum transfer is

$$\langle H(\mathbf{k}) | \theta^{\mu\nu} | H(\mathbf{k})\rangle = 2k^\mu k^\nu, \tag{4.2}$$

where the normalization of states is chosen in accordance with the conventions defined in App. C–3. A vanishing trace would imply zero mass, i.e.,

$$\left\langle H(\mathbf{k}) \left| \theta^\mu_{\ \mu} \right| H(\mathbf{k})\right\rangle = 0 = 2M_H^2. \tag{4.3}$$

This is most obviously a problem in *QCD* where the quark masses are small compared to most composite particle masses.[4] We would not expect the proton mass to vanish if the quark masses were set equal to zero yet the scale-invariance argument implies that it must.

A resolution is suggested by the method which is used to renormalize the theory. In practice, renormalization prescriptions introduce dimensional scales into the theory. Most commonly, there is the momentum scale at which one specifies the running coupling constant to have a particular value, e.g., $\alpha_s(91\ \text{GeV}) \simeq 0.12$. This in turn defines a scale Λ which enters the formula for the running coupling constant, Eq. (II–2.74). Thus, to fully specify *QCD* one needs to specify not only the lagrangian, but also a scale parameter, and the full quantum theory is *not* scale

[4] As can be justified, we neglect here the existence of very heavy quarks, c, b, and t.

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invariant. Although this argument does not, at first sight, seem to nullify the reasoning based on Noether's theorem, it turns out that the trace of the energy-momentum tensor has an anomaly [Cr 72, ChE 72, CoDJ 77], and the specification of a scale and the coefficient of the anomaly are in fact related.

In the following, let us start directly with the path-integral treatment [Fu 81], again in the framework of *QCD*, concentrating on the effect of a single quark. We can introduce an external source coupled to $\theta^{\mu}_{\ \mu}$ into the generating functional

$$Z[h, A^a_\mu] = \int d\psi\, d\bar{\psi} e^{i\int d^4x\left[\mathcal{L}_{QCD}(\psi, A^a_\mu)+h(x)\theta^{\mu}_{\ \mu}\right]}, \tag{4.4}$$

where

$$\theta^{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{D}^\nu \psi. \tag{4.5}$$

As in the case of the chiral anomaly, we can use this as a starting point to explore the nature of the trace $\theta^{\mu}_{\ \mu}$. The key is that if one makes the change of variables

$$\psi(x) = e^{-\alpha(x)/2}\psi'(x), \tag{4.6}$$

one obtains for infinitesimal α

$$\begin{aligned}&\int d^4x \left[\mathcal{L}_{QCD}\left(\psi, A^a_\mu(x)\right) + \alpha(x)\theta^{\mu}_{\ \mu}\right]\\ &\quad = \int d^4x \left[\mathcal{L}_{QCD}\left(\psi', A^a_\mu\right) + \alpha(x) m\bar{\psi}'\psi' + i\bar{\psi}'\gamma_\mu\psi'\partial^\mu\alpha\right].\end{aligned} \tag{4.7}$$

The last term vanishes after an integration by parts. The focus of our calculation can thus be shifted to a jacobian $\mathcal{J}$ by a change of variable,

$$\begin{aligned}Z\left[h+\alpha, A^a_\mu\right] &= \int d\psi\, d\bar{\psi}\; e^{i\int d^4x\left[\mathcal{L}_{QCD}\left(\psi, A^a_\mu\right)+(h+\alpha)\theta^{\mu}_{\ \mu}\right]}\\ &= \int d\psi\, d\bar{\psi}\; e^{i\int d^4x\left[\mathcal{L}_{QCD}\left(\psi', A^a_\mu\right)+h\theta^{\mu}_{\ \mu}+\alpha m\bar{\psi}'\psi'\right]}\\ &= \int d\psi'\, d\bar{\psi}'\; \mathcal{J}\, e^{i\int d^4x\left[\mathcal{L}_{QCD}\left(\psi', A^a_\mu\right)+h\theta^{\mu}_{\ \mu}+\alpha m\bar{\psi}'\psi'\right]}.\end{aligned} \tag{4.8}$$

