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RESEARCH ARTICLE

On the Jones polynomial modulo primes

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Abstract
We derive an upper bound on the density of Jones polynomials of knots modulo a prime number p, within a sufficiently large degree range: $4/p^7$. As an application, we classify knot Jones polynomials modulo two of span up to eight.

1. Introduction

Four decades after the discovery of the Jones polynomial, we have an impressive list of applications, for example, the solution of the famous Tait conjectures on alternating knots, the existence of quantum invariants of knots and manifolds and also new problems concerning the Jones polynomial itself. In particular, it is an open question whether the Jones polynomial detects the trivial knot, and it is unknown which Laurent polynomials $g(t) \in \mathbb{Z}[t^{\pm 1}]$ are realised as Jones polynomials of knots. These two questions set the Jones polynomial far apart from the Alexander polynomial, where the corresponding answers are known. In this note, we take a tiny step towards classifying Jones polynomials of knots with coefficients reduced modulo a prime number p.

Theorem 1. *For all $a, b \in \mathbb{Z}$ with $b - a \geq 7$, the set of Laurent polynomials $g(t) \in \mathbb{F}_p[t^{\pm 1}]$ with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ within the degree range from a to b, that are realised as Jones polynomials of knots, has density at most $4/p^7$.*

As we will see, the bound $4/p^7$ is sharp in the special case $p = 2$, any $a \in \mathbb{Z}$, and $b = a + 8$. In this degree range, there are 2^9 potential Laurent polynomials with coefficients modulo 2, of which $16 = 4/2^7 \cdot 2^9$ are realised.

Corollary 1. *For all $a \in \mathbb{Z}$, there are exactly 16 Jones polynomials of knots modulo two with minimal degree $\geq a$ and maximal degree $\leq a + 8$. All these Laurent polynomials are realised by finite connected sums of 54 prime knots with crossing number 12 or less.*

As Jones observed in his famous publication [4], for any knot K, the difference between the Jones polynomial $V_K(t)$ and 1 is divisible by $(t^3 - 1)(t - 1)$. The proof of Theorem 1 rests on the following refined statement, which does not seem to appear in the literature so far.

Theorem 2. *Let $h(t) = (t^3 - 1)(t - 1)(t^2 + 1)$ and*

$$f(t) = (t^2 - t + 1)h(t) = t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1.$$

For all knots K, there exists a unique polynomial $p(t) \in \mathbb{Z}[t]$ of degree at most seven, belonging to one of the four families below, so that $V_K(t) - p(t)$ is divisible by $f(t)$:

(i) $1 + nh(t)$,
(ii) $V_{3_1}(t) + nh(t)(2t - 1)$,
(iii) $V_{5_1}(t) + nh(t)$,
(iv) $V_{8_{21}}(t) + nh(t)(2t - 1)$.

All these families are parametrised by an integer n satisfying $2n = \pm 1 \pm 3^l$. The symbols $3_1, 5_1, 8_{21}$ refer to knots according to Rolfsen's notation [7].

The membership of a given knot K to one of these families, as well as the value $n \in \mathbb{Z}$, is determined by the pair of values $V_K(i)$, $V_K(\zeta_6)$. The explicit Jones polynomials appearing in Theorem 2 are

$$V_{3_1}(t) = -t^4 + t^3 + t,$$

$$V_{5_1}(t) = -t^7 + t^6 - t^5 + t^4 + t^2,$$

$$V_{8_{21}}(t) = t^7 - 2t^6 + 2t^5 - 3t^4 + 3t^3 - 2t^2 + 2t.$$

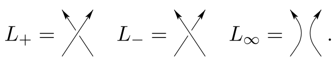
At this point, the reader might already guess that the first theorem is an easy consequence of the second. We will derive Theorems 1 and 2 in Sections 3 and 2, respectively. The corollary relies on the following curious fact: there exist knots – for example the knot $12n237$ in knotinfo notation [5] – whose Jones polynomial is t^{12}, modulo two. This is explained in the fourth and last sections.

2. Listing potential Jones polynomials

The Jones polynomial $V_K(t) \in \mathbb{Z}[t^{\pm 1}]$ of links $K \subset S^3$ admits an elegant recursive definition by a skein relation, which we only paraphrase here, since we will not make explicit use of it:

$$t^{-1}\, V(L_+) - t\, V(L_-) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V(L_0).$$

As usual, the notation L_+, L_-, L_∞ refers to diagrams of links that look locally as follows:



When restricted to knots K, rather than links, the Jones polynomial satisfies the following restrictions in the roots of unity $1, i, \zeta_3, \zeta_6$:

1. $V_K(1) = 1$,
2. $V_K'(1) = 0$,
3. $V_K(\zeta_3) = 1$,
4. $V_K(i) = \pm 1$,
5. $V_K(\zeta_6) = \pm(\sqrt{-3})^m$.

