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Composition of Inner Functions

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Abstract. We study the image of the model subspace K_θ under the composition operator C_φ, where φ and θ are inner functions, and find the smallest model subspace which contains the linear manifold $C_\varphi K_\theta$. Then we characterize the case when C_φ maps K_θ into itself. This case leads to the study of the inner functions φ and ψ such that the composition $\psi \circ \varphi$ is a divisor of ψ in the family of inner functions.

1 Introduction

Given a holomorphic self-map φ of the open unit disk $\mathbb{D}$, the composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in \mathrm{Hol}(\mathbb{D}).$$

It is easily verified that C_φ is continuous with respect to the topology of $\mathrm{Hol}(\mathbb{D})$. Whenever $\mathcal{X}$ is a Banach space inside $\mathrm{Hol}(\mathbb{D})$, we naturally wonder if $\mathcal{X}$ is invariant under C_φ and that the restriction is a bounded operator. Following this fundamental question, several others such as compactness, normality, *etc.*, arise. Generally speaking, it turns out that the operator-theoretic behavior of C_φ is closely related to function-theoretic properties of its symbol φ.

If $\mathcal{X} = H^p(\mathbb{D})$, the Hardy space of the open unit disk, we know that $C_\varphi \colon H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is automatically bounded. This is a consequence of the classical subordination principle of Littlewood [10]. The compactness of C_φ in this setting, however, is a more delicate question. A characterization of compact composition operators on Hardy spaces was obtained by Shapiro [12]; see also the notes in the last chapter of [13]. The study of composition operators on different subclasses of $\mathrm{Hol}(\mathbb{D})$, *e.g.*, on Besov spaces [14], on Bloch spaces [2] or on the Dirichlet space [6, 7, 15], is a very active domain of research. The literature is extremely vast and it is rather impossible to make a comprehensive list.

In his seminal work of 1949, A. Beurling characterized the closed subspaces of H^2 which are invariant under the forward shift operator [1]. They are precisely of the form θH^2 where θ is an inner function. The associated *model subspace* K_θ is defined to be the orthogonal complement of θH^2, *i.e.*, $K_\theta = H^2 \ominus \theta H^2$. If we look at K_θ as a subspace of $H^2(\mathbb{T})$, this relation can be rewritten as

$$K_\theta = H^2 \cap \theta \overline{H^2_0}. \tag{1.1}$$

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Given a set $A \subset H^2$, the smallest S^*-invariant closed subspace of H^2 that contains A will be denoted by $\langle A \rangle$. Therefore, K_θ's are the closed invariant subspaces of the backward shift operator S^*. Moreover, K_θ is a reproducing kernel Hilbert space, *i.e.*, for every $f \in K_\theta$ and $\lambda \in \mathbb{D}$, we have $f(\lambda) = \langle f, k_\lambda^\theta \rangle_2$, where

$$k_\lambda^\theta(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda} z}, \quad z \in \mathbb{D}.$$

In this paper, we study the image of K_θ under the composition operator C_φ, where φ is an inner function. This work is a continuation of [11]. In Theorem 2.1, we show that C_φ maps K_θ into the model subspace K_η, where the inner function η is explicitly determined by φ and θ. The formulation of η depends on the behavior of φ and θ at the origin. Moreover, we show that η is optimal, in the sense that K_η is the *smallest model subspace* of H^2 which contains $C_\varphi K_\theta$. Then in Theorem 4.1, we explore the case when $K_\eta \subset K_\theta$. In other words, we characterize the inner functions θ and φ such that C_φ maps K_θ into itself. In studying this case, we provide non-trivial inner functions φ and ψ such that $\psi \circ \varphi$ is a divisor of ψ in the family of inner functions.

2 What is $\langle C_\varphi K_\theta \rangle$?

In this section, we study the action of C_φ on a given K_θ when φ is an inner function. In particular, we determine the smallest closed S^*-invariant subspace of H^2 that contains $C_\varphi K_\theta$.

Given an inner function φ, we know that f is non-cyclic for S^* if and only if $f \circ \varphi$ is non-cyclic for S^* [5, Theorem 2.4.4]. In other words, if we put

$$K = \bigcup_{\theta \text{ inner}} K_\theta,$$

then for any inner symbol φ, the restricted mappings $C_\varphi \colon K \to K$ and $C_\varphi \colon H^2 \setminus K \to H^2 \setminus K$ are well defined. Theorem 2.1 provides a precise refinement of the first mapping.