Thus, we obtain the identity

$$i\int d^4x\; \theta^{\mu}_{\ \mu}\alpha(x) = \ln\mathcal{J} + i\int d^4x\; m\bar{\psi}\psi\; \alpha(x). \tag{4.9}$$

The form of the jacobian which follows from the work done in Sect. III–3 is

$$\mathcal{J} = \left[\det\left(e^{-\alpha/2}\right)\right]^{-2} = \lim_{M\to\infty} e^{\,\mathrm{Tr}'\int d^4x\;\langle x|\alpha\,\exp -(\not{D}/M)^2|x\rangle}, \tag{4.10}$$

where we have adopted the same regulator as used previously.

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The final result is easily obtained from the general heat-kernel calculation of App. B–1, again using the identities of Eqs. (B–1.17), (B–1.18). After some algebra this becomes

$$\begin{aligned}
&\mathrm{Tr}'\left\langle x\left|\exp -(\not{D}/M)^2\right|x\right\rangle \\
&= \frac{iM^4}{16\pi^2}\,\mathrm{Tr}'\left[1-\frac{g_3^2\lambda^a\lambda^b}{32M^4}\sigma^{\mu\nu}\sigma^{\alpha\beta}F^a_{\mu\nu}F^b_{\alpha\beta}+\frac{[D^\mu,D^\nu][D_\mu,D_\nu]}{12M^4}+\cdots\right] \\
&= \frac{3iM^4}{4\pi^2}+\frac{ig_3^2}{48\pi^2}F^a_{\mu\nu}F^{\mu\nu}_a+\cdots\,. \qquad (4.11)
\end{aligned}$$

Here we have found both a term which is a divergent constant, and one which involves two-gluon field strengths. The divergent constant corresponds to the infinite zero-point energy of the vacuum. This can be seen by noting that if the zero-point energy is defined by the vacuum matrix element

$$\langle 0|\mathcal{H}(x)|0\rangle = \frac{E_0}{V} = \left\langle 0\left|\theta^{00}(x)\right|0\right\rangle, \tag{4.12}$$

then Lorentz covariance requires a nonzero trace

$$\langle 0\,|\theta^{\mu\nu}(x)|\,0\rangle = \frac{E_0}{V}\,g^{\mu\nu} \implies \left\langle 0\left|\theta^\mu_{\ \mu}(x)\right|0\right\rangle = 4\frac{E_0}{V}. \tag{4.13}$$

Thus, a constant in the vacuum matrix element of the trace is just four times the zero-point energy density. It is standard practice to subtract off this zero-point energy, and we shall do so by dropping the constant term. This is similar to the procedure of normal ordering the energy-momentum tensor.

If we now combine these results using Eq. (4.9), we obtain

$$i\int d^4x\,\theta^\mu_{\ \mu}\alpha(x) = i\int d^4x\left[\frac{g_3^2}{48\pi^2}F^a_{\mu\nu}F^{a\mu\nu}+m\bar{\psi}\psi\right]\alpha(x), \tag{4.14}$$

which is equivalent to the operator relation

$$\theta^\mu_{\ \mu} = \frac{\alpha_s}{12\pi}F^a_{\mu\nu}F^{a\mu\nu}+m\bar{\psi}\psi. \tag{4.15}$$

One may also derive the trace anomaly via the calculation of Feynman diagrams, the triangle diagrams of Fig. III–1, but with the axial current replaced by the energy-momentum tensor. The trace anomaly is different from the chiral anomaly in that it receives contributions also from gluons. In the Feynman diagram approach, this arises from the replacement of quark lines by gluons, while in the path-integral context it occurs when one considers scale transformations of the gluon field. A full calculation yields

$$\theta^\mu_{\ \mu} = \frac{\beta_{QCD}}{2g_3}F^a_{\mu\nu}F^{a\mu\nu}+m_u\bar{u}u+m_d\bar{d}d+m_s\bar{s}s+\cdots\,, \tag{4.16}$$

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where β_{QCD} is the beta function of QCD (cf. Eq. (II–2.57b)). The result of our previous calculation, Eq. (4.15), corresponds to the lowest order contribution of a single quark to the beta function.

