Probability in the Engineering and Informational Sciences (2025), 1–18
doi:10.1017/S0269964824000287

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

The bisexual branching processes affected by viral infectivity and with random control functions in random environments

Min Ren and Guanghui Zhang

College of Mathematics and Statistics, Suzhou University, 234000, Suzhou, Anhui, China
Corresponding author: Min Ren; Email: rm0107116@163.com

Keywords: bisexual branching process; limiting behaviors; random control functions; random environments; viral infectivity

MSC: 60G07

Abstract
In this paper, the model of bisexual branching processes affected by viral infectivity and with random control functions in independent and identically distributed (i.i.d.) random environments is established and the Markov property is given firstly. Then the relations of the probability generating functions of this model are studied, and some sufficient conditions for process extinction under common mating functions are presented. Finally, the limiting behaviors of the considered model after proper normalization, such as the sufficient conditions for the convergence in L^1 and L^2 and almost everywhere convergence, are investigated under the condition that the random control functions are super additive.

1. Introduction

In order to accurately describe population models in physics, biology, and chemistry, Daley [2] introduced the bisexual branching process model in 1968. Until now, a lot of scholars have focused on the researches of this model and made intensive studies on it. Alsmeyer, Rösler, González, and Molina discussed the extinction probability, limiting behavior, and statistical inference of the model [1]–[7]. The reproduction of species is affected by many factors such as natural environment and social environment. In order to describe a more complex gender population model, mathematical researchers have modified the basic model established by Daley. The models of super additive bisexual branching processes in varying environments [11], bisexual branching processes in random environments [10], bisexual branching processes with immigration in random environments [12, 16], and bisexual branching process in random environments with random control [15] are introduced, and a lot of research results have been obtained. Li *et al.* [8, 9] studied the limiting behaviors and moment convergence criteria of bisexual branching processes in random environments. Song *et al.* [14] discussed the limiting behaviors of the conditional mean growth rate for a kind of bisexual branching processes in random environments. Ren *et al.* [13] investigated the Markov property, probability generating functions, and extinction probability of bisexual branching processes affected by viral infectivity in random environments. In this paper, a model of bisexual branching processes affected by viral infectivity and with random control functions in a random environment is established, and the Markov property, the relation of the probability generating functions, and extinction probability of the model are discussed. Meanwhile, the limiting behaviors of the model after suitable normalization, such as sufficient conditions for almost everywhere convergence and convergence in L^1 and L^2, are discussed when the random control functions are super additive. There have been many achievements in the study of bisexual branching processes in random environments, but the effects of random control and viral infectivity will produce new properties and require some new conditions and methods to study them. Thus, the theory of bisexual branching process in random environment is generalized.

The remainder of this paper is organized as follows. In Section 2, some notations, definitions, and conventions are introduced. Sections 3–6 are devoted to presenting the main results, including the Markov property, probability generating functions, extinction probability, and the limiting behaviors.

2. Preliminaries

We present some notations, basic definitions, and conventions, which will be used in the remaining of the paper.

Let $(\Omega, \mathfrak{F}, P)$ be a given probability space, (Θ, Σ) be a measurable space, and $N = \{0, 1, 2, \ldots\}, N^+ = \{1, 2, \ldots\}$. Let $\overrightarrow{\xi} = \{\xi_n(w), n \in N\}$ be a sequence of random environment, mapping from $(\Omega, \mathfrak{F}, P)$ to (Θ, Σ). Unless otherwise stated, we assume that $\overrightarrow{\xi} = \{\xi_n, n \in N\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. For fixed $n \in N$, set $\{(f_{ni}, m_{ni}), i \in N^+\}$ be a sequence of i.i.d. random variables mapping from $(\Omega, \mathfrak{F}, P)$ to $N \times N$, representing that the ith mating unit in nth generation of a species reproduces f_{ni} females and m_{ni} males. Let $\{P_j(\xi_n), j \in N\}$ denote the probability of that the ith mating unit in nth generation will reproduce j offspring in environment ξ_n. Let $\{I_{f,ni}, i \in N^+, n \in N\}$ and $\{I_{m,ni}, i \in N^+, n \in N\}$ denote two clusters of random variables sequences on N, representing the virus-infected-trial functions of female and male in the ith mating unit in nth generation, respectively. Let $\{a^x(\theta)(1 - a(\theta))^{1-x}, x = 0 \text{ } or \text{ } 1\}$ and $\{b^x(\theta)(1 - b(\theta))^{1-x}, x = 0 \text{ } or \text{ } 1\}$ be the probability distributions of $\{I_{f,ni}, i \in N^+, n \in N\}$ and $\{I_{m,ni}, i \in N^+, n \in N\}$, respectively.

We denote by $\{(F_n, M_n), n \in N^+\}$ a sequence of random variables mapping from $(\Omega, \mathfrak{F}, P)$ to $N \times N$, where F_n and M_n represents the number of females and males in the nth generation, respectively and generate $Z_n = L(F_n, M_n)$ mating units. Here $L(x, y) : N \times N \longrightarrow N$ is called a mating function, which is assumed to be nondecreasing in each argument and satisfy $L(x, 0) = L(0, y) = 0$, $x \in N, y \in N$. We further assume that the reproduction of each mating unit is independent of the other units in the same generation and other generations. Thus the $\{(F_{n+1}, M_{n+1}), n \in N\}$ individuals are reproduced independently by Z_n mating units and generate $Z_{n+1} = L(F_{n+1}, M_{n+1})$ mating units. $\{\phi_n(k) : n, k \geq 0\}$, which is a cluster of i.i.d. random sequence with respect to n with distribution $Q(\xi_n; k, i) = P(\phi_n(k) = i \mid \overrightarrow{\xi}), i \in N$, is defined as the control function. $\phi_n(k) = i$ means that the number of mating units that can participate in the reproduction is i when there are k mating units in nth generation.

Definition 2.1. *Let $\overrightarrow{X} = \{X_n, n \in N\}$ be a sequence of random variables and $\overrightarrow{\xi} = \{\xi_n, n \in N\}$ be a sequence of random environments. For any $x, n \in N^+$, if*

$$P(X_0 = x \mid \overrightarrow{\xi}) = P(X_0 = x \mid \xi_0), P(X_{n+1} = x \mid X_0, X_1, \ldots, X_n, \overrightarrow{\xi}) = P(\xi_n; X_n, x),$$

then $\overrightarrow{X}$ is called a Markov chain in the random environment $\overrightarrow{\xi}$.

Definition 2.2. *If $\{Z_n, n \geq 0\}$ satisfies*

(i) $Z_0 = 1, (F_{n+1}, M_{n+1}) = \sum_{i=1}^{\phi_n(Z_n)} (f_{ni} I_{f,ni}, m_{ni} I_{m,ni}),$
$Z_{n+1} = L(F_{n+1}, M_{n+1}), n \in N;$

(ii) $P(f_{ni} + m_{ni} = j \mid \overrightarrow{\xi}) = P_j(\xi_n), j \in N, i \in N^+,$
$P(I_{f,ni} = x \mid \overrightarrow{\xi}) = a^x(\xi_n)(1 - a(\xi_n))^{1-x}, x = 0 \text{ } or \text{ } 1, n \in N, i \in N^+,$
$P(I_{m,ni} = x \mid \overrightarrow{\xi}) = b^x(\xi_n)(1 - b(\xi_n))^{1-x}, x = 0 \text{ } or \text{ } 1, n \in N, i \in N^+;$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

(iii) *for* $j_{ni}, k_{ni} \in N, 0 \le n \le l, 1 \le i \le s, l \in N, s \in N^+$,

$$P(f_{ni} = j_{ni}, m_{ni} = k_{ni}, 0 \le n \le l, 1 \le i \le s \mid \overrightarrow{\xi}) = \prod_{n=0}^{l} \prod_{i=1}^{s} P(f_{ni} = j_{ni}, m_{ni} = k_{ni} \mid \overrightarrow{\xi});$$

(iv) *for given* $\overrightarrow{\xi}$, $\{(f_{ni}, m_{ni}), n \in N, i \in N^+\}, \{I_{f,ni}, n \in N, i \in N^+\}$ *and* $\{I_{m,ni}, n \in N, i \in N^+\}$ *are independent; furthermore, for given n, each of them is an identically distributed random variable sequence. For given* $\overrightarrow{\xi}$, *per mating unit in the nth generation produces a female with the probability* $\beta(\xi_n)$*; then* $\{Z_n, n \ge 0\}$ *is called a bisexual branching process under the influence of viral infectivity with random control function in the random environment* $\overrightarrow{\xi}$.