If φ is an inner function, then by a celebrated result of Frostman [8], $\frac{\lambda - \varphi}{1 - \bar{\lambda}\varphi}$ is a Blaschke product for all $\lambda \in \mathbb{D} \setminus \mathcal{E}_\varphi$, where $\mathcal{E}_\varphi$ is an exceptional set of logarithmic capacity zero. We exploit this result in Theorem 2.1 and Corollary 2.4.

Theorem 2.1 *Let φ and θ be inner functions, and let*

$$\eta(z) = \begin{cases} (\theta \circ \varphi)(z), & \text{if } \theta(0) \neq 0 \text{ and } \varphi(0) = 0, \\ z(\theta \circ \varphi)(z), & \text{if } \theta(0) \neq 0 \text{ and } \varphi(0) \neq 0, \\ z\frac{\theta(\varphi(z))}{\varphi(z)}, & \text{if } \theta(0) = 0. \end{cases}$$

Then the mapping $C_\varphi \colon K_\theta \to K_\eta$ is well defined and bounded. Moreover, $\langle C_\varphi K_\theta \rangle = K_\eta$, i.e., *$K_\eta$ is the smallest closed S^*-invariant subspace of H^2 that contains the image of K_θ under C_φ.*

Proof As we agreed, k_λ^θ denotes the reproducing kernel of K_θ at $\lambda \in \mathbb{D}$. Hence, by direct evaluation,

$$C_\varphi k_\lambda^\theta = \frac{1-\overline{\theta(\lambda)}\theta\circ\varphi}{1-\bar{\lambda}\varphi} \in H^\infty \subset H^2. \tag{2.1}$$

According to a theorem of Lindelöf, $\theta\circ\varphi$ is an inner function. Since both φ and $\theta\circ\varphi$ are inner, we get

$$\overline{C_\varphi k_\lambda^\theta(z)} = \frac{\theta\big(\varphi(z)\big)-\theta(\lambda)}{\varphi(z)-\lambda}\cdot\frac{\varphi(z)}{\theta\big(\varphi(z)\big)}, \quad z\in\mathbb{T}. \tag{2.2}$$

For each $\lambda \in \mathbb{D}$ and $f \in H^2$, define

$$Q_\lambda(f)(z) = \frac{f(z)-f(\lambda)}{z-\lambda}, \quad z\in\mathbb{D}.$$

In particular, we have $Q_0 = S^*$. Moreover, the alternative expression $Q_\lambda = (1-\lambda S^*)^{-1}S^*$ shows that $Q_\lambda\colon H^2 \to H^2$ is a bounded operator on H^2. Hence, it follows that

$$\frac{\theta\big(\varphi(z)\big)-\theta(\lambda)}{\varphi(z)-\lambda} = [(C_\varphi\circ Q_\lambda)(\theta)](z),$$

which reveals that

$$\frac{\theta\circ\varphi-\theta(\lambda)}{\varphi-\lambda} \in H^2.$$

To deal with the term $\varphi(z)/\theta\big(\varphi(z)\big)$, we should consider three cases that have been reflected in the definition of η. We just treat the first case in detail. The proofs of the other two cases are similar.

From now on, we suppose that $\theta(0) \neq 0$ and $\varphi(0) = 0$. For this case, we defined $\eta = \theta\circ\varphi$. Thus, by (2.2),

$$\eta\overline{C_\varphi k_\lambda^\theta} = \varphi\times(C_\varphi\circ Q_\lambda)(\theta) \in H_0^2.$$

Hence, by (2.1) and (1.1) and the above relation, we conclude that $C_\varphi k_\lambda^\theta \in K_\eta$ for every $\lambda \in \mathbb{D}$. Therefore, the mapping $C_\varphi\colon K_\theta \to K_\eta$ is a well-defined bounded operator.

It is a bit more delicate to show that $\langle C_\varphi K_\theta\rangle = K_\eta$. Based on the above discussion, we certainly have $\langle C_\varphi K_\theta\rangle \subset K_\eta$. To establish the reverse inclusion, we assume that the function $g \in H^2$ is orthogonal to $\langle C_\varphi K_\theta\rangle$, and then we deduce that $g \in \eta H^2$. This means that $\langle C_\varphi K_\theta\rangle^\perp \subset \eta H^2$, and thus $\langle C_\varphi K_\theta\rangle \supset (\eta H^2)^\perp = K_\eta$.