A feeling of why the beta function enters can be obtained from an extremely simple, but heuristic, derivation of the trace anomaly. Let us rescale the gluon field to $\overline{A}^a_\mu \equiv g_3 A^a_\mu$, such that the massless action becomes

$$\mathcal{L} = -\frac{1}{4g_3^2}\bar{F}^a_{\mu\nu}\bar{F}^{a\mu\nu} + i\overline{\psi}\gamma^\mu \bar{D}_\mu\psi. \tag{4.17}$$

The coupling constant g_3 now enters only as an overall factor in the first term. However in renormalizing the coupling constant, we need to introduce a renormalization scale. If we interpret this coupling as a *running* parameter, the action is no longer invariant under scale transformations. Instead, taking $\lambda = 1 + \delta\lambda$, we find

$$\frac{\delta S}{\delta\lambda} = \int d^4x\, \frac{\partial}{\partial\lambda}\left(-\frac{1}{4g_3^2(\lambda)}\right)\bar{F}^a_{\mu\nu}\bar{F}^{a\mu\nu} = \int d^4x\, \frac{\beta_{QCD}(g_3)}{2g_3}F^a_{\mu\nu}F^{a\mu\nu}, \tag{4.18}$$

where we have changed back to the standard normalization of A^a_μ in the final term. By Noether's theorem, the scale current is no longer conserved, and Eq. (4.16) is reproduced. The need to specify a scale in defining the coupling constant has removed the scale invariance of the theory.

The trace anomaly occupies a significant place in the phenomenology of hadrons because it is the signal for the generation of hadronic masses. Returning to the discussion of masses which began this section, we see that the mass of a state is expressible as a matrix element of the energy-momentum trace. For example, we find for the nucleon state that

$$\begin{aligned} m_N\bar{u}(\mathbf{p})u(\mathbf{p}) &= \langle N(\mathbf{p})|\theta^\mu_{\ \mu}|N(\mathbf{p})\rangle \\ &= \langle N(\mathbf{p})|\frac{\beta_{QCD}}{2g_3}F^a_{\mu\nu}F^{a\mu\nu} + m_s\bar{s}s + m_u\bar{u}u + m_d\bar{d}d|N(\mathbf{p})\rangle. \end{aligned} \tag{4.19}$$

The terms containing the light quark masses m_u, m_d are expected to be small, and indeed the 'σ-term' determined in πN scattering (cf. Sect. III–3) implies that they contribute about only 45 MeV. This leaves the bulk of the nucleon's mass to the gluon and s-quark terms in Eq. (4.19), of which the $F^a_{\mu\nu}F^{a\mu\nu}$ part is expected to be dominant. Although this presents a conceptual problem for the naive quark model interpretation of the proton as a composite of three light quarks, it is nevertheless a central result of QCD.

III–5 Chiral anomalies and vacuum structure

There is a fascinating connection between the axial anomaly described previously in this chapter and the vacuum of QCD. This has important phenomenological

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consequences for both the η' mass and the strong CP problem. Here we present an introductory account of this topic [Pe 89].

The θ vacuum

One is used to considering the effect on gluon fields of 'small' gauge transformations, i.e., those which are connected to the identity operator in a continuous manner. There also exist 'large' gauge transformations which change the color gauge fields in a more drastic fashion. For example the gauge transformation [JaR 76] generated by

$$\Lambda_1(\mathbf{x}) \equiv \frac{\mathbf{x}^2 - d^2}{\mathbf{x}^2 + d^2} + \frac{2id\boldsymbol{\tau}\cdot\mathbf{x}}{\mathbf{x}^2 + d^2}, \tag{5.1}$$

where d is an arbitrary parameter and $\boldsymbol{\tau}$ is an $SU(2)$ Pauli matrix in any $SU(2)$ subgroup of $SU(3)$, transforms the null potential $\mathbf{A}(\mathbf{x}) = 0$ into

$$\begin{aligned}\mathbf{A}_j^{(1)}(\mathbf{x}) &= -\frac{i}{g_3}\left(\nabla_j\Lambda_1(\mathbf{x})\right)\Lambda_1^{-1}(\mathbf{x}) \\ &= -\frac{2d}{g_3\left(\mathbf{x}^2+d^2\right)^2}\left[\boldsymbol{\tau}_j(d^2-\mathbf{x}^2) + 2\mathbf{x}_j(\boldsymbol{\tau}\cdot\mathbf{x}) - 2d(\mathbf{x}\times\boldsymbol{\tau})_j\right].\end{aligned} \tag{5.2}$$