We further suppose $I_{m,ni} = 0$ when the male in ith mating unit in nth generation died of a viral infection, that is, the ith mating unit in nth generation lost the ability to reproduce; $I_{m,ni} = 1$ when the male in ith mating unit in nth generation didn't have the virus or was cured of it, that is, the ith mating unit in nth generation can reproduce normally. Likewise, we define $I_{f,ni} = 0$ and $I_{f,ni} = 1$ for the female in ith mating unit in nth generation.

For ease of exposition, we present some notations.

Let $\mathfrak{F}_n(\overrightarrow{\xi}) = \sigma(Z_0, Z_1, \ldots, Z_n, \overrightarrow{\xi}), n \in N$;

For any $k, s \in N$, $B_1 = \{(r_l, a_l, b_l, j_l) \mid \sum_{l=1}^{h} r_l \ge k + s, \sum_{l=1}^{h} (a_l j_l, b_l(r_l - j_l)) = (k, s), r_l \ge 0, a_l, b_l = 0 \; or \; 1, 0 \le j_l \le r_l, l = 1, \ldots, h\}$;

For any $k, l \in N^+$, $B_2 = \{(r_v, a_v, b_v, j_v) \mid \sum_{v=1}^{s} r_v \ge k + l, \sum_{v=1}^{s} (a_v j_v, b_v(r_v - j_v)) = (k, l), r_v \ge 0, a_v, b_v = 0 \; or \; 1, 0 \le j_v \le r_v, v = 1, \cdots, s\}$;

$P_{kj}(\xi_n) = P(f_{ni} I_{f,ni} = k, m_{ni} I_{m,ni} = j \mid \overrightarrow{\xi})$ represents the conditional probability of that k females and j males in the offspring of ith mating unit in nth generation will survival under the influence of the virus.

We further introduce some conventions, which will be used in the proofs of some theorems.

(A_1) To avoid triviality, for any $\theta \in \Theta$, assume that $\beta(\theta), a(\theta), b(\theta) \in (0,1), 0 < P_0(\theta) + P_1(\theta) < 1, a.s.$

(A_2) There exists a constant $c \in (0,1)$ such that $P(cj \le \phi_n(j) \le j \mid \overrightarrow{\xi}) = 1$.

(A_3) For any $n, x, y \in N$, it holds that $L(x, y)$ and $\mathfrak{F}_n(\overrightarrow{\xi})$ are independent and $L(x, y)$ is super additive.

(A_4) When $\overrightarrow{\xi}$ is given, $\{\phi_n(k), n, k \in N\}$, $\{(f_{ni}, m_{ni}), i \ge 1\}_{n \ge 0}$ and $\{I_{f,ni}, i \in N^+, n \in N\}$ are conditional independent.

3. Markov property

Theorem 3.1. $\{Z_n, n \ge 0\}$ *is a Markov chain in the random environment* $\overrightarrow{\xi}$*, and the one-step transition probabilities are*

$$P(\xi_n; i, j) = \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \sum_{h=0}^{\infty} Q(\xi_n; i, h) P(L(k, s) = j) \left\{ \sum_{(r_l, a_l, b_l, j_l) \in B_1} [\prod_{l=1}^{h} P_{r_l}(\xi_n) \cdot a^{a_l}(\xi_n)(1 - a(\xi_n))^{1-a_l} b^{b_l}(\xi_n)(1 - b(\xi_n))^{1-b_l} C_{r_l}^{j_l} \beta^{j_l}(\xi_n)(1 - \beta(\xi_n))^{r_l - j_l}] \right\}.$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

Proof. By the definition of $\{Z_n, n \geq 0\}$, we have

$$P(Z_0 = 1 \mid \overrightarrow{\xi}) = P(Z_0 = 1 \mid \xi_0).$$

From conventions (A_3) and (A_4), and the fact that $\{(f_{ni}I_{f,ni}, m_{ni}I_{m,ni}), i \geq 1\}$ are i.i.d., for any $i_1, i_2, \ldots i_{n-1}, i, j \in N^+$, one can derive that

$$\begin{aligned}
&P(Z_{n+1} = j \mid Z_1 = i_1, Z_2 = i_2, \cdots, Z_{n-1} = i_{n-1}, Z_n = i, \overrightarrow{\xi}) \\
&= \frac{P(Z_{n+1} = j, Z_1 = i_1, Z_2 = i_2, \cdots, Z_{n-1} = i_{n-1}, Z_n = i \mid \overrightarrow{\xi})}{P(Z_1 = i_1, Z_2 = i_2, \cdots, Z_{n-1} = i_{n-1}, Z_n = i \mid \overrightarrow{\xi})} \\
&= \frac{P(L(\sum_{l=1}^{\phi_n(Z_n)} (f_{nl}I_{f,nl}, m_{nl}I_{m,nl})) = j, Z_1 = i_1, \cdots, Z_{n-1} = i_{n-1}, Z_n = i \mid \overrightarrow{\xi})}{P(Z_1 = i_1, \cdots, Z_{n-1} = i_{n-1}, Z_n = i \mid \overrightarrow{\xi})} \\
&= \frac{\sum_{h=0}^{\infty} P(L(\sum_{l=1}^{h} (f_{nl}I_{f,nl}, m_{nl}I_{m,nl})) = j, \phi_n(i) = h, Z_1 = i_1, \cdots, Z_n = i \mid \overrightarrow{\xi})}{P(Z_1 = i_1, \cdots, Z_{n-1} = i_{n-1}, Z_n = i \mid \overrightarrow{\xi})} \\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\sum_{h=0}^{\infty} P(L(k,s) = j, \sum_{l=1}^{h}(f_{nl}I_{f,nl}, m_{nl}I_{m,nl}) = (k,s), \phi_n(i) = h \mid \overrightarrow{\xi}) \\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\sum_{h=0}^{\infty} Q(\xi_n; i, h) P(L(k,s) = j) \left\{ \sum_{(r_l, a_l, b_l, j_l) \in B_1} \left[\prod_{l=1}^{h} P_{r_l}(\xi_n)(1 - a(\xi_n))^{1-a_l} \right.\right. \\
&\left.\left. \cdot a^{a_l}(\xi_n) b^{b_l}(\xi_n)(1 - b(\xi_n))^{1-b_l} C_{r_l}^{j_l} \beta^{j_l}(\xi_n)(1 - \beta(\xi_n))^{r_l - j_l} \right] \right\}.
\end{aligned}$$

□

By Definition 2.1, we have that $\{Z_n, n \geq 0\}$ is a Markov chain in the random environment $\overrightarrow{\xi}$ with the desired one-step transition probabilities.

Theorem 3.2. *$\{(F_n, M_n), n \geq 1\}$ is a Markov chain in random environment $\overrightarrow{\xi}$, and the one-step transition probabilities are*

$$\begin{aligned}
P(\xi_n; (i,j), (k,l)) = &\sum_{s=0}^{\infty}\sum_{h=0}^{\infty} P(L(i,j) = h) Q(\xi_n; h, s) \left\{ \sum_{(r_v, a_v, b_v, j_v) \in B_2} \left[\prod_{v=1}^{s} P_{r_v}(\xi_n) \right.\right. \\
&\left.\left. \cdot a^{a_v}(\xi_n)(1 - a(\xi_n))^{1-a_v} b^{b_v}(\xi_n)(1 - b(\xi_n))^{1-b_v} C_{r_v}^{j_v} \beta^{j_v}(\xi_n)(1 - \beta(\xi_n))^{r_v - j_v} \right] \right\}.
\end{aligned}$$

Proof. By $(F_1, M_1) = (f_{01}I_{f,01}, m_{01}I_{m,01})$, for any $(i,j) \in N^+ \times N^+$, we have

$$\begin{aligned}
P((F_1, M_1) = (i,j) \mid \overrightarrow{\xi}) = &\sum_{\substack{r_1 \geq i+j, a_1, b_1 = 0 \ or 1, \\ (a_1 j_1, b_1(r_1 - j_1)) = (i,j)}} \{P_{r_1}(\xi_0) a^{a_1}(\xi_0)(1 - a(\xi_0))^{1-a_1} \\
&\cdot b^{b_1}(\xi_0)(1 - b(\xi_0))^{1-b_1} C_{r_1}^{j_1} \beta^{j_1}(\xi_0))(1 - \beta(\xi_0))^{r_1 - j_1}\} \\
&= P((F_1, M_1) = (i,j) \mid \xi_0).
\end{aligned}$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