The assumption $g \perp \langle C_\varphi K_\theta\rangle$ means that

$$\langle g, S^{*n}C_\varphi k_\lambda^\theta\rangle_{H^2} = 0$$

for every $\lambda \in \mathbb{D}$ and $n \geq 0$. But

$$\langle g, S^{*n}C_\varphi k_\lambda^\theta\rangle_{H^2} = \langle S^n g, C_\varphi k_\lambda^\theta\rangle_{H^2} = \langle z^n, \bar{g}C_\varphi k_\lambda^\theta\rangle_{L^2}.$$

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Hence, we conclude

$$\overline{C_\varphi k_\lambda^\theta} \quad g \in H_0^2,$$

or equivalently, by (2.2),

$$\frac{\varphi(z)}{\theta\big(\varphi(z)\big)} \cdot \frac{\theta\big(\varphi(z)\big) - \theta(\lambda)}{\varphi(z) - \lambda} \cdot g(z) \in H_0^2. \tag{2.3}$$

According to Frostman's theorem [8], we can pick a $\lambda \in \mathbb{D}$ such that

$$\frac{(\theta \circ \varphi)(z) - \theta(\lambda)}{1 - \overline{\theta(\lambda)}(\theta \circ \varphi)(z)}$$

and

$$\frac{\varphi(z) - \lambda}{1 - \bar{\lambda}\varphi}$$

are both Blaschke products,

$$\theta(\lambda) \neq \theta(0) \quad \text{and} \quad \theta(\lambda) \neq 0.$$

In fact, this is true for all $\lambda \in \mathbb{D}$, except on a set of logarithmic capacity zero. The zeros of the first Blaschke product come from the equation $\theta\big(\varphi(z)\big) = \theta(\lambda)$ while the zeros of the second Blaschke product satisfy the equation $\varphi(z) = \lambda$. Hence, the zeros of the first product include the zeros of the second one, and the two zero sets are not necessarily equal in general. Note that the first Blaschke product does not vanish at the origin. Moreover, the denominators of both quotients are outer functions that are bounded from below by $\min\{1-|\lambda|, 1-|\varphi(\lambda)|\}$ and from above by 2. Therefore, the quotient

$$\frac{\theta\big(\varphi(z)\big) - \theta(\lambda)}{\varphi(z) - \lambda}$$

is the product of an outer function that is bounded from above and below on $\mathbb{D}$ and possibly a Blaschke product that is non-null at the origin. Therefore, Smirnov's theorem applied to (2.3) ensures that

$$\frac{\varphi(z)}{\theta\big(\varphi(z)\big)} \cdot g(z) \in H_0^2. \tag{2.4}$$

Now, the key point is that the inner functions φ and $\theta \circ \varphi$ have no common non-constant inner factor. Remember that $\theta(0) \neq 0$. Hence, in the first place, φ and $\theta \circ \varphi$ cannot have a common Blaschke factor. Secondly, if φ has a singular part, for the corresponding singular measure we can choose a non-empty carrier $A \subset \mathbb{T}$ such that for each $\zeta \in A$ we have $\lim_{r\to 1} \varphi(r\zeta) = 0$. Then at any such point we would have

$$\lim_{r\to 1} (\theta \circ \varphi)(r\zeta) = \theta(0) \neq 0.$$

Hence, if $\theta \circ \varphi$ has a singular part, the carrier of its corresponding singular measure can be taken to be disjoint from A. Thus, φ and $\theta \circ \varphi$ have no non-constant inner factors. Back to (2.4) and the fact that the inner-outer factorization of H^2 functions is unique up to a unimodular multiplicative constant, we conclude that $\theta \circ \varphi$ must divide g. In other words, $g \in (\theta \circ \varphi)H^2 = \eta H^2$. ∎

Remark 2.2. Since φ is inner, by [4, Theorem 3.8], $C_\varphi K_\theta$ is a closed subspaces of H^2. However, $C_\varphi K_\theta$ may fail to be invariant under S^*. For example, take $\theta(z) = \varphi(z) = z^2$. Then $\eta(z) = z\theta\big(\varphi(z)\big)/\varphi(z) = z^3$, $K_\theta = \operatorname{span}\{1, z\}$, $K_\eta = \operatorname{span}\{1, z, z^2\}$, but $C_\varphi K_\theta = \operatorname{span}\{1, z^2\}$.

In the hierarchy of inner functions, for a given φ and θ, the largest inner function η among the three possible situations in Theorem 2.1 is $z \times \theta \circ \varphi$, and this gives the largest model subspace $K_{z\theta\circ\varphi}$ among the three possible cases. Hence, if we are not keen about the smallest possible model subspace which contains the image of C_φ, we obtain the following result.