The first four conditions were already derived by Jones [4]; the fifth one by Przytycki [6]. In fact, these restrictions admit generalisations to links, giving rise to similar divisibility results for the Jones polynomial of links; we will not discuss these here. The exponent m in condition (5) coincides with the rank of the first homology of the double branched cover $M_2(K)$ with coefficients in $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$, and can also be interpreted as the dimension of the 3-colouring invariant of K, as described in [6]. The sign in condition (4) is determined by the Arf invariant of K: $V_K(i) = (-1)^{\mathrm{Arf}(K)}$. In terms of Vassiliev invariants, the first two conditions reflect the fact that knots admit no non-constant finite type invariants of order zero and one [1]. Interestingly, this implies that no monomial other than 1 is the Jones polynomial of a knot [3]. Here is a remarkable consequence of the first three conditions together: $V_K(t) - 1$ is divisible by $(t-1)^2(t^2+t+1) = (t^3-1)(t-1)$.

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Even better, suppose $p(t) \in \mathbb{Z}[t^{\pm 1}]$ admits the same values as $V_K(t)$, for $t = 1, i, \zeta_3, \zeta_6$, and satisfies $p'(1) = 0$. Then the difference $V_K(t) - p(t)$ is divisible by $(t-1)^2$ times the product of the minimal polynomials of i, ζ_3, ζ_6:

$$\begin{aligned} f(t) &= (t-1)^2(t^2+1)(t^2+t+1)(t^2-t+1) \\ &= t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1. \end{aligned}$$

Therefore, all we need in order to derive Theorem 2 is finding a suitable set of reference polynomials $p(t)$, with $p'(1) = 0$, covering all the possible values of knot Jones polynomials at $t = 1, i, \zeta_3, \zeta_6$. This is easy enough.

Proof of Theorem 2. First, we observe that all the four families of polynomials listed in Theorem 2 satisfy $p(1) = 1$, $p'(1) = 0$ and $p(\zeta_3) = 1$. Here, we use the fact that $h(t) = (t^3 - 1)(t - 1)(t^2 + 1)$ has a double root at $t = 1$ and a single root at $t = \zeta_3$.

Next, we observe that all the polynomials of families (i) and (iv) listed in Theorem 2 satisfy $p(i) = 1$, and all the polynomials of families (ii) and (iii) satisfy $p(i) = -1$. Here, we use that $h(t)$ also has a single root at $t = i$.

Last, we take care of the value $p(\zeta_6)$, which should cover all the complex numbers of the form $\pm(\sqrt{-3})^m$. The values of

$$p(t) = 1, V_{3_1}(t), V_{5_1}(t), V_{8_{21}}(t)$$

at $t = \zeta_6$ are $1, \sqrt{3}i, -1, \sqrt{3}i$, respectively. Furthermore, we have $h(\zeta_6) = 2$ and $h(\zeta_6)(2\zeta_6 - 1) = 2\sqrt{3}i$. This implies that the polynomials of families (i) and (iii) cover all the odd integers at $t = \zeta_6$, while the polynomials of families (ii) and (iv) cover all the odd multiples of $\sqrt{3}i$ at $t = \zeta_6$. Altogether, the four families listed in Theorem 2 cover all the possible combinations of values of knot Jones polynomial at $t = 1, i, \zeta_3, \zeta_6$, including the double root at $t = 1$. This finishes the proof of Theorem 2. □

3. Jones polynomial modulo primes

The goal of this section is to derive Theorem 1 by reducing Theorem 2 modulo a fixed prime number p. We use the notation $\bar{f}(t) \in \mathbb{F}_p[t^{\pm 1}]$ for the reduction of $f(t) \in \mathbb{Z}[t^{\pm 1}]$ modulo p. Theorem 2 remains valid modulo p, with the additional feature that the parameter n is in $\mathbb{F}_p$. From this, we deduce that the number of Jones polynomials of knots modulo p in the degree range $[0, 7]$ is at most $4p$. This is in accordance with the ratio $4/p^7$, since there are exactly p^8 polynomials modulo p in the degree range $[0, 7]$. We will refer to these $4p$ potential Jones polynomials as reference polynomials $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{4p} \in \mathbb{F}_p[t^{\pm 1}]$.

