ON CONVERGENCE OF CLOSED SETS IN A METRIC SPACE
AND DISTANCE FUNCTIONS

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Let $CL(X)$ denote the nonempty closed subsets of a metric space
X. We answer the following question: in which spaces X is
the Kuratowski convergence of a sequence $\{C_n\}$ in $CL(X)$ to a
nonempty closed set C equivalent to the pointwise convergence
of the distance functions for the sets in the sequence to the
distance function for C? We also obtain some related results
from two general convergence theorems for equicontinuous families
of real valued functions regarding the convergence of graphs and
epigraphs of functions in the family.

Let $\langle X, d \rangle$ be a metric space, and let $CL(X)$ denote the collection
of closed nonempty subsets of X. For each $C \in CL(X)$ the *distance
function* $d(\cdot, C) : X \to [0, \infty)$ for C is defined by the formula
$d(x, C) = \inf\{d(x, z) : z \in C\}$. In this note we examine the relationship
between the convergence of a sequence of such distance functions and the
convergence of the sequence of the underlying sets themselves.

DEFINITION. Let $\{C_n\}$ be a sequence of closed sets in a metric
space. Then $\operatorname{Li} C_n$ (respectively $\operatorname{Ls} C_n$) is the set of points y each
neighborhood of which meets all but finitely (respectively meets

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infinitely) many sets C_n .

Clearly $\text{Li } C_n \subseteq \text{Ls } C_n$, and both $\text{Li } C_n$ and $\text{Ls } C_n$ are closed sets (although possibly empty). We say $\{C_n\}$ Kuratowski *converges* [9] to a closed set C if $\text{Li } C_n = \text{Ls } C_n = C$; equivalently, $\text{Ls } C_n \subseteq C \subseteq \text{Li } C_n$. When X is separable and locally compact, Kuratowski convergence is actually convergence with respect to a certain metrizable topology on $\text{CL}(X)$ [8], but we will proceed more generally here. Suppose $\{C, C_1, C_2, \ldots\} \subseteq \text{CL}(X)$ and $\{d(\cdot, C_n)\}$ converges pointwise to $d(\cdot, C)$. For each x in C the condition $\lim_{n \to \infty} d(x, C_n) = 0$ implies $x \in \text{Li } C_n$, so that $C \subseteq \text{Li } C_n$. On the other hand, if $x \in \text{Ls } C_n$, there exists a sequence $\{x_k\}$ convergent to x and an increasing sequence of integers $\{n_k\}$ such that, for each k , $x_k \in C_{n_k}$. We have $\lim_{k \to \infty} d(x, C_{n_k}) = 0$, and it follows from the pointwise convergence of the distance functions that $x \in C$, that is, $\text{Ls } C_n \subseteq C$. Thus, pointwise convergence of $\{d(\cdot, C_n)\}$ to $d(\cdot, C)$ forces Kuratowski convergence of $\{C_n\}$ to C . The converse obviously fails.

EXAMPLE 1. Let $X = (0, 2)$ as a subspace of the line, and let $C_n = (0, 1] \cup \{2-(1/n)\}$. Clearly $\{C_n\}$ Kuratowski converges to $(0, 1]$, whereas $\lim_{n \to \infty} d(7/4, C_n) = 1/4 < 3/4 = d(7/4, (0, 1])$.

What conditions on X ensure the equivalence of the two modes of convergence?

THEOREM 1. *Let $\langle X, d \rangle$ be a metric space. The following are equivalent:*

(1) *whenever $\{C_n\}$ is a sequence in $\text{CL}(X)$ Kuratowski convergent to a closed nonempty set C , then $\{d(\cdot, C_n)\}$ converges pointwise to $d(\cdot, C)$;*

(2) *for each p in X , whenever $\{x_n\}$ is a sequence in X*

> *with no cluster point, then, for each x in X ,*
> $d(p, x) \le \lim \inf d(p, x_n)$.

Proof. $(1) \to (2)$. Suppose (2) fails. Choose points p and x in X and a sequence $\{x_n\}$ in X with no cluster point for which $\lim \inf d(p, x_n) < d(p, x)$. By passing to a subsequence we may assume that for some $\epsilon > 0$ and each n , $d(p, x_n) < d(p, x) - \epsilon$. For each n let $C_n = \{x, x_n\}$. Clearly Li C_n = Ls $C_n = \{x\}$. However, with $C = \{x\}$,

$$\lim \sup d(p, C_n) = \lim \sup d(p, x_n)$$
$$\le d(p, x) - \epsilon$$
$$= d(p, C) - \epsilon .$$

Thus $\{d(\cdot, C_n)\}$ fails to converge pointwise to $d(\cdot, C)$.