Corollary 2.3 *For inner functions φ and θ, the composition operator $C_\varphi \colon K_\theta \to K_{z\theta\circ\varphi}$ is well defined and bounded.*

If the inner function θ is a Blaschke product, then we can say a bit more about the behavior of C_φ on K_θ. This is mainly because K_θ is generated by the kernels k_λ^θ, where λ's are the zeros of θ (we do not need to consider all $\lambda \in \mathbb{D}$).

Corollary 2.4 *Let φ be inner with $\varphi(0) = 0$, and denote its exceptional set by $\mathcal{E}_\varphi$. Let $(d_n)_{n\geq 1}$ be a sequence of positive integers, and let $(\lambda_n)_{n\geq 1}$ be a sequence of distinct points in $\mathbb{D} \setminus (\{0\} \cup \mathcal{E}_\varphi)$ such that*

$$\sum_{n\geq 1} d_n(1 - |\lambda_n|) < \infty.$$

Then the zeros of all equations $\varphi(z) = \lambda_n$, $n \geq 1$ (multiplicities counted) form a Blaschke sequence, and for the corresponding Blaschke product B, we have

$$\frac{1}{(1 - \overline{\lambda}_n\varphi)^{j_n}} \in K_B, \quad n \geq 1,\ 1 \leq j_n \leq d_n.$$

Moreover, K_B is the smallest model subspace of H^2 which contains all elements $1/(1 - \overline{\lambda}_n\varphi)^{j_n}$, $n \geq 1$, $1 \leq j_n \leq d_n$.

Proof Put

$$\theta(z) = \prod_{n\geq 1}\Big(\frac{|\lambda_n|}{\lambda_n}\frac{\lambda_n - z}{1 - \overline{\lambda}_n z}\Big)^{d_n}.$$

Then

$$K_\theta = \operatorname{Span}\big\{\big(1/(1 - \overline{\lambda}_n z)\big)^{j_n} : n \geq 1,\ 1 \leq j_n \leq d_n\big\}$$

and

$$C_\varphi K_\theta = \operatorname{Span}\big\{\big(1/(1 - \overline{\lambda}_n \varphi)\big)^{j_n} : n \geq 1,\ 1 \leq j_n \leq d_n\big\}.$$

Note that we implicitly applied [4, Theorem 3.8]. Therefore, by Theorem 2.1,

$$\frac{1}{(1 - \overline{\lambda}_n\varphi)^{j_n}} \in K_\eta, \quad n \geq 1,\ 1 \leq j_n \leq d_n,$$

where

$$\eta(z) = (\theta \circ \varphi)(z) = \prod_{n \geq 1} \Big(\frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - \varphi(z)}{1 - \overline{\lambda}_n \varphi(z)} \Big)^{d_n},$$

and moreover

$$\langle 1/(1 - \overline{\lambda}_n \varphi)^{j_n} : n \geq 1, 1 \leq j_n \leq d_n \rangle = K_\eta.$$

Using again Frostman's theorem for each term in the product above, we see that η is a Blaschke product if none of the λ_n's is taken from the exceptional set of φ. Hence, η becomes a Blaschke product which we denoted by B. ■

If θ is a single Blaschke factor, *i.e.*,

$$\theta(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}, \quad \lambda \in \mathbb{D} \setminus \{0\},$$

then $K_\theta = \mathbb{C} k_\lambda$, and, by the above corollary,

$$\frac{1}{1 - \overline{\lambda}\varphi} \in K_{\frac{\lambda - \varphi}{1 - \overline{\lambda}\varphi}},$$

and moreover $1/(1 - \overline{\lambda}\varphi)$ is S^*-cyclic for $K_{\frac{\lambda - \varphi}{1 - \overline{\lambda}\varphi}}$. Corollary 2.4 was obtained by analyzing the first case of η in Theorem 2.1. It is straightforward to obtain similar results corresponding to the other two cases.

3 Discussion on a Schröder-type Equation

In studying the inner functions φ for which C_φ maps K_θ into itself, we will face with the functional equation

$$\psi\big(\varphi(z)\big) \times \omega(z) = \psi(z), \quad z \in \mathbb{D}, \tag{3.1}$$

where all the functions ψ, φ and ω are inner. A variation of (3.1) is known as *Schröder's equation* and has a very long and rich history. A detailed discussion of this topic can be found in [3]. We study the general version (3.1) in this section. Since φ is a self map of $\mathbb{D}$, it has a fix point in $\overline{\mathbb{D}}$. To proceed, depending on the location of the fixed point of φ in $\mathbb{D}$ or on $\mathbb{T}$, we consider two cases. While the first case is rather easy and straightforward, the latter is dramatically complex.