Here, we are using the matrix notation

$$\mathbf{A}_\mu \equiv A_\mu^a\frac{\boldsymbol{\lambda}^a}{2}. \tag{5.3}$$

This potential lies in an $SU(2)$ subgroup of the full color $SU(3)$ group, and is 'large' in the sense that it cannot be brought continuously into the identity. The $\boldsymbol{\tau}\cdot\mathbf{x}$ factor couples the internal color indices to the spatial position such that a path in coordinate space implies a corresponding path in the $SU(2)$ color subspace. All gauge potentials $\mathbf{A}_\mu$ carry a conserved topological charge called the *winding number*,

$$n = \frac{ig_3^3}{24\pi^2}\int d^3x\,\mathrm{Tr}\left(\mathbf{A}_i(x)\mathbf{A}_j(x)\mathbf{A}_k(x)\right)\epsilon^{ijk}. \tag{5.4}$$

As can be demonstrated by direct substitution, the gauge field of Eq. (5.2) corresponds to the value $n = 1$. Fields with any integer value of the winding number n can be obtained by repeated applications of $\Lambda_1(x)$, viz.,

$$\Lambda_n(\mathbf{x}) = [\Lambda_1(\mathbf{x})]^n. \tag{5.5}$$

All gauge potentials can be classified into disjoint sectors labeled by their winding number.

The existence of these distinct classes has interesting consequences. For example, consider a configuration of the gluon field that starts off at $t = -\infty$ as the zero potential $\mathbf{A}(\mathbf{x}) = 0$, has some interpolating $\mathbf{A}(x, t)$ for intermediate times, and ends up at $t = +\infty$ lying in the gauge-equivalent configuration $\mathbf{A}(x) = \mathbf{A}^{(1)}(x)$.[5] Then the following integral can be shown to be nonvanishing:

$$\frac{g_3^2}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \qquad \left(\tilde{F}^{a\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^a_{\alpha\beta}\right). \tag{5.6}$$

This is surprising because the integrand is a total divergence. As noted previously in Eq. (II–2.23), $F\tilde{F}$ can be written as

$$F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = \partial_\mu K^\mu, \quad K^\mu = \epsilon^{\mu\nu\lambda\sigma}\left[A^a_\nu F^a_{\lambda\sigma} + \frac{1}{3}g_3 f_{abc}A^a_\nu A^b_\lambda A^c_\sigma\right], \tag{5.7}$$

and thus the integral can be written as a surface integral at $t = \pm\infty$. For the field configuration under consideration, this reduces to the winding-number integral

$$\begin{aligned}
\frac{g_3^2}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a\mu\nu} &= \frac{g_3^2}{32\pi^2}\int d^4x\, \partial_\mu K^\mu \\
&= \frac{g_3^2}{32\pi^2}\int d^3x\, K_0\Big|_{t=-\infty}^{t=\infty} \\
&= \frac{g_3^3}{24\pi^2}i\int d^3x\, \epsilon^{ijk}\,\mathrm{Tr}\left(\mathbf{A}_i^{(1)}(x)\mathbf{A}_j^{(1)}(x)\mathbf{A}_k^{(1)}(x)\right) \\
&= 1.
\end{aligned} \tag{5.8}$$

More generally, the integral of $F\tilde{F}$ gives the *change* in the winding number

$$\frac{g_3^2}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = \frac{g_3^2}{32\pi^2}\int d^3x\, K^0\Big|_{t=-\infty}^{t=\infty} = n_+ - n_- \tag{5.9}$$

between asymptotic gauge-field configurations.