$(2) \to (1)$. Let $\{C_n\}$ be a sequence in $CL(X)$ Kuratowski convergent to $C \ne \emptyset$. Fix $p \in X$. Since $C \subset$ Li C_n , it is evident that $\lim \sup d(p, C_n) \le d(p, C)$ with no assumptions on X . It remains to show $d(p, C) \le \lim \inf d(p, C_n)$. For each n choose $x_n \in C_n$ for which $d(p, x_n) < d(p, C_n) + 1/n$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which $\lim_{k \to \infty} d(p, x_{n_k}) = \lim \inf d(p, C_n)$. If $\{x_{n_k}\}$ has a cluster point x , then $x \in$ Ls $C_n = C$, and by the continuity of d , $d(p, C) \le d(p, x) = \lim \inf d(p, C_n)$. Otherwise, for each x in X , $d(p, x) \le \lim \inf d(p, x_n) = \lim \inf d(p, C_n)$. In particular, this is true for each x in C , whence $d(p, C) \le \lim \inf d(p, C_n)$.

It is easy to check that if X satisfies condition (2) above, then X must be both locally compact and complete. Also, compactness of each closed and bounded subset of X guarantees (2): each sequence $\{x_n\}$ in X without a cluster point eventually lies outside of each ball, whence, for each p in X , $\lim \inf d(x_n, p) = \infty$. The class of spaces for which (2) holds also includes the zero-one metric spaces and others.

   EXAMPLE 2. For each $j \in Z^+$ let e_j be the sequence that takes the value one at j and is zero otherwise. Let X be the metric subspace of l^∞ defined by $X = \{e_j : j \in Z^+\} \cup \{(1/j)e_j : j \in Z^+\} \cup \{0\}$. For each sequence $\{x_n\}$ in X without a cluster point and for each p in X , we have $d(p, x_n) = 1$ eventually. Since the diameter of X is one, for each x in X we have $d(p, x) \leq 1 = \lim \inf d(p, x_n)$. If we replace $\{e_j : j \in Z^+\}$ in X by $\{(j/(j+1))e_j : j \in Z^+\}$, the space still satisfies condition (2); however, if we replace it by $\{((j+1)/j)e_j : j \in Z^+\}$, it no longer does.

   Does the pointwise convergence of the distance functions corresponding to a sequence of closed sets bear any relationship to the Kuratowski convergence of the functions, *viewed as sets themselves*? Most naturally, we can identify a real valued function f on X with its graph in $X \times R$. With this identification, the Kuratowski convergence of a sequence of continuous functions $\{f_n\}$ to a continuous function f can be expressed as follows:

   (1) for each x in X there exists $\{x_n\}$ convergent to x
        for which $f(x) = \lim\limits_{n \to \infty} f_n(x_n)$, and

   (2) whenever $\{(x_k, f_{n_k}(x_k))\}$ converges to (x, y) , then
        $y = f(x)$.

Indeed, condition (1) is equivalent to $f \subset \text{Li } f_n$, whereas (2) is equivalent to Ls $f_n \subset f$. However, this is not the only identification that can be made, and from the point of view of applications, it is often not the most useful (see, for example, [1] or [11]). As an alternative, we can associate a continuous f with its *epigraph*, the closed set defined by

$$\text{epi } f = \{(x, \alpha) : \alpha \geq f(x)\} .$$

It is an easy exercise to show that Kuratowski convergence of epigraphs is equivalent to the following pair of conditions:

condition (1) above, and

(2) for each x in X , whenever $\{x_n\} \to x$, then

$f(x) \leq \lim \inf f_n(x_n)$.

Of course, continuous functions are not the only ones with closed
epigraphs; this larger class of functions is precisely the class of lower
semicontinuous functions on X [2]. The study of Kuratowski convergence
of epigraphs originated with Wijsman [19], and has its roots in convex
analysis. This mode of convergence is now usually called *epiconvergence* in
the literature (see, for example, [10] or [15]).

