witty asides and footnotes and much wise advice. For example (page 54), on the definition $gf(x) = g\big(f(x)\big)$, "This is the Big-endian definition. I regret to inform you that there is also a Little-endian definition where fg is computed as $\big((x)f\big)g$. I cannot stop this sort of thing, but at least I can warn you of the hazards ahead"; (page 183, on Euler's formula), "Some authors are so worried by not being able to prove the above results rigorously at this stage that they simply *define* $e^{i\theta} = \cos\theta + i\sin\theta$. One wonders how such people can sleep at night"; and (page 204), "The sequence—complex numbers, real numbers, rational numbers, integers and natural numbers—represents a progressive simplification in our notion of number and a progressive complication in the algebra that results.".

Algebra and geometry successfully meets its aims. It has a reassuringly large overlap with familiar ideas from school mathematics but reappraises them in a readable yet rigorous manner. It introduces readers to the style of abstract reasoning that will be the staple of pure mathematics courses at university. It also includes plenty of nuggets that can be savoured after a first reading (such as the construction of the real numbers via equivalence classes of Cauchy sequences of rationals, and the proofs of the generalised associativity and Cantor-Schröder-Bernstein theorems). I shall happily recommend this book to prospective undergraduate mathematicians and warmly welcome it to the growing shelf of recent bridging texts such as [1, 2, 3, 4].

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Anachronisms in the history of mathematics edited by Niccolò Guicciardini, pp. 366, £110 (hard), ISBN 978-1-10883-496-4, Cambridge University Press (2021)

Mainstream historians learn early the dangers of applying present-day thinking and concepts to the study of the past. To what extent do the same dangers apply to the history of mathematics? After all, a common, if not uncontroversial, claim for the special status of mathematics is that its content is independent of time or culture. This book derives from a symposium held in Pasadena in 2018; in what is a relatively new discipline, there is a welcome sense of freshness to the papers, although there are also quite a lot of 'isms'.

The editor Niccolò Guicciardini argues that, although anachronistic thinking has obvious dangers, it is not always wrong or unhelpful in mathematics:

> I do not share the skepticism, sometimes even derision, so commonly felt by professional historians when they accuse mathematicians who turn to history of producing work that is hopelessly naïve because it is often based on anachronistic translations and evaluations.

And in his introduction:

> The important thing is to make sure that, by manipulating the glass [lens] of historical research too clumsily, we do not "turn that glass into a mirror", and end up seeing our face reflected in it.

Texts from the past were written by and for actors (as they are called here) with mind-sets different from those of today, and in most historical study it is essential to consider such differences. Words and concepts change their meanings and accumulate baggage; Newton's or Leibnitz's concepts of calculus have been developed and refined ever since they were written. Translation adds a further level of possible distortion, and for mathematicians this includes the use of modern notation. Yet mathematicians generally feel that they are building directly on the work of previous generations, so it is not unreasonable to seek where and when concepts and methods originated. It is good for learners in particular to see how concepts developed. At the same time, awareness of anachronism gives us insights into the past as a foreign country. Guicciardini sets out the need to "strike a balance between familiarizing and foreignizing past texts, between recognition and wonder."

It seems to me that the most important questions raised by this book are of the form "to what extent could it be said that X anticipated the modern concept of Y?" Kim Plofker discusses the "error" of division by zero as it is discussed in mediaeval Sanskrit algebra and argues that modern concepts positively help the understanding of what is written. She then considers the Sanskrit derivation of what we would now write as $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$, using approximations derived from Pythagoras's theorem (the name is a cultural misplacement); the translated text says "the error will not be large". Is this calculus *avant la lettre*? Discussion is not helped, in my view, by the US style of including all aspects of analysis under the heading 'calculus', so I fully concur with Plofker's "We must begin by inquiring more critically what calculus is." She concludes that "it is not any specific breakthrough, concept or technique that constitutes calculus so much as the eventual cohesion of these breakthroughs … into a particular strand."

Guiccardini himself discusses the "deceptive familiarity" of Johann Bernoulli's solution of a problem involving differential equations. No one disputes the importance of Newton in mechanics, even though his *Principia* is notoriously couched in geometric terms which are now difficult for most modern readers to follow. But when Bernoulli considered the differential equation governing motion under a central force, his *solution* was a geometrical construction; the concept of function, and in particular a "developed notation and calculus for trigonometric functions", were not yet in existence. The differential equation Bernoulli considered is, in modern notation,

$$d\theta = \frac{l\,dr}{\sqrt{2Er^4 - 2r^4 \int_r^\infty F\,dr - l^2r^2}}.$$

With $F = a^2g/r^2$, Bernoulli uses standard substitutions to get as far as $\frac{dz}{a} = \frac{dt}{\sqrt{h^2 - t^2}}$. But instead of solving this algebraically as would any competent A-level student today, Bernoulli produces a geometrical construction of a curve on a diagram. Guiccardini's point is not that Bernoulli was not doing modern mechanics but that the thought of the time did not yet include the modern concept of function.

Guiccardini also explains that Bernoulli did not write the differential equation with the familiar "modern" constants quoted above, but in the form

$$dz = \frac{aac\,dx}{\sqrt{abx^4 - 2x^4 \int F\,dx - aaccxx}}.$$

The extra constants here make all quantities homogeneous, in a way that validates the geometrical construction method; this was basic to Bernoulli's thinking.

One very obvious way in which mathematics has changed is in the development of rigour. Morris Kline is quoted as saying:

> It is safe to say that no proof given at least up to 1800 in any area of mathematics … would be regarded as satisfactory by the standards of 1900.