Thus, the vacuum state vector will be characterized by configurations of gluon fields, which fall into classes labeled by the winding number. Moreover, there will exist a corespondence between the gauge transformations $\{\Lambda_n\}$ and unitary operators $\{U_n\}$, which transform the state vectors. For example, a vacuum state dominated by field configurations in the zero winding class ('near' to $A_\mu = 0$) would be transformed by U_1 into configurations with a dominance of $n = 1$ configurations, or more generally,

$$U_1|n\rangle = |n+1\rangle. \tag{5.10}$$

[5] Such configurations are known to exist [Co 85].

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This implies that a gauge-invariant vacuum state requires contributions from *all* classes, such as the coherent superposition

$$|\theta\rangle = \sum_n e^{-in\theta}|n\rangle, \tag{5.11}$$

where θ is an arbitrary parameter. It follows from Eq. (5.10) that this θ*-vacuum* is gauge-invariant up to an overall phase

$$U_1|\theta\rangle = e^{i\theta}|\theta\rangle. \tag{5.12}$$

The *QCD* vacuum must contain contributions from all topological classes.

The θ term

Given this nontrivial vacuum structure, one requires three ingredients to completely specify *QCD*: (1) the *QCD* lagrangian, (2) the coupling constant (i.e. Λ_{QCD}), and (3) the vacuum label θ. How can we account for the different vacua corresponding to different choices of θ? In a path-integral representation, the $\theta = 0$ vacuum would imply generic transition elements of the form

$$_{\text{out}}\langle\theta = 0|X|\theta = 0\rangle_{\text{in}} = \int [dA_\mu][d\psi]\,[d\bar{\psi}]\; X e^{iS_{QCD}} = \sum_{n,m} {}_{\text{out}}\langle m|X|n\rangle_{\text{in}}. \tag{5.13}$$

The presence of a nonzero θ leads to an extra phase,

$$_{\text{out}}\langle\theta|X|\theta\rangle_{\text{in}} = \sum_{n,m} e^{i(m-n)\theta}\;\; {}_{\text{out}}\langle m|X|n\rangle_{\text{in}}. \tag{5.14}$$

However, this phase can be accounted for in the path integral by the addition of a new term to S_{QCD}. In particular we have, through the use of Eq. (5.9),

$$\begin{aligned} {}_{\text{out}}\langle\theta|X|\theta\rangle_{\text{in}} &= \int [dA_\mu][d\psi][d\bar{\psi}]\; X\; e^{iS_{QCD}+i\frac{g_3^2}{32\pi^2}\theta\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a\mu\nu}} \\ &= \sum_{n,m} e^{i(m-n)\theta}\;\; {}_{\text{out}}\langle m|X|n\rangle_{\text{in}}, \end{aligned} \tag{5.15}$$

where X is some operator. We see that the quantity $(m - n)$ given by the winding-number difference of the fields contributing to the path integral is equivalent to a new exponential factor containing $F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$. Thus, a correct procedure for doing calculations involving θ vacua is to follow the ordinary path-integral methods but with a *QCD* lagrangian containing the new term

$$\mathcal{L}_{QCD} = \mathcal{L}^{(\theta=0)}_{QCD} + \theta\frac{g_3^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}. \tag{5.16}$$

The parameter θ is to be considered a coupling constant. Since the operator $F\tilde{F}$ is P-odd and T-odd, a nonzero θ can induce measurable T violation. In Sect. IX–4, we shall show how to connect θ to physical observables. There is an important distinction between the various θ vacua of QCD and the many possible vacuum states of a spontaneously broken symmetry such as the Higgs sector of the electroweak theory. In the latter case, the various possible vacuum expectation values of the Higgs field label different states within the same theory. In contrast, each value of θ corresponds to a different theory, just as each value of Λ_{QCD} would label a different theory. Specifying θ and Λ_{QCD} then specifies the content of the version of QCD used by Nature.