We remind that

$$\tau_p(z) = \frac{p - z}{1 - \overline{p} z}, \quad z \in \mathbb{D},$$

and

$$\rho_\lambda(z) = \lambda z, \quad \lambda \in \mathbb{T}, z \in \mathbb{D}.$$

Case I: φ has a fixed point inside $\mathbb{D}$ Suppose that $\varphi(p) = p$, for some $p \in \mathbb{D}$, and that ψ has a zero of order $m \geq 0$ at p. Then we can write

$$\psi(z) = \Big(\frac{p-z}{1-\bar{p}z}\Big)^m \psi_0(z),$$

where ψ_0 is an inner function with $\psi_0(p) \neq 0$. Hence, (3.1) becomes

$$\Big(\frac{p-\varphi(z)}{1-\bar{p}\varphi(z)}\Big)^m \psi_0\big(\varphi(z)\big)\,\omega(z) = \Big(\frac{p-z}{1-\bar{p}z}\Big)^m \psi_0(z).$$

Divide by $(p-z)^m$ and then let $z \to p$ to deduce

$$\big(\varphi'(p)\big)^m \omega(p) = 1.$$

Hence, $\varphi'(p)$ and $\omega(p)$ are both unimodular constants. Thus, by the maximum principle and Schwarz's lemma, $\varphi = \tau_p \circ \rho_\lambda \circ \tau_p$ for some $\lambda \in \mathbb{T}$, and since $\lambda = \varphi'(p)$, $\omega \equiv \bar{\lambda}^m$. Moreover, ψ_0 must satisfy

$$\psi_0 \circ \tau_p \circ \rho_\lambda \circ \tau_p = \psi_0.$$

By induction, if we repeatedly compose both sides with $\tau_p \circ \rho_\lambda \circ \tau_p$, we obtain

$$\psi_0 \circ \tau_p \circ \rho_{\lambda^k} \circ \tau_p = \psi_0, \quad k \geq 1, \tag{3.2}$$

which we rewrite as

$$(\psi_0 \circ \tau_p)(\lambda^k z) = (\psi_0 \circ \tau_p)(z), \quad k \geq 1,\ z \in \mathbb{D}. \tag{3.3}$$

Now, depending on λ being a root of unity or not, we should consider some subcases.

Category I: there is no integer $n \geq 1$ such that $\lambda^n = \big(\varphi'(p)\big)^n = 1$ If λ is not a root of unity, then (3.3) and the uniqueness theorem for analytic functions force $\psi_0 \circ \tau_p$ to be a unimodular constant, and thus $\psi_0 \equiv \gamma$, for some $\gamma \in \mathbb{T}$. Hence, the inner function

$$\psi = \gamma(\tau_p)^m, \quad \gamma \in \mathbb{T}, p \in \mathbb{D}, m \geq 0,$$

and the hyperbolic rotations

$$\varphi = \tau_p \circ \rho_\lambda \circ \tau_p,$$

satisfy the functional equation

$$\psi \circ \varphi = \lambda^m \psi. \tag{3.4}$$

A trivial but important special case of this category is $\psi(z) = \gamma z^m$ and $\varphi(z) = \lambda z$.

Note that, after all, we avoid emphasizing that λ is not a root of unity. Firstly, (3.4) is fulfilled with all $\lambda \in \mathbb{T}$. Secondly, this is because the extra solutions are obtained as a particular situation of the third category which is treated below. But we integrate them here.

Category II: $\lambda = \varphi'(p) = 1$ This means $\varphi(z) = z$ and hence it is a trivial case.

Category III: there is an integer $n \geq 2$ **such that** $\lambda^n = \big(\varphi'(p)\big)^n = 1$ Take n to be the smallest such integer. Then with a proper choice of $k \geq 1$, we have $\lambda^k = e^{i2\pi/n}$. Hence, by (3.2), ψ_0 must satisfy

$$\psi_0 \circ \tau_p \circ \rho_{e^{i2\pi/n}} \circ \tau_p = \psi_0.$$

In this situation, there are plenty of nonconstant solutions for ψ_0. (Constant solutions are counted in category I.) The above restriction on ψ_0 is equivalent to say that the Taylor expansion of $\psi_0 \circ \tau_p$ is of the form

$$(\psi_0 \circ \tau_p)(z) = \sum_{k=0}^{\infty} a_k z^{kn}, \quad z \in \mathbb{D}.$$