The distance functions corresponding to a collection of closed sets in
a metric space are each Lipschitz with a common Lipschitz constant.
Familiar weaker requirements are next described.

DEFINITION. A collection Ω of real valued functions on a metric
space X is called *pointwise equicontinuous* if for each p in X and
$\varepsilon > 0$ there exists $\delta > 0$, depending on ε and p , such that whenever
$d(x, p) < \delta$ then $|f(x)-f(p)| < \varepsilon$ for all $f \in \Omega$. If the same δ can
be chosen for each p in X , we call Ω *equicontinuous*.

The relationship between pointwise convergence of distance functions
and the convergence of distance functions as sets in $X \times R$ falls out of
the next theorem.

THEOREM 2. *Let $\{f_n\}$ be a pointwise equicontinuous sequence of real
valued continuous functions on a metric space X , and let $f : X \to R$ be
continuous. The following are equivalent:*

(1) *whenever $\{x_n\}$ is a sequence in X convergent to x , then*

$\lim_{n \to \infty} f_n(x_n) = f(x)$;

(2) $\{f_n\}$ *converges to f uniformly on compact subsets of X ;*

(3) $\{f_n\}$ *converges pointwise to f ;*

(4) $\{f_n\}$ *Kuratowski converges to f ;*

(5) $\{f_n\}$ *epiconverges to f .*

Proof. The equivalence of (1) (usually called *continuous convergence*) and (2) is well known [7], and each condition alone implies pointwise equicontinuity so that this assumption is superfluous here. In view of our local description of (4) and (5), it is evident that (1) implies (3), (4), and (5), again, without an equicontinuity assumption. To see that each of these implies continuous convergence, fix x in X and let $\{x_n\}$ be a sequence convergent to x. If any of the last three conditions hold, there exists $\{w_n\}$ convergent to x for which $f(x) = \lim_{n \to \infty} f_n(w_n)$.

Choose $\delta > 0$ such that if $d(x, z) < \delta$, then, for each n, $|f_n(x) - f_n(z)| < \varepsilon/3$. For all n sufficiently large, we can guarantee that $|f_n(w_n) - f(x)| < \varepsilon/3$, $d(x_n, x) < \delta$, and $d(w_n, x) < \delta$, whence $|f_n(x_n) - f(x)| < \varepsilon$.

With no assumptions on the sequence of functions, none of (3), (4), or (5) need imply either of the other two, even if X is compact. It is easy to modify the standard construction [13] of a sequence of nonnegative characteristic functions on $[0, 1]$ that converges in measure to the zero function but nowhere converges pointwise to yield a sequence of piecewise linear (continuous) functions with the same property; this sequence will satisfy (5) but neither (3) nor (4). If $f_n : [0, 1] \to R$ has as its graph the line segments joining $(0, 0)$ to $(1/2n, -1)$, $(1/2n, -1)$ to $(1/n, 0)$, and $(1/n, 0)$ to $(1, 0)$, then $\{f_n\}$ satisfies (3) but neither (4) nor (5). To see that (4) may hold without either (3) or (5), let $X = \{0\} \cup \{1/k : k \in Z^+\}$ as a subspace of the line, and for each $n \in Z^+$ define $f_n : X \to R$ to be $-n$ times the characteristic function of $\{0\} \cup \{1/k : k \geq n\}$. Note that the only point at which $\{f_n\}$ fails to converge is $x = 0$. However, one can produce a sequence $\{f_n\}$ on the Cantor set that satisfies (4) but converges nowhere pointwise [5]. One might guess that if X were connected, then (4) would force (3) and/or (5), but Example 1 of [5] shows this need not occur. What is needed is local connectedness. With this assumption on X, (4) not only yields (3) and (5), it also buys (2), and thus (1) [5]. On the other hand, if X is compact *and* connected, (4) will also buy (1) [4]. Finally, (3) and (5) are equivalent with just a pointwise equi-lower semicontinuity assumption [15].