Connection with chiral rotations

There is a connection between the axial anomaly and the presence of a θ vacuum ['tH 76a,b]. It involves the matrix element of $F\tilde{F}$ as follows. Consider the limit of N_f massless quarks. The $U(1)$ axial current

$$J_{5\mu}^{(0)} = \sum_{j=1}^{N_f} \bar{\psi}_j \gamma_\mu \gamma_5 \psi_j \tag{5.17}$$

is not conserved due to the anomaly,

$$\partial^\mu J_{5\mu}^{(0)} = \frac{N_f \alpha_s}{4\pi} F_{\mu\nu}^a \tilde{F}^{a\mu\nu}. \tag{5.18}$$

However, because of the fact that $F\tilde{F}$ is a total divergence, one can define a new conserved current

$$\tilde{J}_{5\mu} = J_{5\mu}^{(0)} - \frac{N_f \alpha_s}{4\pi} K_\mu. \tag{5.19}$$

While $\tilde{J}_{5\mu}$ does form a conserved charge,

$$\tilde{Q}_5 = \int d^3x \; \tilde{J}_{5,0}(x), \tag{5.20}$$

neither $\tilde{Q}_5$ nor $\tilde{J}_{5\mu}$ is gauge-invariant. In fact, under the gauge transformation Λ_1 of Eq. (5.1), it follows from Eq. (5.8) that the operator $\tilde{Q}_5$ changes by a c-number integer

$$U_1 \tilde{Q}_5 U_1^{-1} = \tilde{Q}_5 - 2N_f. \tag{5.21}$$

This tells us that in the world of massless quarks, the different θ-vacua are related by a chiral $U(1)$ transformation,

$$U_1 \, e^{i\alpha \tilde{Q}_5} |\theta\rangle = U_1 e^{i\alpha \tilde{Q}_5} U_1^{-1} \, U_1 |\theta\rangle = e^{i(\theta - 2N_f \alpha)} e^{i\alpha \tilde{Q}_5} |\theta\rangle, \tag{5.22}$$

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or, from Eq. (5.12),

$$e^{i\alpha\tilde{Q}_5}|\theta\rangle = |\theta - 2N_f\alpha\rangle, \tag{5.23}$$

where α is a constant. Therefore, in the limit of massless quarks, when $\tilde{Q}_5$ is a conserved quantity, all of the θ vacua are equivalent and one can transform away the θ dependence by a chiral $U(1)$ transformation. The same is not true if quarks have mass, as the mass terms in $\mathcal{L}_{QCD}$ are not invariant under a chiral transformation. We shall return to this topic in Sect. IX–4.

To summarize, one finds that the existence of topologically nontrivial gauge transformations, and of field configurations which make transitions between the different topological sectors of the theory, leads to the existence of nonvanishing effects from a new term in the *QCD* action. Chiral rotations can change the value of θ, allowing it to be rotated away if any of the quarks are massless. However, for massive quarks, the net effect is a measurable *CP*-violating term in the *QCD* lagrangian.

III–6 Baryon- and lepton-number violation in the Standard Model

An even more dramatic effect arises from an anomaly in the current for the total baryon plus lepton number $(B + L)$. Baryon number appears to be a conserved quantity when Noether's theorem is applied to the lagrangian of the Standard Model, as is total lepton number.[6] The invariances are

$$q \to e^{i\varphi_B} q, \quad \ell \to e^{i\varphi_L} \ell \tag{6.1}$$

for all quarks q and leptons ℓ. The corresponding currents involve the sum over all quarks and leptons

$$\begin{aligned} J_B^\mu &= \frac{1}{3}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d + \cdots) \\ J_L^\mu &= \bar{e}\gamma^\mu e + \bar{\nu}_{eL}\gamma^\mu \nu_{eL} + \cdots , \end{aligned} \tag{6.2}$$

where the normalization of the baryon current is chosen to give a baryon a charge of $+1$.

The baryon current is vectorial, and naively might not be expected to have an anomaly. However, the coupling of the quarks to the $SU(2)_L$ and $U(1)_Y$ gauge bosons violates parity, so that there are VVA triangle diagrams involving the baryon current with two gauge currents. For example, the triangle diagram

[6] If there are neutrino Majorana masses, lepton number will be violated. However, this is independent of the anomaly effect discussed in this section. Majorana masses will be discussed in Chap. VI.