Using conventions (A_3) and (A_4) and the fact that for any $n \in N$, $\{(f_{ni}I_{f,ni}, m_{ni}I_{m,ni}), i \geq 1\}$ are i.i.d., it is deduced that, for $(i_1,j_1), (i_2,j_2), \ldots (i_{n-1},j_{n-1}), (i,j), (k,l) \in N^+ \times N^+$,

$$\begin{aligned}
&P((F_{n+1}, M_{n+1}) = (k,l) \mid (F_n, M_n) = (i,j), \ldots, (F_1, M_1) = (i_1, j_1), \overrightarrow{\xi}) \\
&= \frac{P((F_{n+1}, M_{n+1}) = (k,l), (F_n, M_n) = (i,j), \cdots, (F_1, M_1) = (i_1, j_1) \mid \overrightarrow{\xi})}{P((F_n, M_n) = (i,j), \cdots, (F_1, M_1) = (i_1, j_1) \mid \overrightarrow{\xi})} \\
&= \frac{P(\sum_{v=1}^{\phi_n(L(i,j))} (f_{nv}I_{f,nv}, m_{nv}I_{m,nv}) = (k,l), \cdots, (F_1, M_1) = (i_1, j_1) \mid \overrightarrow{\xi})}{P((F_n, M_n) = (i,j), \cdots, (F_1, M_1) = (i_1, j_1) \mid \overrightarrow{\xi})} \\
&= \sum_{s=0}^{\infty} \sum_{h=0}^{\infty} P(L(i,j) = h, \phi_n(h) = s, \sum_{v=1}^{s} (f_{nv}I_{f,nv}, m_{nv}I_{m,nv}) = (k,l) \mid \overrightarrow{\xi}) \\
&= \sum_{s=0}^{\infty} \sum_{h=0}^{\infty} P(L(i,j) = h) Q(\xi_n; h, s) \{ \sum_{(r_v, a_v, b_v, j_v) \in B_2} [\prod_{v=1}^{s} P_{r_v}(\xi_n)(1 - a(\xi_n))^{1-a_v} \\
&\cdot a^{a_v}(\xi_n) b^{b_v}(\xi_n)(1 - b(\xi_n))^{1-b_v} C_{r_v}^{j_v} \beta^{j_v}(\xi_n)(1 - \beta(\xi_n))^{r_v - j_v}] \}.
\end{aligned}$$

By Definition 2.1, we obtain that $\{(F_n, M_n), n \geq 1\}$ is a Markov chain in the random environment $\overrightarrow{\xi}$ with the desired one-step transition probabilities. □

4. Probability generating functions

For fixed $n \in N$, by the independence of $\{f_{ni}\}, \{m_{ni}\}, \{I_{f,ni}\}$, and $\{I_{m,ni}\}, i \in N^+$, we denote

$$\varphi_{\xi_n}(s,t) = E\{s^{f_{ni}I_{f,ni}} t^{m_{ni}I_{m,ni}} \mid \overrightarrow{\xi}\}, \ \Pi_n(s,t) = E\{s^{F_n} t^{M_n}\}, 0 \leq s, t \leq 1,$$

$$\varphi_{\xi_n}(s) = E\{s^{f_{ni}I_{f,ni} + m_{ni}I_{m,ni}} \mid \overrightarrow{\xi}\}, \ \Psi_n(s) = E(s^{Z_n}), 0 \leq s \leq 1.$$

Lemma 4.1. [13] *For any* $0 \leq s, t \leq 1, n \in N$, *it holds that*

$$\varphi_{\xi_n}(s,t) = \varphi_{\xi_n}(\beta(\xi_n) a(\xi_n) s + (1 - \beta(\xi_n)) b(\xi_n) t).$$

Theorem 4.2. *For any* $0 \leq s, t \leq 1, n \in N$, *it holds that*

$$E(s^{F_{n+1}} t^{M_{n+1}} \mid Z_n, \overrightarrow{\xi}) = \{\varphi_{\xi_n}(s,t)\}^{\phi_n(Z_n)}, \Pi_{n+1}(s,t) = E[(\varphi_{\xi_n}(s,t))^{\phi_n(Z_n)}].$$

Proof. For fixed $n \in N$, by the fact that $\{f_{ni}\}, \{m_{ni}\}, \{I_{f,ni}\}, \{I_{m,ni}\}, i \in N^+$ are independent and each of them is identically distributed, we have, for $0 \leq s, t \leq 1, n, k \in N$,

$$\begin{aligned}
E(s^{F_{n+1}} t^{M_{n+1}} \mid Z_n = k, \overrightarrow{\xi}) &= E\{s^{\sum_{l=1}^{\phi_n(Z_n)} f_{nl}I_{f,nl}} t^{\sum_{l=1}^{\phi_n(Z_n)} m_{nl}I_{m,nl}} \mid Z_n = k, \overrightarrow{\xi}\} \\
&= E\{\prod_{l=1}^{\phi_n(k)} s^{f_{nl}I_{f,nl}} t^{m_{nl}I_{m,nl}} \mid \overrightarrow{\xi}\} \\
&= \prod_{l=1}^{\phi_n(k)} E(s^{f_{nl}I_{f,nl}} t^{m_{nl}I_{m,nl}} \mid \overrightarrow{\xi}) \\
&= \{\varphi_{\xi_n}(s,t)\}^{\phi_n(k)}.
\end{aligned}$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

Thus

$$E(s^{F_{n+1}}t^{M_{n+1}} \mid Z_n, \overrightarrow{\xi}) = \{\varphi_{\xi_n}(s,t)\}^{\phi_n(Z_n)}.$$

Then it follows that

$$\Pi_{n+1}(s,t) = E[E(s^{F_{n+1}}t^{M_{n+1}} \mid Z_n, \overrightarrow{\xi})] = E[(\varphi_{\xi_n}(s,t))^{\phi_n(Z_n)}].$$

□

Corollary 4.3. *For any* $0 \le s,t \le 1, n,k \in N$*, the following equalities hold*

(1) $P(F_{n+1} = 0, M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi}) = [\varphi_{\xi_n}(0,0)]^{\phi_n(k)}$.
(2) $E(s^{F_{n+1}} \mid Z_n = k, \overrightarrow{\xi}) = [\varphi_{\xi_n}(s,1)]^{\phi_n(k)}$.
(3) $P(M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi}) = [\varphi_{\xi_n}(1,0)]^{\phi_n(k)}$.
(4) $\sum_{i=0}^{\infty} P(F_{n+1} = i, M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi})s^i = [\varphi_{\xi_n}(s,0)]^{\phi_n(k)}$.
(5) $E(s^{M_{n+1}} \mid Z_n = k, \overrightarrow{\xi}) = [\varphi_{\xi_n}(1,s)]^{\phi_n(k)}$.

Proof. (1) For any $0 \le s,t \le 1, n,k \in N$, using Theorem 4.2 gives

$$\begin{aligned}
[\varphi_{\xi_n}(s,t)]^{\phi_n(k)} &= E(s^{F_{n+1}}t^{M_{n+1}} \mid Z_n = k, \overrightarrow{\xi}) \\
&= \sum_{i,j\ge 0} s^i t^j P(F_{n+1} = i, M_{n+1} = j \mid Z_n = k, \overrightarrow{\xi}) \\
&= P(F_{n+1} = 0, M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi}) \\
&\quad + \sum_{j\ge 1} s^0 t^j P(F_{n+1} = 0, M_{n+1} = j \mid Z_n = k, \overrightarrow{\xi}) \\
&\quad + \sum_{i\ge 1} s^i t^0 P(F_{n+1} = i, M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi}) \\
&\quad + \sum_{i,j\ge 1} s^i t^j P(F_{n+1} = i, M_{n+1} = j \mid Z_n = k, \overrightarrow{\xi}) \\
&= P(F_{n+1} = 0, M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi}) \\
&\quad + \sum_{j\ge 1} t^j P(F_{n+1} = 0, M_{n+1} = j \mid Z_n = k, \overrightarrow{\xi}) \\
&\quad + \sum_{i\ge 1} s^i P(F_{n+1} = i, M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi}) \\
&\quad + \sum_{i,j\ge 1} s^i t^j P(F_{n+1} = i, M_{n+1} = j \mid Z_n = k, \overrightarrow{\xi}).
\end{aligned}$$

Therefore, we have

$$[\varphi_{\xi_n}(0,0)]^{\phi_n(k)} = P(F_{n+1} = 0, M_{n+1} = 0 \mid Z_n = k, \overrightarrow{\xi}).$$

In a similar way as above, we can obtain (2)–(5) in Corollary 4.3. □

Below the average number of females and males of the $(n+1)$th generation will be given by the probability generating function.

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

Theorem 4.4.