Equivalently, we have

$$(\psi_0 \circ \tau_p)(z) = \psi_1(z^n), \quad z \in \mathbb{D},$$

where ψ_1 is any arbitrary nonconstant inner function. Hence, with this last formula for ψ_0, the inner function ψ becomes

$$\psi = \gamma(\tau_p)^m \psi_1\big((\tau_p)^n\big), \quad \psi_1 \text{ inner and nonconstant}, p \in \mathbb{D}, \gamma \in \mathbb{T}, m \geq 0, n \geq 2,$$

and *only* with the hyperbolic rotations

$$\varphi = \tau_p \circ \rho_{e^{i2k\pi/n}} \circ \tau_p, \quad 1 \leq k \leq n-1,$$

they fulfills the functional equation

$$\psi \circ \varphi = e^{i2mk\pi/n} \psi.$$

Considering the above categories, we can say that, up to a hyperbolic rotation and a unimodular constant, the solutions of (3.1) that have a fixed point inside $\mathbb{D}$ are of the forms

$$\psi(z) = z^m \quad \text{with } \varphi(z) = \lambda z,$$

or

$$\psi(z) = z^m \psi_0(z^n) \quad \text{with } \varphi(z) = e^{i2k\pi/n} z.$$

Note that, in case I, the inner factor ω is always a unimodular constant.

Case II: φ has no fixed point inside $\mathbb{D}$ In this case, φ has its Denjoy–Wolff point on $\mathbb{T}$. If ω is a unimodular constant, then the equation (3.1) becomes

$$\psi\big(\varphi(z)\big) = \lambda\psi(z), \quad z \in \mathbb{D}, \tag{3.5}$$

where $\lambda \in \mathbb{T}$. But, if ω is not a unimodular constant, then by replacing z by $\varphi(z)$ in (3.1) and using induction, we see that

$$\psi\big(\varphi^{[N+1]}(z)\big) \times \prod_{n=0}^{N} \omega\big(\varphi^{[n]}(z)\big) = \psi(z), \quad N \geq 1,$$

where $\varphi^{[0]}(z) = z$ and $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$, $n \geq 1$. Hence, considering the fact that $\prod_{n=0}^{N} \omega\big(\varphi^{[n]}(z)\big)$ is an increasing sequence of divisors of ψ, we deduce that the product

$$\psi_1(z) = \prod_{n=0}^{\infty} \omega\big(\varphi^{[n]}(z)\big) \tag{3.6}$$

is a well-defined non-constant inner function and, moreover, ψ_1 is a divisor of ψ which itself satisfies (3.1). Since, for each $z \in \mathbb{D}$, the sequence $\varphi^{[n]}(z)$, $n \geq 1$, tends nontangentially to p and the above product is convergent, ω must have the radial limit 1 at p. Define $\psi_2 = \psi/\psi_1$. Thus, putting $\psi = \psi_1\psi_2$ in (3.1) and simplifying the equation reveal that ψ_2 must fulfil

$$\psi_2\big(\varphi(z)\big) = \psi_2(z), \quad z \in \mathbb{D}. \tag{3.7}$$

Therefore, the solutions of (3.1) are of the form

$$\psi = \psi_1\psi_2,$$

where ψ_1 is given by (3.6), and ψ_2 satisfies (3.7), which is a special case of (3.5).

In (3.6), the convergence of the product is the main required condition. If it converges, then surely the outcome is a solution of (3.1). The following lemma provides a set of sufficient conditions to achieve this goal.

Lemma 3.1 *Suppose that φ and ω are inner functions with the following properties:*

(i) *p, the Denjoy–Wolff point of φ, is on $\mathbb{T}$;*
(ii) *$|\varphi'(p)| < 1$;*
(iii)
$$\Big|\frac{1-\omega(z)}{p-z}\Big|^2 \leq \frac{C}{1-|z|^2}, \quad z \in \mathbb{D},$$
where C is a constant (e.g., *if ω has finite angular derivative in the sense of Carathéodory at p with $\omega(p) = 1$, the above inequality is fulfilled).*

Then
$$\psi(z) = \prod_{n=0}^{\infty} \omega\big(\varphi^{[n]}(z)\big)$$
is a well-defined non-constant inner function that satisfies the equation (3.1).