A classic instance of the consequences is studied by Craig Fraser and Andrew Schroter. Euler produced astonishing insights by methods that cause many today to throw up their hands in horror. (When I showed Euler's derivation of $\zeta(2) = \pi^2/6$ to a class, one pupil asked me, "On a scale of 1 to 10, how illegal is that?") But Euler's results obtained from divergent series have been 'rigorized' (some would say 'given meaning') by later writers, such as Hardy in *Divergent Series*; does this mean that Euler had insights that anticipated later times, that he had an "intuitive familiarity with the concept [of summability]", or merely that he had intuitions that could not be justified at the time but turned out to be correct? The treatment of an expression such as 1 – 1 + 1 – 1 … was for Euler purely formal, governed by whether it gave useful results. Fraser and Schroter discuss the issues by contrasting Euler's search for the (presumably unique) "meaning" of such an expression with the later approach of "defining" it; at the same time they acknowledge that an anachronistic approach has its uses in the classroom, for instance in explaining why convergence matters. They conclude:

> The significance of Euler's formal approach … is not in the way it foreshadowed modern theories … but rather in the latitude it provided him to obtain actual numerical values. … The modern concept [is] very different from Euler's own beliefs and outlooks. Claims that Euler grasped invariance, or was a summabilist, thus are anachronistic".

At the same time we can concur with Goldstine that "it is truly in keeping with Euler's genius that he should have worked with ideas that were only to be satisfactorily and completely discussed in modern times."

Some of the contributors criticise earlier historians of mathematics, or consider the effect of cultural difference. The latter is often sensitive—can we mention Pascal's triangle when discussing Chinese mathematics? However, these chapters seem to me to offer fewer insights into the central concerns of the book, and I give an over-simplified outline. Karine Chemla and Martina R. Schneider discuss aspects of Chinese mathematics. Schneider discusses the 'Chinese remainder theorem' in the light of the reconstruction by Ludwig Matthiesson (1874). Chemla argues that a method for finding square roots numerically, given as a single problem in a text of c. 400 CE, should be viewed as an algorithm, in Donald Knuth's sense. This discussion is hard for non-specialists to follow as the method is not given—at least, not until the very last chapter in the book, where the same method is considered by Joseph W. Dauben, who shows that it is typical of an almost kinaesthetic approach to such problems. Dauben also considers the work of C.S. Pierce in relation to transfinite set theory and non-standard analysis. Jacqueline Feke argues the need to rethink the relationship between the mathematical sciences and philosophy in ancient Greece, seeing the modern bifurcation between the 'two cultures' as misleading. Robin Goulding considers the unfortunate effects of a misdating of the fifth-century mathematician Proclus by Petrus Ramus (1515-1572), the first modern writer of the history of mathematics, Jemma Lorenat the history of non-metric projective geometry, and Jeremy Gray the division of techniques into 'elementary' and 'advanced' by Roberto Bonola (1906), in the context of non-Euclidean geometry.

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Why do we study the history of mathematics? The late Ivor Grattan-Guinness identified the crucial point, summarised here by Fraser and Schroter:

> [There is] a disjunction between *heritage* (our tracking of a particular concept's journey along the "royal road" from the past to the present) and *history* (our attempt to explain why a certain mathematical development happened). The "heritage approach" evaluates past mathematics in the light of recent theories, looking for similarities that reveal the gradual unveiling of a mathematical concept. Conversely, "history" instinctively looks for differences and discontinuities.

Guicciardini and his team ask key questions and offer some answers. Although there may be a slight risk of emulating the centipede which, asked which foot was being moved, became unable to move at all, anyone writing, or seriously interested in, the history of mathematics should read this important book.

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Abstract algebra, a comprehensive introduction by John W. Lawrence and Frank A. Zorzitto, pp. 619, £64.99 (hard), ISBN 978-1-10883-665-4, Cambridge University Press (2021)

This book is aimed at "senior undergraduate students" and "those more gifted in mathematics", as well as "beginning graduate students who need a refresher". It assumes knowledge of injections and surjections, elementary matrix properties and linear transformations, but it starts with a "refresher" chapter on basic number theory, up to Fermat's Little Theorem. There are then two chapters on groups; the first goes as far as quotient groups and external and internal products, the second covers Cauchy's and Sylow's Theorems and chains of solvable groups. Chapter 4 covers rings, including maximal ideals; Chapter 5, on primes and unique factorisation, is largely concerned with Noetherian domains, and Chapters 6 and 7 are on Galois theory, from algebraic field extensions to the insolvability of the general quintic, with a mention of the inverse Galois problem. The last two chapters cover principal ideal domains and division algorithms, ending with an extensive treatment of Gröbner bases. An appendix discusses infinite sets, including Zorn's lemma, cardinality and the algebraic closure of a field. The subtitle 'comprehensive introduction' is indeed accurate.

The treatment is concise and rigorous but approachable in style, with plenty of helpful advice and motivation. For example, from the introduction to the section on Cauchy's and Sylow's theorems:

> If m is the power of a single prime and m divides the order of the group, subgroups of order m will exist, and quite a bit can be said about them. That is what the upcoming results are about. It takes quite a bit of slogging to work through the ensuing ideas, but the reward will be a more profound understanding of finite groups.

Another characteristic of the authors is to write proofs formally in content but not in style, often using constructions such as "Well, …": "What are the conjugates of τ? Well, the group relations yield $\sigma\tau\sigma^{-1} = \sigma^2\tau$." This approach is very user-friendly, although readers need to get used to the fact that when the authors say that something 'seems' to be the case, they are asserting it, and not indicating that it is a