(1) $\frac{\partial \varphi_{\xi_n}(s,1)}{\partial s} |_{s=1}= a(\xi_n)E(f_{ni} \mid \overrightarrow{\xi})$, $\frac{\partial \varphi_{\xi_n}(1,t)}{\partial t} |_{t=1}= b(\xi_n)E(m_{ni} \mid \overrightarrow{\xi})$;
(2) *For any $i \in N^+, n \in N$, if $E[f_{ni}] < \infty$ and $E[m_{ni}] < \infty$, then it holds that*

$$E[F_{n+1}] = E\{\phi_n(Z_n)a(\xi_n)E(f_{ni} \mid \overrightarrow{\xi})\},\ E[M_{n+1}] = E\{\phi_n(Z_n)b(\xi_n)E(m_{ni} \mid \overrightarrow{\xi})\}.$$

Proof. To prove (1), by the definitions of $\varphi_{\xi_n}(s,t)$ and $\Pi_n(s,t)$, one derives

$$\begin{aligned} \varphi_{\xi_n}(s,t) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} s^k t^j P(f_{ni}I_{f,ni} = k, m_{ni}I_{m,ni} = j \mid \overrightarrow{\xi}) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} s^k t^j P_{kj}(\xi_n). \end{aligned} \tag{4.1}$$

Letting $t = 1$ and taking partial derivative with respect to s in (4.1), we have

$$\frac{\partial \varphi_{\xi_n}(s,1)}{\partial s} = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} ks^{k-1}P_{kj}(\xi_n) = \sum_{k=1}^{\infty} ks^{k-1}P(f_{ni}I_{f,ni} = k \mid \overrightarrow{\xi}).$$

Since f_{ni} and $I_{f,ni}, i \geq 1$ are independent when $\overrightarrow{\xi}$ is given, we obtain

$$\frac{\partial \varphi_{\xi_n}(s,1)}{\partial s} |_{s=1}= \sum_{k=1}^{\infty} kP(f_{ni}I_{f,ni} = k \mid \overrightarrow{\xi}) = a(\xi_n)E(f_{ni} \mid \overrightarrow{\xi}). \tag{4.2}$$

Likewise, we have

$$\frac{\partial \varphi_{\xi_n}(1,t)}{\partial t} |_{t=1}= b(\xi_n)E(m_{ni} \mid \overrightarrow{\xi}).$$

Now we proceed to the proof of (2). It follows from Theorem 4.2 that

$$\Pi_{n+1}(s,1) = E\{[\varphi_{\xi_n}(s,1)]^{\phi_n(Z_n)}\}. \tag{4.3}$$

Owing to $E[f_{ni}] < \infty$ and $E[m_{ni}] < \infty, i = 1,2,3,\ldots, n = 0,1,2,3,\ldots$, taking partial derivative with respect to s on both sides of (4.3), letting $s = 1$ and combining with dominated convergence theorem and (4.2), we deduce that

$$\begin{aligned} E(F_{n+1}) &= \frac{\partial \Pi_{n+1}(s,1)}{\partial s} |_{s=1}= \frac{\partial\{E\{[\varphi_{\xi_n}(s,1)]^{\phi_n(Z_n)}\}\}}{\partial s} |_{s=1} \\ &= E\{\frac{\partial[(\varphi_{\xi_n}(s,1))^{\phi_n(Z_n)}]}{\partial s} |_{s=1}\} \\ &= E\{[\phi_n(Z_n)(\varphi_{\xi_n}(s,1))^{\phi_n(Z_n)-1}\varphi'_{\xi_n}(s,1))] |_{s=1}\} \\ &= E\{\phi_n(Z_n)a(\xi_n)E(f_{ni} \mid \overrightarrow{\xi})\}. \end{aligned}$$

Likewise, we have

$$E[M_{n+1}] = E\{\phi_n(Z_n)b(\xi_n)E(m_{ni} \mid \overrightarrow{\xi})\}.$$

□

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

5. Extinction probability

Set $q = \lim_{n\to\infty} P(Z_n = 0)$, then q is the extinction probability of $\{Z_n, n \geq 0\}$. We denote

$$g_{\xi_n}(s) = \varphi_{\xi_n}(\beta(\xi_n)a(\xi_n)s + (1-\beta(\xi_n))b(\xi_n)),$$

$$\overline{g}_{\xi_n}(s) = \varphi_{\xi_n}(\beta(\xi_n)a(\xi_n) + (1-\beta(\xi_n))b(\xi_n)s^d), 0 \leq s \leq 1, n \in N, d \in N^+.$$

Lemma 5.1 ([13]). *For given $\overrightarrow{\xi}$ and any $n \in N, s \in [0,1]$, $g_{\xi_n}(s)$ and $\overline{g}_{\xi_n}(s)$ are probability generating functions.*

Lemma 5.2 ([8]). *Suppose $\overrightarrow{\xi}$ is an i.i.d. random environment, and $h_{\xi_n}(s), s \in [0,1]$ is a probability generating function. If $E[h'_{\xi_0}(1)] \leq 1$, then*

$$\lim_{n\to\infty} E[h_{\xi_0}(h_{\xi_1}(\cdots(h_{\xi_n}(0))\cdots))] = 1.$$

Below we will discuss the extinction conditions for processes under several given mating functions.
(H_1) $L(x,y) = x \cdot \min\{1,y\}$ (polyandry, such as Bronze-winged Jacana Metopidius);
(H_2) $L(x,y) = \min\{x, dy\}$, $d \in N^+$
($d=1$: Monogamy, such as swans; $d \geq 2$: Polygamy, such as mandarin ducks);
(H_3) $L(x,y) = x$ (Parthenogenetic reproduction, such as stick insects).

Theorem 5.3. *Let $L(x,y) = x\min\{1,y\}$. If $E[\beta(\xi_0)a(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1-\beta(\xi_0)b(\xi_0)))] \leq 1$, then $q=1$.*

Proof. By the definition of $L(\cdot,\cdot)$, for any $s \in (0,1)$, we have

$$\begin{aligned}
\Psi_{n+1}(s) = E(s^{Z_{n+1}}) &= E\{\sum_{k=0}^{\infty} P(Z_{n+1} = k \mid \overrightarrow{\xi})s^k\} \\
&= E\{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P(Z_{n+1} = k, Z_n = j \mid \overrightarrow{\xi})s^k\} \\
&= E\{\sum_{j=0}^{\infty}[\sum_{k=0}^{\infty} P(Z_{n+1} = k \mid Z_n = j, \overrightarrow{\xi})s^k]P(Z_n = j \mid \overrightarrow{\xi})\} \\
&= E\{\sum_{j=0}^{\infty} P(Z_n = j \mid \overrightarrow{\xi})[\sum_{k=0}^{\infty} P(F_{n+1} = k \mid Z_n = j, \overrightarrow{\xi})s^k \\
&\quad - \sum_{k=0}^{\infty} P(F_{n+1} = k, M_{n+1} = 0 \mid Z_n = j, \overrightarrow{\xi})s^k + P(M_{n+1} = 0 \mid Z_n = j, \overrightarrow{\xi})]\}.
\end{aligned}$$

From Corollary 4.3, Lemma 4.1, and the definition of $g_{\xi_n}(s)$, we get

$$\begin{aligned}
\Psi_{n+1}(s) =& E\{\sum_{j=0}^{\infty} P(Z_n = j \mid \overrightarrow{\xi})[(\varphi_{\xi_n}(s,1))^{\phi_n(j)} - (\varphi_{\xi_n}(s,0))^{\phi_n(j)} \\
&+ (\varphi_{\xi_n}(1,0))^{\phi_n(j)}]\} \\
=& E\{E[(\varphi_{\xi_n}(s,1))^{\phi_n(Z_n)} \mid \overrightarrow{\xi}] - E[(\varphi_{\xi_n}(s,0))^{\phi_n(Z_n)} \mid \overrightarrow{\xi}]
\end{aligned}$$

$$
\begin{aligned}
&+E[(\varphi_{\xi_n}(1,0))^{\phi_n(Z_n)} \mid \overrightarrow{\xi}]\} \\
=&E[(\varphi_{\xi_n}(s,1))^{\phi_n(Z_n)}] - E[(\varphi_{\xi_n}(s,0))^{\phi_n(Z_n)}] + E[(\varphi_{\xi_n}(1,0))^{\phi_n(Z_n)}] \\
=&E[(g_{\xi_n}(s))^{\phi_n(Z_n)}] - E[(\varphi_{\xi_n}(\beta(\xi_n)a(\xi_n)s))^{\phi_n(Z_n)}] \\
&+E[(\varphi_{\xi_n}(\beta(\xi_n)a(\xi_n)))^{\phi_n(Z_n)}].
\end{aligned}
$$