A stronger type of convergence for closed sets is convergence with respect to Hausdorff distance. We pause for a moment to describe this notion. For each point x in X let $B_\varepsilon[x]$ denote the closed ball of radius ε with center x , and if $C \in CL(X)$, denote $\bigcup\limits_{x \in C} B_\varepsilon[x]$ by $B_\varepsilon[C]$. If K is another nonempty closed set, then the *Hausdorff distance* H_d between C and K is given by

$$H_d(C,\ K) = \inf\{\varepsilon \ :\ B_\varepsilon[C] \supset K \text{ and } B_\varepsilon[K] \supset C\} \ .$$

The distance H_d defines an infinite valued metric on $CL(X)$. If d' is another metric on X which defines the same uniformity (in other words, the identity map from $\langle X,\ d \rangle$ to $\langle X,\ d' \rangle$ is bi-uniformly continuous), then H_d and $H_{d'}$ are equivalent on $CL(X)$. Hausdorff metric convergence always forces Kuratowski convergence in $CL(X)$; the converse holds precisely in those metric spaces that are compact. For example, if $C_n = [0,\ 1] \cup \{n\}$, then $\{C_n\}$ is a Kuratowski convergent sequence in the line which fails to converge in the Hausdorff metric. More facts about Hausdorff distance can be found in [2] and in [6].

Now let $C(X,\ R)$ denote the continuous real valued functions on X . Equip $X \times R$ with the *box metric*, defined by $\rho[(x_1,\ \alpha_1),\ (x_2,\ \alpha_2)] = \max\{d(x_1,\ x_2),\ |\alpha_1 - \alpha_2|\}$. We consider three infinite valued metrics on the function space $C(X,\ R)$:

$$D_1(f,\ g) = \sup\{|f(x) - g(x)| \ :\ x \in X\} \ ;$$

$$D_2(f,\ g) = H_\rho(f,\ g) \ ;$$

$$D_3(f,\ g) = H_\rho(\text{epi } f, \text{ epi } g) \ .$$

With no assumptions on X , D_2 convergence in $C(X,\ R)$ ensures pointwise convergence, whereas D_3 convergence does not, even if $X = [0,\ 1]$. Note that metrics equivalent to D_2 and D_3 result if we define distance in $X \times R$ to be $\left[d(x_1,\ x_2)^2 + (\alpha_1 - \alpha_2)^2\right]^{\frac{1}{2}}$, and so on. It is well known [2] that the mapping $C \to d(\cdot,\ C)$ is an isometry from $\langle CL(X), H_d \rangle$ into

$\langle C(X, R), D_1 \rangle$. Thus Hausdorff metric convergence of a sequence of closed sets in X is equivalent to the uniform convergence of their distance functions.

THEOREM 3. *Let Ω be an equicontinuous family of real functions on a metric space $\langle X, d \rangle$. Then D_1, D_2 and D_3 are equivalent as infinite valued metrics on Ω .*

Proof. We first show that D_1 is at least as strong as D_3 , in fact, that $D_3(f, g) \leq D_1(f, g)$. Suppose $D_1(f, g) = \lambda$. Let (x, α) be an arbitrary point of epi f and set $\beta = \alpha - f(x)$. Then $(x, g(x)+\beta) \in$ epi g , and $\rho[(x, g(x)+\beta), (x, \alpha)] = |g(x)-f(x)| \leq \lambda$. This shows epi $f \subset B_\lambda[\text{epi } g]$. Similarly, epi $g \subset B_\lambda[\text{epi } f]$.

We next show that D_3 is at least as strong as D_2 . Let ϵ be positive; we produce $\lambda > 0$ such that for all f, g in Ω , $D_3(f, g) < \lambda$ implies $D_2(f, g) \leq \epsilon$. Choose $\delta > 0$ such that for each $h \in \Omega$ whenever $d(w, z) < \delta$ then $|h(w)-h(z)| < \epsilon/3$, and set $\lambda = (1/2)\min\{\delta, \epsilon/3\}$. Suppose $D_3(f, g) < \lambda$ but $D_2(f, g) > \epsilon$. Without loss of generality we can assume there exists x for which $(x, f(x)) \notin B_\epsilon[g]$. Choose $(z, \alpha) \in$ epi g for which $\rho[(z, \alpha), (x, f(x))] < \lambda$. We must have $g(z) < f(x) - \epsilon$, or else $\rho[(z, g(z)), (x, f(x))] \leq \epsilon$. Since $\lambda < \delta$, we have $g(x) < (f(x)-\epsilon) + \epsilon/3 = f(x) - (2/3)\epsilon$. For the same reason whenever $d(x, w) \leq \lambda$, we have $f(w) > f(x) - \epsilon/3$. Hence, if $d(x, w) \leq \lambda$ and $(w, \theta) \in$ epi f , we obtain

$$\rho[(x, g(x)), (w, \theta)] = \max\{d(x, w), \theta-g(x)\}$$
$$\geq \max\{d(x, w), \epsilon/3\}$$
$$= \epsilon/3 > \lambda .$$

We have shown $(x, g(x)) \notin B_\lambda[\text{epi } f]$, in violation of $D_3(f, g) < \lambda$.