Proof If $f\colon \mathbb{D} \to \mathbb{D}$ has finite angular derivative in the sense of Carathéodory at $\alpha \in \mathbb{T}$, then Julia's inequality says

$$\frac{|f(\alpha)-f(z)|^2}{1-|f(z)|^2} \le |f'(\alpha)| \frac{|\alpha - z|^2}{1-|z|^2}, \quad z \in \mathbb{D}.$$

We apply this inequality to φ. Since $\varphi(p) = p$, by induction, we obtain

$$\frac{|p-\varphi^{[n]}(z)|^2}{1-|\varphi^{[n]}(z)|^2} \le |\varphi'(p)|^n \frac{|p - z|^2}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Hence, replacing z by $\varphi^{[n]}(z)$ in (iii) and exploiting the inequality before, we obtain

$$\left|1-\omega\big(\varphi^{[n]}(z)\big)\right|^2 \le \Big(C\frac{|p-z|^2}{1-|z|^2}\Big)|\varphi'(p)|^n, \quad z \in \mathbb{D}.$$

The assumption $|\varphi'(p)| < 1$ now ensures the convergence of the product in the definition of ψ. ■

As a special case, we can take $\omega = \bar{p}\varphi^{[n_0]}$, $n_0 \ge 0$, in Lemma 3.1. Hence, we obtain

$$\psi(z) = \prod_{n=n_0}^{\infty} \overline{p}\varphi^{[n]}(z)$$

which is a non-constant inner function satisfying the equation

$$\psi\big(\varphi(z)\big) \times \overline{p}\varphi^{[n_0]}(z) = \psi(z).$$

At the end, let us directly give an explicit non-trivial solution of (3.1). For $0 < \alpha < 1$, define

$$\varphi_\alpha(z) = \frac{z+\alpha}{1+\alpha z}.$$

It is straightforward to verify that

$$\varphi_\alpha \circ \varphi_\beta = \varphi_{\frac{\alpha+\beta}{1+\alpha\beta}}.$$

Hence, $\varphi_\alpha^{[n]} = \varphi_\alpha \circ \cdots \circ \varphi_\alpha = \varphi_{\alpha_n}$, where α_n satisfies the recursive relation

$$\alpha_n = \frac{\alpha + \alpha_{n-1}}{1+\alpha\alpha_{n-1}}, \qquad n \ge 2.$$

Thus

$$1-\alpha_n = \frac{(1-\alpha)(1-\alpha_{n-1})}{1+\alpha\alpha_{n-1}} \le (1-\alpha)(1-\alpha_{n-1}),$$

which, by induction, implies

$$1-\alpha_n \le (1-\alpha)^n, \quad n \ge 1.$$

For this inequality, we could have appealed to Lemma 3.1. Therefore, the sequence $(\alpha_n)_{n\ge1}$ is a Blaschke sequence, and

$$\psi = \prod_{n=1}^{\infty} \varphi_\alpha^{[n]} = \prod_{n=1}^{\infty} \varphi_{\alpha_n}$$

is a well-defined non-constant Blaschke product. Moreover, ψ satisfies the functional equation $\psi \circ \varphi_\alpha \times \varphi_\alpha = \psi$.

4 When is $C_\varphi \in \mathcal{L}(K_\theta)$?

Since in the mapping $C_\varphi\colon K_\theta \to K_\eta$, given in Theorem 2.1, the choice of η is optimal we naturally wonder when the inclusion $K_\eta \subset K_\theta$ holds in order to obtain a composition operator which maps K_θ into itself.

Theorem 4.1 *Let φ and θ be inner functions on $\mathbb{D}$. Then the mapping $C_\varphi\colon K_\theta \to K_\theta$ is well defined and bounded if and only if one of the the following situations holds:*

(i) $\varphi(z) = z$ *and any inner* θ*;*

(ii) $\theta(z) = \gamma z$, $\gamma \in \mathbb{T}$, *and any inner* φ*;*

(iii) $\theta(z) = \vartheta(z^n)$, *for some integer* $n \geq 2$ *and an arbitrary inner function* ϑ *with* $\vartheta(0) \neq 0$, *and*

$$\varphi = \rho_{e^{i2k\pi/n}}, \quad 1 \leq k \leq n;$$

(iv) $\theta(z) = \gamma z\big(\tau_p(z)\big)^m$, *where* $\gamma \in \mathbb{T}$, $p \in \mathbb{D}$, $m \geq 1$, *and any hyperbolic rotation*

$$\varphi = \tau_p \circ \rho_\lambda \circ \tau_p, \quad \lambda \in \mathbb{T};$$

(v) $\theta(z) = z\big(\tau_p(z)\big)^m \psi\Big(\big(\tau_p(z)\big)^n\Big)$, *where* $p \in \mathbb{D}$, $m \geq 0$, $n > 1$, ψ *is a non-constant inner function, and*

$$\varphi = \tau_p \circ \rho_{e^{i2k\pi/n}} \circ \tau_p, \quad 1 \leq k \leq n;$$