Using the convention (A_2) and the properties of probability generating function gives

$$
\begin{aligned}
\Psi_{n+1}(s) \geq& E[(g_{\xi_n}(s))^{Z_n}] - E[(\varphi_{\xi_n}(\beta(\xi_n)a(\xi_n)s))^{\phi_n(Z_n)}] \\
&+E[(\varphi_{\xi_n}(\beta(\xi_n)a(\xi_n)))^{\phi_n(Z_n)}] \\
\geq& E[(g_{\xi_n}(s))^{Z_n}] = E[\Psi_n(g_{\xi_n}(s))] \geq E[\Psi_n(g_{\xi_n}(0))],
\end{aligned} \tag{5.1}
$$

that is

$$
\Psi_{n+1}(s) \geq E[\Psi_n(g_{\xi_n}(0))].
$$

By using the recursion of (5.1), we obtain

$$
\Psi_{n+1}(s) \geq E[\Psi_0(g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots)))] = E[g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots))].
$$

According to Lemma 5.2, if $E[g'_{\xi_0}(1)] \leq 1$, that is,

$$
E[\beta(\xi_0)a(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1-\beta(\xi_0)b(\xi_0)))] \leq 1,
$$

then

$$
q = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} \Psi_{n+1}(0) \geq \lim_{n\to\infty} E(g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots))) = 1.
$$

□

Theorem 5.4. *Let* $L(x,y) = \min\{x, dy\}, d \in N^+$. *If* $\min\{E[\beta(\xi_0)a(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1 - \beta(\xi_0))b(\xi_0))], E[d(1-\beta(\xi_0))b(\xi_0)\cdot \varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1-\beta(\xi_0))b(\xi_0))]\} \leq 1$, *then* $q = 1$.

Proof. It follows from the definitions of $L(\cdot,\cdot)$ and $g_{\xi_n}(s)$ and corollary 4.3 that

$$
P(Z_{n+1} \leq k \mid Z_n, \overrightarrow{\xi}) \geq P(F_{n+1} \leq k \mid Z_n, \overrightarrow{\xi})
$$

and

$$
\begin{aligned}
&\frac{E(s^{Z_{n+1}} \mid Z_n, \overrightarrow{\xi})}{1-s} = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} P(Z_{n+1} = j \mid Z_n, \overrightarrow{\xi})s^{j+m} \\
&= \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} P(Z_{n+1} = j \mid Z_n, \overrightarrow{\xi})s^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k} P(Z_{n+1} = j \mid Z_n, \overrightarrow{\xi})s^k \\
&= \sum_{k=0}^{\infty} P(Z_{n+1} \leq k \mid Z_n, \overrightarrow{\xi})s^k \geq \sum_{k=0}^{\infty} P(F_{n+1} \leq k \mid Z_n, \overrightarrow{\xi})s^k \\
&= \frac{E(s^{F_{n+1}} \mid Z_n, \overrightarrow{\xi})}{1-s} = \frac{[\varphi_{\xi_n}(s,1)]^{\phi_n(Z_n)}}{1-s} = \frac{[g_{\xi_n}(s)]^{\phi_n(Z_n)}}{1-s}, s \in (0,1).
\end{aligned}
$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

According convention (A_2) and the properties of the probability generating functions, we obtain

$$\begin{aligned}\Psi_{n+1}(s) &= E\{E(s^{Z_{n+1}} \mid Z_n, \overrightarrow{\xi})\} \geq E\{[g_{\xi_n}(s)]^{\phi_n(Z_n)}\} \\ &\geq E\{[g_{\xi_n}(s)]^{Z_n}\} = E[\Psi_n(g_{\xi_n}(s))] \geq E[\Psi_n(g_{\xi_n}(0))],\end{aligned}$$

that is,

$$\Psi_{n+1}(s) \geq E[\Psi_n(g_{\xi_n}(0))]. \tag{5.2}$$

Using the recursion of (5.2), we obtain

$$\Psi_{n+1}(s) \geq E[\Psi_0(g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots)))] = E[g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots))].$$

According to Lemma 5.2, if $E[g'_{\xi_0}(1)] \leq 1$, that is, $E[\beta(\xi_0)a(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0)+(1-\beta(\xi_0))b(\xi_0))] \leq 1$, then

$$q = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} \Psi_{n+1}(0) \geq \lim_{n\to\infty} E(g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots))) = 1.$$

Similarly, we get $E(s^{Z_{n+1}} \mid Z_n, \overrightarrow{\xi}) \geq [\overline{g}_{\xi_n}(s)]^{Z_n}$, and therefore

$$\begin{aligned}\Psi_{n+1}(s) &= E(s^{Z_{n+1}}) \geq E[\Psi_0(\overline{g}_{\xi_0}(\overline{g}_{\xi_1}(\cdots(\overline{g}_{\xi_n}(0))\cdots)))] \\ &= E[\overline{g}_{\xi_0}(\overline{g}_{\xi_1}(\cdots(\overline{g}_{\xi_n}(0))\cdots))].\end{aligned}$$

Owing to Lemma 5.2, if $E[\overline{g}'_{\xi_0}(1)] \leq 1$, that is, $E[d(1-\beta(\xi_0))b(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1-\beta(\xi_0))b(\xi_0))]\} \leq 1$, then

$$q = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} \Psi_{n+1}(0) \geq \lim_{n\to\infty} E(\overline{g}_{\xi_0}(\overline{g}_{\xi_1}(\cdots(\overline{g}_{\xi_n}(0))\cdots))) = 1.$$

In summary, if $\min\{E[\beta(\xi_0)a(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1-\beta(\xi_0))b(\xi_0))], E[d(1-\beta(\xi_0))b(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1-\beta(\xi_0))b(\xi_0))]\} \leq 1$, then $q = 1$. □

Theorem 5.5. *Let $L(x, y) = x$. If $E[\beta(\xi_0)a(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0) + (1-\beta(\xi_0))b(\xi_0))] \leq 1$, then $q = 1$.*

Proof. By the definition of $L(\cdot,\cdot)$ and Corollary 4.3, we have

$$E(s^{Z_{n+1}} \mid Z_n, \overrightarrow{\xi}) = E(s^{F_{n+1}} \mid Z_n, \overrightarrow{\xi}) = [\varphi_{\xi_n}(s, 1)]^{\phi_n(Z_n)}.$$

From Lemma 5.2, convention (A_2), the definition of $g_{\xi_n}(s)$, and the properties of probability generating functions, it follows that

$$E(s^{Z_{n+1}} \mid Z_n, \overrightarrow{\xi}) = [g_{\xi_n}(s)]^{\phi_n(Z_n)} \geq [g_{\xi_n}(s)]^{Z_n}.$$

Hence

$$\begin{aligned}\Psi_{n+1}(s) &= E\{E(s^{Z_{n+1}} \mid Z_n, \overrightarrow{\xi})\} \geq E\{[g_{\xi_n}(s)]^{\phi_n(Z_n)}\} \\ &\geq E\{[g_{\xi_n}(s)]^{Z_n}\} = E[\Psi_n(g_{\xi_n}(s))] \geq E[\Psi_n(g_{\xi_n}(0))].\end{aligned} \tag{5.3}$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

By the recursion of (5.3), we obtain

$$\Psi_{n+1}(s) \geq E[\Psi_0(g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots)))] = E[g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots))].$$

By Lemma 5.2, if $E[g'_{\xi_0}(1)] \leq 1$, that is, $E[\beta(\xi_0)a(\xi_0)\varphi'_{\xi_0}(\beta(\xi_0)a(\xi_0)+(1-\beta(\xi_0))b(\xi_0))] \leq 1$, then

$$q = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} \Psi_{n+1}(0) \geq \lim_{n\to\infty} E(g_{\xi_0}(g_{\xi_1}(\cdots(g_{\xi_n}(0))\cdots))) = 1.$$

□

6. Limiting behaviors

Definition 6.1. *Suppose $\{Z_n, n \geq 0\}$ is a bisexual branching process affected by viral infectivity and with random control functions in the random environment $\overrightarrow{\xi}$, when the nth generation has k mating units,*

$$r_k(\theta) = \frac{E(Z_{n+1} \mid Z_n = k, \xi_n = \theta)}{k}$$

is defined to be the mean growth rate of per mating unit in nth generation.