involving the baryon current with the $U(1)_Y$ hypercharge current has a VVA triangle involving the quantum number sum

$$\mathrm{Tr}(B(Y_L + Y_R)(Y_L - Y_R)) = -2 \tag{6.3}$$

where $B = 1/3$ for quarks and $B = 0$ for leptons. These diagrams then yield an anomaly. Because the axial current of this triangle is a gauge current, any gauge-invariant regularization of the triangle diagram will place the anomaly in global baryon-number current even though it is vectorial (see the discussion surrounding Eq. (3.22)). Similar anomalies occur in the lepton number current.[7] The anomalies cancel if we take the difference of the baryon and lepton currents, with the resulting anomaly equations

$$\begin{aligned} \partial_\mu (J_B^\mu - J_L^\mu) &= 0 \\ \partial_\mu (J_B^\mu + J_L^\mu) &= \frac{3}{32\pi^2}\left(g_2^2 F_{\mu\nu}^i \tilde{F}_i^{\mu\nu} - g_1^2 B_{\mu\nu}\tilde{B}^{\mu\nu}\right). \end{aligned} \tag{6.4}$$

Here we see that, because of the anomaly, baryon number is in fact not conserved in the Standard Model, although $B - L$ is.

However, the baryon-number violation due to the anomaly is unmeasurably small at low temperature. Any transition that would change baryon number is nonperturbative in nature, as it is not seen in the usual perturbative Feynman rules. In weakly coupled field theory, such nonperturbative phenomena are suppressed in rate by a factor ['tH 76b]

$$[e^{-8\pi^2/g_2^2}]^2 \sim 10^{-160}, \tag{6.5}$$

so that such transitions are unobservable.

At high temperatures the situation is different [KuRS 85]. The classical solution mediating a transition which changes baryon number, a *sphaleron* [KlM 84], is known in the limit $\theta_w \to 0$ and the corrections due to a nonzero θ_w can be estimated. The solution has an energy around $E_{\mathrm{sph}} \sim 10$ TeV, taking into account the measured Higgs-boson mass. At high temperature, thermal effects can cause transitions with a Boltzmann factor $e^{-E_{\mathrm{sph}}/T}$, and at very high temperatures all suppressions disappear and the rate per unit volume scales with the temperature $\Gamma/V \sim T^4$.

This has an important consequence – at equilibrium in the early Universe an initial excess of baryons can disappear. More precisely, the equilibrium value of $B + L$ is zero at high temperature. However, $B - L$ is still conserved, so that an initial excess of $B - L$ will be preserved.

[7] Because possible right-handed neutrinos have no gauge couplings, their presence would not modify the anomaly.

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It is natural to ask if a sufficiently large baryon asymmetry in the Universe can be generated by out-of-equilibrium processes near the electroweak phase transition, using only Standard Model interactions. The answer appears to be negative [GaHOP 94], as the necessary *CP* violation within the Standard Model is too small and the phase transition is not strong enough. New interactions near the weak scale could provide the needed extra physics. Alternatively, the residual baryon asymmetry may arise from a net $B - L$ generated in the Universe before the electroweak epoch. Within the context of the Standard Model interactions, the simplest such possibility is *leptogenesis* involving heavy right-handed neutrinos with Majorana masses. This mechanism will be discussed in Sect. VI–6.

Problems

(1) **Currents and anomalies**

(a) Verify that all currents coupled to gauge bosons in the Standard Model are anomaly free.

(b) Find the relative strength of the anomaly coupling of the baryon number current to the $SU(2)_L$ and $U(1)_R$ gauge bosons.

(2) **Trace anomaly in *QED***

In d dimensions, the trace of the energy-momentum tensor does not vanish classically, except at $d = 4$. For example, in massless *QED* the energy-momentum tensor,

$$\theta^{\mu\nu} = -F_{\lambda}^{\mu} F^{\lambda\nu} + \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma} + \frac{i}{2} \bar{\psi} \gamma^{\mu} \overleftrightarrow{D}^{\nu} \psi,$$

has trace $\theta_{\mu}^{\mu} = \frac{d-4}{4} F^{\lambda\sigma} F_{\lambda\sigma}$. In the renormalization of the operator $F^{\lambda\sigma} F_{\lambda\sigma}$, one encounters a renormalization constant which diverges as $d \to 4$. Use this feature to calculate the *QED* trace anomaly using dimensional regularization.

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