Proof of Theorem 1. Suppose we are given a degree range $[a, b]$ with $b - a \geq 7$ and a knot K with Jones polynomial $\bar{V}_K(t)$ in that degree range. By Theorem 2, there exists a reference polynomial $\bar{f}_i$, so that $\bar{V}_K(t) - \bar{f}_i$ is divisible by

$$\bar{f}(t) = t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1 \in \mathbb{F}_p[t^{\pm 1}].$$

Denote the minimal and maximal degree of $\bar{V}_K(t) - \bar{f}_i$ by α and β, respectively. Then there exist unique coefficients

$$c_\alpha, c_{\alpha+1}, \ldots, c_{\beta-8} \in \mathbb{F}_p,$$

satisfying the following equation:

$$\bar{V}_K(t) - \bar{f}_i = \bar{f}(t)\left(c_\alpha t^\alpha + c_{\alpha+1}t^{\alpha+1} + \cdots + c_{\beta-8}t^{\beta-8}\right).$$

The polynomial $\bar{V}_K(t)$ is therefore determined by $\beta - \alpha - 7$ parameters in $\mathbb{F}_p$. However, since $\bar{V}_K(t)$ is in the degree range $[a, b]$, all the coefficients c_γ with $\gamma \notin [a, b-8]$ are determined by $\bar{f}_i$ alone. In other words, only the coefficients $c_a, c_{a+1}, \ldots, c_{b-8}$ change if we vary $\bar{V}_K(t)$ in the given degree range. Since there are $4p$ reference polynomials $\bar{f}_i$, this allows for a maximum of $4p$ times p^{b-a-7} potential Jones

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polynomials, out of a total of p^{b-a+1} polynomials with coefficients in $\mathbb{F}_p$ in the degree range $[a, b]$. The resulting ratio is again $4/p^7$, as claimed. □

For odd primes $p \geq 5$, the bound $4/p^7$ is never sharp, since the parameter n appearing in Theorem 2, case (i), satisfies $1 + 2n = \pm 3^l$. In particular, $2n$ cannot be $-1 \pmod p$, since 3^l cannot be zero $(\mathrm{mod}\, p)$. The knot table at our disposition (knotinfo, up to 12 crossings [5]) is too small to draw any conclusion about the sharpness of the bound $4/p^7$ for $p = 3$. This leaves us with the case $p = 2$, which is most interesting and deserves its own section.

4. Jones polynomial modulo two

The list of $4p$ potential Jones polynomials in the degree range $[0, 7]$, called reference polynomials in the previous section, boils down to eight polynomials for $p = 2$. These are in fact realised by the following knots: the trivial knot O, 3_1, 5_1, 5_2, 8_{21}, 9_{43}, 10_{140}, 10_{160}. The corresponding Jones polynomials (mod 2) are

$$1,\ t + t^3 + t^4,\ t^2 + t^4 + t^5 + t^6 + t^7,\ t + t^2 + t^4 + t^5 + t^6,$$

$$t^3 + t^4 + t^7,\ 1 + t + t^7,\ 1 + t + t^2 + t^3 + t^5 + t^6 + t^7,\ 1 + t^2 + t^3 + t^5 + t^6.$$

In order to prove Corollary 1, we need to find 16 knot Jones polynomials in the degree range $[a, a + 8]$, for all $a \in \mathbb{Z}$, which appears rather difficult. Luckily, a single knot comes at our rescue: $12n237$.

As mentioned above, no monomial other than 1 is the Jones polynomial of a knot. Indeed, no polynomial of the form $p(t) = at^n$, except 1, satisfies $p(1) = 1$ and $p'(1) = 0$. In contrast, the Jones polynomial of the knot $12n237$ is a non-trivial monomial modulo 2:

$$\bar{V}_{12n237}(t) = t^{12} \pmod 2.$$

Remark 1. *The connected sum of the knot* $12n237$ *with its mirror image has trivial Jones polynomial modulo* 2*. The existence of non-trivial knots with that property, even prime ones, was known before* [2]*. Likewise, for odd primes p, the monomial* t^{12p} *is a potential Jones polynomial modulo p, since* $t^{12p} - 1$ *is divisible by* $f(t) = (t^2 - t + 1)(t^3 - 1)(t - 1)(t^2 + 1)$ *in* $\mathbb{F}_p[t^{\pm 1}]$*. We do not know whether* t^{12p} *(modulo p) is the Jones polynomial of an actual knot.*