Lemma 6.2. *Let $\phi_n(\cdot)$ and $L(\cdot,\cdot)$ be super additive, then for any $n \in N, j \in N^+$, it holds that $\inf_{j\geq 1} r_j(\xi_n)$ exists.*

Proof. By the super additivity of mating function $L(\cdot,\cdot)$ and the condition $P(cZ_n \leq \phi_n(Z_n) \leq Z_n \mid \overrightarrow{\xi}) = 1$, we get

$$\begin{aligned} r_j(\xi_n) &= j^{-1}E(Z_{n+1} \mid Z_n = j, \overrightarrow{\xi}) \\ &= j^{-1}E\{L(\sum_{i=1}^{\phi_n(Z_n)}(f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid Z_n = j, \overrightarrow{\xi}\} \\ &= j^{-1}E\{L(\sum_{i=1}^{\phi_n(j)}(f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid \overrightarrow{\xi}\} \\ &\geq j^{-1}E\{\sum_{i=1}^{\phi_n(j)} L((f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid \overrightarrow{\xi}\}, j \in N^+. \end{aligned}$$

For given $\overrightarrow{\xi}$ and fixed $n \in N$, $\{(f_{ni}I_{f,ni}, m_{ni}I_{m,ni}), i \geq 1\}$ are i.i.d., so we have

$$r_j(\xi_n) \geq cE(L(f_{n1}I_{f,n1}, m_{n1}I_{m,n1}) \mid \overrightarrow{\xi}), j \in N^+.$$

According to the supremum and infimum principle, it holds that $\inf_{j\geq 1} r_j(\xi_n)$ exists.

Writing $R(\xi_n) = \inf_{j\geq 1} r_j(\xi_n)$, we have

$$R(\xi_n) \leq cE(L(f_{n1}I_{f,n1}, m_{n1}I_{m,n1}) \mid \overrightarrow{\xi}).$$

□

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

Lemma 6.3. *Let $\overrightarrow{\xi}$ be an i.i.d. random environment, $L(.,.)$ and $\phi_n(\cdot)$ be super additive, then it holds that*

$$\lim_{j\to\infty} r_j(\xi_n) = \sup_{j\geq 0} r_j(\xi_n) \doteq r(\xi_n), n \in N.$$

Proof. Using the definition of conditional mean growth rate, the super additivity of $\phi_n(\cdot)$ and $L(\cdot,\cdot)$ and the fact that for given $\overrightarrow{\xi}$ and any $n \in N$, $\{(f_{ni}I_{f,ni}, m_{ni}I_{m,ni}), i \geq 1\}$ are i.i.d., it suffices to show that

$$\begin{aligned}
&(k+j)r_{k+j}(\xi_n) = E(Z_{n+1} \mid Z_n = k+j, \overrightarrow{\xi})\\
&= E\{L(\sum_{i=1}^{\phi_n(Z_n)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid Z_n = k+j, \overrightarrow{\xi}\}\\
&\geq E\{L(\sum_{i=1}^{\phi_n(k)+\phi_n(j)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid \overrightarrow{\xi}\}\\
&\geq E\{L(\sum_{i=1}^{\phi_n(k)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid \overrightarrow{\xi}\}\\
&+ E\{L(\sum_{i=\phi_n(k)+1}^{\phi_n(k)+\phi_n(j)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid \overrightarrow{\xi}\}\\
&= E\{L(\sum_{i=1}^{\phi_n(k)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid \overrightarrow{\xi}\} + E\{L(\sum_{i=1}^{\phi_n(j)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid \overrightarrow{\xi}\}\\
&= E\{L(\sum_{i=1}^{\phi_n(Z_n)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid Z_n = k, \overrightarrow{\xi}\}\\
&+ E\{L(\sum_{i=1}^{\phi_n(Z_n)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni})) \mid Z_n = j, \overrightarrow{\xi}\}\\
&= kr_k(\xi_n) + jr_j(\xi_n).
\end{aligned}$$

Namely $kr_k(\xi_n)$ is super additive, so we have

$$\lim_{j\to\infty} r_j(\xi_n) = \sup_{j\geq 0} r_j(\xi_n) \doteq r(\xi_n).$$

□

Corollary 6.4. *For any $n \in N$, it holds that*

$$\prod_{k=0}^{n-1} R(\xi_k) \leq E(Z_n \mid \overrightarrow{\xi}) \leq \prod_{k=0}^{n-1} r(\xi_k). \tag{6.1}$$

Proof. We shall prove this result by induction. For $n = 1$, using the definition of conditional mean growth rate, convention (A_2) and Lemma 6.3 gives

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

$$R(\xi_0) \le cE(L((f_{01}I_{f,01}, m_{01}I_{m,01}) \mid \xi_0)) \le E(Z_1 \mid \overrightarrow{\xi}) = E(Z_1 \mid Z_0 = 1, \overrightarrow{\xi})$$
$$= E(L(\sum_{i=1}^{\phi_0(1)} (f_{0i}I_{f,0i}, m_{0i}I_{m,0i})) \mid \overrightarrow{\xi}) \le r(\xi_0).$$

Namely inequality (6.1) holds for $n=1$. Supposing inequality (6.1) holds for $n = s \in N^+$, below we prove it holds for $n = s + 1$

$$E(Z_{s+1} \mid \overrightarrow{\xi}) = E[E(Z_{s+1} \mid Z_s, \overrightarrow{\xi}) \mid \overrightarrow{\xi}] = E(Z_s r_{Z_s}(\xi_s) \mid \overrightarrow{\xi}) \le E(Z_s r(\xi_s) \mid \overrightarrow{\xi})$$
$$= r(\xi_s)E(Z_s \mid \overrightarrow{\xi}) \le \prod_{k=0}^{s} r(\xi_k).$$

On the other hand, we can also obtain

$$E(Z_{s+1} \mid \overrightarrow{\xi}) = E[Z_s r_{Z_s}(\xi_s) \mid \overrightarrow{\xi}] \ge E(Z_s R(\xi_s) \mid \overrightarrow{\xi})$$
$$= R(\xi_s)E(Z_s \mid \overrightarrow{\xi}) \ge \prod_{k=0}^{s} R(\xi_k),$$

which completes the proof. □

In what follows, we let $S_n = \prod_{k=0}^{n-1} r(\xi_k), I_n = \prod_{k=0}^{n-1} R(\xi_k), n \in N^+, S_0 = I_0 = 1, \widehat{W}_n = S_n^{-1} \cdot Z_n, \overline{W}_n = I_n^{-1} \cdot Z_n, n \in N$.

Theorem 6.5. *Let $\overrightarrow{\xi}$ be an i.i.d. random environment, $\phi_n(\cdot)$ and $L(\cdot,\cdot)$ be super additive, then there exists a nonnegative, finite random variable $\widehat{W}$ such that*

$$\lim_{n\to\infty} \widehat{W}_n = \widehat{W} \quad a.s.$$

Proof. For any $n \in N$, it holds that

$$E(Z_{n+1} \mid \mathfrak{F}_n(\overrightarrow{\xi})) = E(Z_{n+1} \mid Z_n, \overrightarrow{\xi}) = Z_n r_{Z_n}(\xi_n) \le Z_n r(\xi_n).$$

From the definition of $\widehat{W}_n$ and Lemma 6.3, we deduce that

$$E(\widehat{W}_{n+1} \mid \mathfrak{F}_n(\overrightarrow{\xi})) = [\prod_{k=0}^{n} r(\xi_k)]^{-1} E(Z_{n+1} \mid \mathfrak{F}_n(\overrightarrow{\xi})) \le [\prod_{k=0}^{n-1} r(\xi_k)]^{-1} Z_n = \widehat{W}_n.$$

Namely $\{\widehat{W}_n, \mathfrak{F}_n(\overrightarrow{\xi}), n \ge 0\}$ is a nonnegative supermartingale. Since

$$E(\widehat{W}_{n+1}) = E[E(\widehat{W}_{n+1} \mid \mathfrak{F}_n(\overrightarrow{\xi}))] \le E(\widehat{W}_n) \le \cdots \le E(\widehat{W}_0) = 1 \ , \ n \in N,$$

according to the martingale convergence theorem, there exists a nonnegative, finite random variable $\widehat{W}$ such that

$$\lim_{n\to\infty} \widehat{W}_n = \widehat{W} \quad a.s.$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

The proof ends. □

In what follows, for any $n \in N, j \in N^+$, we denote $\sigma_j(\xi_n) \doteq j^{-2}Var(Z_{n+1} \mid Z_n = j, \overrightarrow{\xi}), d_j(\xi_n) \doteq E(Z_{n+1}^2 \mid Z_n = j, \overrightarrow{\xi})$, then we have the following.