Finally we show that D_2 is at least as strong as D_1 . Let ϵ be positive, and choose $\delta > 0$ such that for each h in Ω , whenever $d(z, w) < \delta$ then $|h(z)-h(w)| < \epsilon/2$. Set $\lambda = (1/2)\min\{\delta, \epsilon\}$. Suppose $D_2(f, g) < \lambda$ and $x \in X$ is arbitrary. There exists $w \in X$ for which

both $d(x, w) < \lambda$ and $\left| f(x)-g(w) \right| < \lambda$.   Since $\lambda < \delta$ we get
$\left| g(x)-g(w) \right| < \varepsilon/2$, whence $\left| f(x)-g(x) \right| < \lambda + \varepsilon/2 \le \varepsilon$.   As a result,
$D_1(f, g) \le \varepsilon$.

   COROLLARY.   *Let $\langle X, d \rangle$ be a metric space, and let*
$\{C, C_1, C_2, \ldots\} \subset \mathrm{CL}(X)$.   *Let ρ be a metric on $X \times R$ compatible with*
the product uniformity.   The following are equivalent:

   (1) $\{C_n\}$ *converges to C with respect to H_d ;*

   (2) $\{d(\cdot, C_n)\}$ *converges to $\{d(\cdot, C)\}$ with respect to H_ρ ;*

   (3) $\{\mathrm{epi}\, d(\cdot, C_n)\}$ *converges to $\{\mathrm{epi}\, d(\cdot, C)\}$ with respect*
         to H_ρ .

   Proof.   The family $\{d(\cdot, C_n) : n \in Z^+\} \cup \{d(\cdot, C)\}$ is equi-
continuous, and the box metric is compatible with the product uniformity.
Any other such metric, by definition, is uniformly equivalent to the box
metric, and thus determines the same Hausdorff metric topology on
$\mathrm{CL}(X \times R)$.

   We note that the proof of Theorem 3 shows that D_1, D_2 , and D_3 are
uniformly equivalent metrics on each equicontinuous family Ω .   With no
assumptions on Ω , D_1 is at least as strong as D_2 , and D_2 is at
least as strong as D_3 .   If Ω is merely pointwise equicontinuous, D_2
may be weaker than D_1 , and D_3 may be weaker than D_2 .

   EXAMPLE   3.   Let $X = \{n+(1/n) : n = 2, 3, 4, \ldots\} \cup Z^+$, as a sub-
space of the line.   Let $f = \chi_{Z^+}$.   If, for each $n > 1$, $f_n : X \to R$ is
defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x = n + 1/n \ , \\ 0 & \text{if } x = n \ , \\ f(x) & \text{otherwise,} \end{cases}$$

then $D_2(f_n, f) = 1/n$.   However, for each n , $D_1(f_n, f) = 1$.   Instead,
if g is the zero function on X and for each $n > 1$, $g_n = \chi_{\{n\}}$, then

$D_3(g_n, g) = 1/n$ whereas $D_2(g_n, g) = 1$. Clearly $\{f, f_2, f_3, \ldots\}$ and $\{g, g_2, g_3, \ldots\}$ are both pointwise equicontinuous families.

If X is compact, then the proof of $(2) \to (1)$ in Theorem 3 can be adapted to show that D_2-convergence in $C(X, R)$ forces D_1-convergence. Thus, in this case, D_1 and D_2 are equivalent (but not uniformly equivalent) finite valued metrics [12]. In fact, if X has finitely many components, the equivalence of D_1 and D_2 on $C(X, R)$ characterizes compactness for the domain [4]. A comprehensive study of the approximation of functions with respect to D_2 has been underway for over twenty years by Sendov, Popov, Veselinov and their associates in Sofia (see, for example, [16] or [17]), whereas this author has produced analogs of some of the fundamental approximation theorems of Stone [18] with respect to D_3 (see, for example, [3]).

In closing we mention that the convergence of sequences of closed convex sets in a normed linear space and their associated support functionals has been particularly well studied (see [14] and the references therein).

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