Proof of Corollary 1. Using the knot $12n237$ with Jones polynomial t^{12} modulo 2, we can reduce the realisation problem to finitely many values of $a \in \mathbb{Z}$. Indeed, suppose we find 16 Jones polynomials in a fixed degree range $[a, a + 8]$, realised by the knots $K_1, K_2, \ldots, K_{16}$. Then, by adding k copies of the knot $12n237$ to the knots K_i, we obtain 16 Jones polynomials in the degree range $[a + 12k, a + 12k + 8]$. This also works for negative integers k, by adding $|k|$ copies of the mirror image of the knot $12n237$ to the K_i. Hence, in order to cover all degree ranges, it is sufficient to consider the cases $-9 \leq a \leq 2$. In fact, it is even enough to consider the cases $-4 \leq a \leq 2$, by the symmetry $V_K(t) = V_{K^*}(t^{-1})$ between the Jones polynomial of a knot K and its mirror image K^*. Based on Rolfsen's table [7] and knotinfo [5], we found 53 prime knots, plus the trivial knot O, which provide 16 Jones polynomials in all degree ranges of the form $[a, a + 8]$, $a \in \{-4, -3, -2, -1, 0, 1, 2\}$. These knots include all knots with crossing number ≤ 8, except the knots 8_9, 8_{13}, 8_{16}, 8_{18} (whose Jones polynomials modulo 2 coincide with the ones of $4_1\#4_1$, 8_4, 8_{10}, 8_{12}, in this order), as well as the following knots:

$$9_{42}, 9_{43}, 9_{44}, 10_{124}, 10_{126}, 10_{127}, 10_{128}, 10_{133}, 10_{136}, 10_{140}, 10_{143}, 10_{145},$$

$$10_{146}, 10_{147}, 10_{160}, 10_{163}, 10_{165}, 11n63, 11n71, 11n99, 11n118, 11n173.$$

The table below indicates the degree range of their corresponding Jones polynomials modulo 2. Our convention here is chosen so that K has higher maximal degree than K^*. By taking suitable connected

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Table 1. *Knots with Jones polynomials of span ≤8*

Degree range	Knots
$[-4, 4]$	O, 3_1, 3_1^*, 4_1, 6_1, 6_1^*, 6_3, 7_7, 7_7^*, $4_1\#4_1$, 8_3, 8_{12}, 8_{17}, 9_{42}, 10_{136}, 10_{136}^*
$[-3, 5]$	O, 3_1, 4_1, 6_1, 6_2, 6_3, 7_7, 8_4, 8_8, 8_{20}, 9_{42}, 9_{44}, 10_{136}, 10_{146}, 10_{147}, 10_{163}
$[-2, 6]$	O, 3_1, 4_1, 5_2, 6_1, 6_2, $3_1\#4_1$, 7_6, $3_1^*\#5_1$, 8_1, 8_7, 8_{10}, 8_{20}, 9_{44}, 10_{160}, 10_{163}
$[-1, 7]$	O, 3_1, 5_1, 5_2, 6_2, $3_1\#4_1$, 7_6, 8_6, 8_{11}, 8_{14}, 8_{20}, 8_{21}, 9_{43}, 10_{140}, 10_{160}, $11n173$
$[0, 8]$	O, 3_1, 5_1, 5_2, $3_1\#3_1$, 7_2, 7_4, 8_2, 8_5, 8_{19}, 8_{21}, 9_{43}, 10_{126}, 10_{140}, 10_{143}, 10_{160}
$[1, 9]$	3_1, 5_1, 5_2, $3_1\#3_1$, 7_2, 7_3, 7_4, 7_5, 8_{19}, 8_{21}, 10_{133}, 10_{165}, $11n77$, $11n99$, $11n118$, $4_1\#8_{21}$
$[2, 10]$	5_1, $3_1\#3_1$, 7_1, 7_3, 7_5, $3_1\#5_2$, 8_{15}, 8_{19}, 8_{21}, 10_{124}, 10_{127}, 10_{128}, 10_{145}, 10_{165}, $11n63$, $11n118$

sums of these knots, together with the knot $12n237$ (making it a total of 54 prime knots), and all their mirror images, we find 16 Jones polynomials in every degree range of the form $[a, a+8]$, as stated in Corollary 1. □

Remark 2. *We do not know to what extent the statement of Corollary 1 can be generalised. For example, we found* 64 *knot Jones polynomials modulo two in the degree ranges* $[-5, 5]$ *and* $[0, 10]$*, all realised by knots with* 12 *or fewer crossings.*

We invite the reader to answer the following concluding questions.

Question 1. *Let p be an odd prime. Is there a knot $K \subset S^3$ with*

$$\bar{V}_K(t) = t^{12p} \pmod{p}?$$

Question 2. *Does every degree range $[a, b]$ with $b - a \geq 7$ contain 2^{b-a-4} Jones polynomials modulo 2, as predicted by Theorem 1?*

Question 3. *Is every Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ satisfying conditions (1)–(5) the Jones polynomial of a knot?*

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