Lemma 6.6. *For any $n \in N$, let $\phi_n(\cdot)$ and $L(\cdot,\cdot)$ be super additive, then there exists a $\sigma(\xi_n)$ such that $\sigma_j(\xi_n) \le \sigma(\xi_n), j \in N^+$ when $\overrightarrow{\xi}$ is given.*

Proof. From Definition (2.2), the super additivity of $L(\cdot,\cdot)$ and $\phi_n(\cdot)$, and the fact that $\{(f_{ni}I_{f,ni}, m_{ni}I_{m,ni}), i \ge 1\}$ are i.i.d. when n is given, it follows that

$$\begin{aligned} E(Z_{n+1}^2 \mid Z_n = k+j, \overrightarrow{\xi}) &= E\{[L(\sum_{i=1}^{\phi_n(k+j)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni}))]^2 \mid \overrightarrow{\xi}\} \\ &\ge E\{[L(\sum_{i=1}^{\phi_n(k)+\phi_n(j)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni}))]^2 \mid \overrightarrow{\xi}\} \\ &\ge E\{[L(\sum_{i=1}^{\phi_n(k)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni}))]^2 \mid \overrightarrow{\xi}\} \\ &\quad + E\{[L(\sum_{i=1}^{\phi_n(j)} (f_{ni}I_{f,ni}, m_{ni}I_{m,ni}))]^2 \mid \overrightarrow{\xi}\} \\ &= E(Z_{n+1}^2 \mid Z_n = k, \overrightarrow{\xi}) + E(Z_{n+1}^2 \mid Z_n = j, \overrightarrow{\xi}). \end{aligned}$$

So $d_j(\xi_n) = E(Z_{n+1}^2 \mid Z_n = j, \overrightarrow{\xi})$ is super additive, then it holds

$$\lim_{j\to\infty} j^{-1}d_j(\xi_n) = \sup_{j>0} j^{-1}d_j(\xi_n) = \sup_{j>0} j^{-1}E(Z_{n+1}^2 \mid Z_n = j, \overrightarrow{\xi}).$$

Since $j^{-2}d_j(\xi_n) = j^{-2}E(Z_{n+1}^2 \mid Z_n = j, \overrightarrow{\xi}) \le j^{-1}E(Z_{n+1}^2 \mid Z_n = j, \overrightarrow{\xi})$, then $\sigma_j(\xi_n) = j^{-2}d_j(\xi_n) - r_j^2(\xi_n) \le \sup_{j>0} j^{-1}E(Z_{n+1}^2 \mid Z_n = j, \overrightarrow{\xi}) - R^2(\xi_n) \doteq \sigma(\xi_n)$, $j \in N^+$, which completes the proof of Lemma 6.6. □

Lemma 6.7 ([11]). *Let $R^+ = (0, +\infty)$. For given $\overrightarrow{\xi}$, it follows that*

(1) *For any given $n \in N$, if $\{A_j(\xi_n), j \ge 1\}$ is a nonincreasing sequence, then there exists a nonincreasing function $V_{\xi_n}(\cdot)$ on R^+ such that $V_{\xi_n}(j) \ge A_j(\xi_n), j \in N^+$ and $V_{\xi_n}^*(x) = x \cdot V_{\xi_n}(x), x \ge 1$ and $\widehat{V}_{\xi_n}^*(x) = x \cdot V_{\xi_n}^2(x^{\frac{1}{2}}), x \ge 1$ are concave.*
(2) *For any given $n \in N$, if $\{A_j(\xi_n), j \ge 1\}$ is a nondecreasing sequence, then there exists a nondecreasing function $\psi_{\xi_n}(\cdot)$ on R^+ such that $\psi_{\xi_n}(j) \le A_j(\xi_n), j \in N^+$ and $\psi_{\xi_n}^*(x) = x \cdot \psi_{\xi_n}(x), x > 0$ is convex.*

Theorem 6.8. *Let $\overrightarrow{\xi}$ be an i.i.d. random environment, $\phi_n(\cdot)$ and $L(\cdot,\cdot)$ be super additive. If*

$$\sum_{k=0}^{\infty} E[r^{-2}(\xi_k)\sigma(\xi_k)] < \infty$$

and

$$\sum_{k=0}^{\infty} E[1 - r^{-1}(\xi_k)\psi_{\xi_k}(E(Z_k \mid \overrightarrow{\xi}))] < \infty,$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

then it holds that $\{\widehat{W}_n, n \geq 0\}$ *converges in* L^1*, as* $n \to \infty$*, to a nonnegative finite random variable* $\widehat{W}$ *with* $P(\widehat{W} > 0) > 0$*.*

Proof. For any $n \in N$, by the definition of $\widehat{W}_n$ and Lemmas 6.3 and 6.6, we have

$$\begin{aligned} E(\widehat{W}_{n+1}^2 \mid \overrightarrow{\xi}) &= S_{n+1}^{-2} E[E(Z_{n+1}^2 \mid \overrightarrow{\xi}) \mid \overrightarrow{\xi}] = S_{n+1}^{-2} E(d_{Z_n}(\xi_n) \mid \overrightarrow{\xi}) \\ &= S_{n+1}^{-2} E[Z_n^2(\sigma_{Z_n}(\xi_n) + r_{Z_n}^2(\xi_n)) \mid \overrightarrow{\xi}] \\ &= E(\widehat{W}_n^2 \mid \overrightarrow{\xi})(\sigma_{Z_n}(\xi_n) r^{-2}(\xi_n) + r_{Z_n}^2(\xi_n) r^{-2}(\xi_n)) \\ &\leq E(\widehat{W}_n^2 \mid \overrightarrow{\xi})(1 + \sigma(\xi_n) r^{-2}(\xi_n)), \end{aligned}$$

namely,

$$E(\widehat{W}_{n+1}^2 \mid \overrightarrow{\xi}) \leq E(\widehat{W}_n^2 \mid \overrightarrow{\xi})(1 + \sigma(\xi_n) r^{-2}(\xi_n)). \tag{6.2}$$

By the recursion of (6.2), we have

$$E(\widehat{W}_{n+1}^2 \mid \overrightarrow{\xi}) \leq \prod_{k=0}^{n} (1 + \sigma(\xi_k) r^{-2}(\xi_k)) \leq \prod_{k=0}^{\infty} (1 + \sigma(\xi_k) r^{-2}(\xi_k)).$$

Since $\sum_{k=0}^{\infty} E[r^{-2}(\xi_k)\sigma(\xi_k)] < \infty$ and $\overrightarrow{\xi}$ is i.i.d., it follows that

$$E(\widehat{W}_{n+1}^2) = E[E(\widehat{W}_{n+1}^2 \mid \overrightarrow{\xi})] \leq \prod_{k=0}^{\infty} [1 + E(\sigma(\xi_k) r^{-2}(\xi_k))] < \infty.$$

Namely $\{\widehat{W}_n, n \geq 0\}$ is bounded in L^2, and therefore $\{\widehat{W}_n, n \geq 0\}$ is uniformly integrable. It follows from Theorem 6.5 that

$$\lim_{n \to \infty} E\widehat{W}_n = E(\lim_{n \to \infty} \widehat{W}_n) = E\widehat{W} < \infty.$$

Below we prove $P(\widehat{W} > 0) > 0$. By the fact that $r_{Z_n}(\xi_n)$ is nondecreasing and Lemma 6.7, it suffices to show that there exists a nondecreasing function $\psi_{\xi_n}(\cdot)$ and a convex function $\psi_{\xi_n}^*(x) = x \cdot \psi_{\xi_n}(x)$ on R^+ such that

$$\begin{aligned} E(\widehat{W}_{n+1} \mid \overrightarrow{\xi}) &= S_{n+1}^{-1} E[E(Z_{n+1} \mid \overrightarrow{\xi}) \mid \overrightarrow{\xi}] = S_{n+1}^{-1} E(Z_n r_{Z_n}(\xi_n) \mid \overrightarrow{\xi}) \\ &\geq S_{n+1}^{-1} E[Z_n \psi_{\xi_n}(Z_n)] = S_{n+1}^{-1} E[\psi_{\xi_n}^*(Z_n) \mid \overrightarrow{\xi}] \\ &\geq S_{n+1}^{-1} \psi_{\xi_n}^*[E(Z_n \mid \overrightarrow{\xi})] = S_{n+1}^{-1} E(Z_n \mid \overrightarrow{\xi}) \psi_{\xi_n}[E(Z_n \mid \overrightarrow{\xi})] \\ &= r^{-1}(\xi_n) E(\widehat{W}_n \mid \overrightarrow{\xi}) \psi_{\xi_n}[E(Z_n \mid \overrightarrow{\xi})], \end{aligned}$$

namely,

$$E(\widehat{W}_{n+1} \mid \overrightarrow{\xi}) \geq r^{-1}(\xi_n) E(\widehat{W}_n \mid \overrightarrow{\xi}) \psi_{\xi_n}[E(Z_n \mid \overrightarrow{\xi})]. \tag{6.3}$$

The recursion of (6.3) implies

$$E(\widehat{W}_{n+1} \mid \overrightarrow{\xi}) \geq \prod_{k=0}^{n} r^{-1}(\xi_k) \psi_{\xi_k}[E(Z_k \mid \overrightarrow{\xi})], n \in N.$$

Combining $\sum_{k=0}^{\infty} E[1-r^{-1}(\xi_k)\psi_{\xi_k}(E(Z_k \mid \overrightarrow{\xi}))] < \infty$ with Lemma 6.7, one derives that

$$\lim_{n\to\infty} E(\widehat{W}_n \mid \overrightarrow{\xi}) \geq \prod_{k=0}^{\infty} r^{-1}(\xi_k)\psi_{\xi_k}[E(Z_k \mid \overrightarrow{\xi})] > 0.$$

By the Dominated convergence theorem, we have

$$E\widehat{W} = \lim_{n\to\infty} E\widehat{W}_n = E[\lim_{n\to\infty} E(\widehat{W}_n \mid \overrightarrow{\xi})] > 0,$$

which completes the proof. □

Let $\varepsilon_k(\xi_n) = r(\xi_n) - r_k(\xi_n) > 0, k \in N$. By Lemma 6.7 (i) and the fact that $r_k(\xi_n), k \in N$ is nondecreasing, there exists a nonincreasing function $V_{\xi_n}(\cdot)$ satisfying Lemma 6.7 (i) such that $V_{\xi_n}(k) \geq \varepsilon_k(\xi_n), k \in N^+$ and $V^*_{\xi_n}(x) = xV^2_{\xi_n}(x^{\frac{1}{2}}), x \geq 1$ are convex.