(vi) p, *the Denjoy–Wolff point of* φ, *is on* $\mathbb{T}$, *and* θ *is of the form* $\theta(z) = z\psi(z)$, *where* ψ *fulfills*

$$\psi\big(\varphi(z)\big) = \lambda\psi(z), \quad z \in \mathbb{D},$$

for some unimodular constant λ*;*

(vii) p, *the Denjoy–Wolff point of* φ, *is on* $\mathbb{T}$, *and*

$$\theta(z) = \gamma z\psi(z) \prod_{n=0}^{\infty} \omega\big(\varphi^{[n]}(z)\big),$$

where ω *is a non-constant inner function such that the product is convergent, and* ψ *fulfills*

$$\psi\big(\varphi(z)\big) = \psi(z), \quad z \in \mathbb{D}.$$

Proof If $\theta(z) = \gamma z$, then $K_\theta = \mathbb{C}$, for which each C_φ is a well-defined operator on K_θ. It is also trivial that $\varphi(z) = z$ gives the composition operator $C_\varphi = \mathrm{id}$ on each K_θ.

According to Theorem 2.1, $C_\varphi \in \mathcal{L}(K_\theta)$ if and only if $K_\eta \subset K_\theta$, and the latter happens if and only if η divides θ in the family of inner functions, *i.e.*,

$$\eta(z)\theta_1(z) = \theta(z), \quad z \in \mathbb{D},$$

where θ_1 is an inner function. To treat this equation, we should naturally consider three cases corresponding to the different definitions of η which were given in Theorem 2.1:

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(i) If $\theta(0) \neq 0$ and $\varphi(0) = 0$, then $\eta = \theta \circ \varphi$ and we must have

$$\theta\big(\varphi(z)\big)\,\theta_1(z) = \theta(z), \quad z \in \mathbb{D}.$$

This is the Case I, Category III, with $m = 0$ and $p = 0$. Hence, there is an integer $n \geq 1$ and an inner function ϑ, with $\vartheta(0) \neq 0$, such that $\theta(z) = \vartheta(z^n)$ and $\varphi = \rho_{e^{i2k\pi/n}}$ for $1 \leq k \leq n$.

(ii) If $\theta(0) \neq 0$ and $\varphi(0) \neq 0$, then $\eta(z) = z\theta\big(\varphi(z)\big)$ and we must have

$$z\theta\big(\varphi(z)\big)\,\theta_1(z) = \theta(z), \quad z \in \mathbb{D}.$$

Put $z = 0$ to see that this is impossible.

(iii) If $\theta(0) = 0$, then $\eta(z) = z\theta\big(\varphi(z)\big)/\varphi(z)$ and we must have

$$\frac{\theta\big(\varphi(z)\big)}{\varphi(z)}\theta_1(z) = \frac{\theta(z)}{z}, \quad z \in \mathbb{D}.$$

Put $\theta_2(z) = \theta(z)/z$. Hence, the above becomes

$$\theta_2\big(\varphi(z)\big)\,\theta_1(z) = \theta_2(z), \quad z \in \mathbb{D}.$$

According to the Grand Iteration Theorem, φ has a fixed point p in $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. Hence, we have the following three possibilities.

(a) If $p \in \mathbb{D}$, Category I gives

$$\theta(z) = \gamma z\big(\tau_p(z)\big)^m$$

where $\gamma \in \mathbb{T}$ and $m \geq 1$, and

$$\varphi = \tau_p \circ \rho_\lambda \circ \tau_p,$$

where $\lambda \in \mathbb{T}$.

(b) If $p \in \mathbb{D}$, Category II gives

$$\theta(z) = \gamma z\big(\tau_p(z)\big)^m \psi\Big(\big(\tau_p(z)\big)^n\Big)$$

where $\gamma \in \mathbb{T}$, $m \geq 1$, $n > 1$, ψ is a nonconstant inner function, and

$$\varphi = \tau_p \circ \rho_{e^{i2k\pi/n}} \circ \tau_p, \quad 1 \leq k \leq n.$$

(c) If $p \in \mathbb{T}$, then we are in Case II. Thus, θ is either of the form $\theta(z) = z\theta_2(z)$, where θ_2 fulfills

$$\theta_2\big(\varphi(z)\big) = \lambda\theta_2(z), \quad z \in \mathbb{D},$$

for some unimodular constant λ, or

$$\theta(z) = \gamma z\theta_2(z) \prod_{n=0}^{\infty} \theta_1\big(\varphi^{[n]}(z)\big),$$

where the product is convergent and θ_2 fulfills

$$\theta_2\big(\varphi(z)\big) = \theta_2(z), \quad z \in \mathbb{D}. \qquad \blacksquare$$

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