Theorem 6.9. *If* $\sum_{n=0}^{\infty} E[r^{-2}(\xi_n)\sigma(\xi_n)] < \infty$ *and* $\sum_{n=0}^{\infty}\{E[r^{-2}(\xi_n)V^2_{\xi_n}(I_n)]\}^{\frac{1}{2}} < \infty$, *then* $\{\widehat{W}_n, n \in N\}$ *converges in* L^2 *to a nonnegative random variable* $\widehat{W}$.

Proof. By Theorem 6.8, if $\sum_{n=0}^{\infty} E[r^{-2}(\xi_n)\sigma(\xi_n)] < \infty$, then $\{\widehat{W}_n, n \in N\}$ is bounded in L^2, that is, there exists a constant $C \geq 0$ such that $E\widehat{W}_n^2 \leq C, n \in N$. Since $\{\widehat{W}_n, \mathfrak{F}_n(\overrightarrow{\xi}), n \in N\}$ is a nonnegative supermartingale, it follows from the Doob martingale decomposition theorem that $\widehat{W}_n = Y_n - T_n, n \in N$, where $\{Y_n, \mathfrak{F}_n(\overrightarrow{\xi}), n \in N\}$ is a martingale with $T_0 = 0$ and

$$T_n = \sum_{k=0}^{n-1}(\widehat{W}_k - E(\widehat{W}_{k+1} \mid \mathfrak{F}_n(\overrightarrow{\xi}))) = \sum_{k=0}^{n-1} \widehat{W}_k r^{-1}(\xi_k)\varepsilon_{Z_k}(\xi_k). \quad a.s.$$

Below we show that $\{T_n, n \in N\}$ is bounded in L^2

$$\begin{aligned}
\| T_n \|_2 &= \| \sum_{k=0}^{n-1} \widehat{W}_k r^{-1}(\xi_k)\varepsilon_{Z_k}(\xi_k) \|_2 \\
&\leq \sum_{k=0}^{n-1} \| \widehat{W}_k r^{-1}(\xi_k)\varepsilon_{Z_k}(\xi_k) \|_2 \\
&= \sum_{k=0}^{n-1} \{E[\widehat{W}_k^2 r^{-2}(\xi_k)\varepsilon^2_{Z_k}(\xi_k)]\}^{\frac{1}{2}} \\
&\leq \sum_{k=0}^{\infty} \{E[E(\widehat{W}_k^2 r^{-2}(\xi_k)\varepsilon^2_{Z_k}(\xi_k) \mid \overrightarrow{\xi})]\}^{\frac{1}{2}}.
\end{aligned}$$

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

Lemma 6.7 implies that $xV^2_{\xi_n}(x^{\frac{1}{2}})$ is convex, and we deduce from Jensens inequality and Corollary 6.4 that

$$\begin{aligned}
\| T_n \|_2 &\leq \sum_{k=0}^{\infty} \{E[S_{k+1}^{-2} E(Z_k^2 \varepsilon^2_{Z_k}(\xi_k) \mid \overrightarrow{\xi})]\}^{\frac{1}{2}} \leq \sum_{k=0}^{\infty} \{E[S_{k+1}^{-2} E(Z_k^2 V^2_{\xi_k}(Z_k) \mid \overrightarrow{\xi})]\}^{\frac{1}{2}} \\
&\leq \sum_{k=0}^{\infty} \{E[S_{k+1}^{-2} (E(Z_k \mid \overrightarrow{\xi}))^2 V^2_{\xi_k}(E(Z_k \mid \overrightarrow{\xi})]\}^{\frac{1}{2}} \\
&= \sum_{k=0}^{\infty} \{E[\widehat{W}_k^2 r^{-2}(\xi_k) V^2_{\xi_k}(E(Z_k \mid \overrightarrow{\xi})]\}^{\frac{1}{2}} \\
&\leq \sum_{k=0}^{\infty} \sqrt{C} \{E[r^{-2}(\xi_k) V^2_{\xi_k}(I_k)]\}^{\frac{1}{2}}.
\end{aligned}$$

An immediate consequence of the assumptions of Theorem 6.9 is that $\| \quad T_n \quad \|_2 \leq \sum_{k=0}^{\infty} \sqrt{C}\{E[r^{-2}(\xi_k)V^2_{\xi_k}(I_k)]\}^{\frac{1}{2}} < \infty$, namely $\{T_n, n \in N\}$ is bounded in L^2, and therefore $\{T_n, n \in N\}$ converges in L^2 . Since $Y_n = \widehat{W}_n + T_n$, then $\{Y_n, n \in N\}$ is a martingale bounded in L^2. It follows that from the martingale convergence theorem that $\{Y_n, n \in N\}$ converges in L^2. In summary, we get that $\{\widehat{W}_n, n \in N\}$ converges to $\widehat{W}$ in L^2. □

Theorem 6.10. *If* $\sum_{k=0}^{\infty}[E(r(\xi_k)R^{-1}(\xi_k)) - 1] < \infty$*, then there exists a nonnegative finite random variable* $\overline{W}$ *such that*

$$\lim_{n\to\infty} \overline{W}_n = \overline{W} \quad a.s.$$

Proof. By Definition 2.2, we have

$$\begin{aligned}
E(\overline{W}_{n+1} \mid \mathfrak{F}_n(\overrightarrow{\xi})) &= I_{n+1}^{-1} E(Z_{n+1} \mid \mathfrak{F}_n(\overrightarrow{\xi})) = I_{n+1}^{-1} E(Z_{n+1} \mid Z_n, \xi_n) \\
&= I_{n+1}^{-1} Z_n r_{Z_n}(\xi_n) \geq \overline{W}_n \quad a.s.
\end{aligned}$$

Namely $\{\overline{W}_{n+1}, \mathfrak{F}_n(\overrightarrow{\xi}), n \geq 0\}$ is a nonnegative submartingale. Corollary 6.4 implies

$$E(\overline{W}_n \mid \overrightarrow{\xi}) = I_n^{-1} E(Z_n \mid \overrightarrow{\xi}) \leq \prod_{k=0}^{n-1} r(\xi_k) R^{-1}(\xi_k).$$

Since $\overrightarrow{\xi}$ is an i.i.d. random environment and $R(\xi_n) \leq r(\xi_n)$, we have

$$E(\overline{W}_n) = E[E(\overline{W}_n \mid \overrightarrow{\xi})] \leq \prod_{k=0}^{n-1} E[r(\xi_k) R^{-1}(\xi_k)] \leq \prod_{k=0}^{\infty} E[r(\xi_k) R^{-1}(\xi_k)].$$

From $\sum_{k=0}^{\infty}[E(r(\xi_k)R^{-1}(\xi_k)) - 1] < \infty$, we have $\sup_{n\geq 0} E(\overline{W}_n) < \infty$. Thus, by the Doob convergence theorem, it follows that there exists a nonnegative finite random variable $\overline{W}$ such that $\lim_{n\to\infty} \overline{W}_n = \overline{W} \quad a.s.$ □

7. Conclusion

So far, there are few results on the the model of bisexual branching processes affected by viral infectivity and with random control functions in i.i.d. random environments. In this paper, based on the model, we

https://doi.org/10.1017/S0269964824000287 Published online by Cambridge University Press

discussed the Markov property, the relations of the probability generating functions of this model, and some sufficient conditions for process extinction under common mating functions as well as the limiting behaviors. The results of classical bisexual branching process are generalized and its application scope is broadened.

Acknowledgments. The authors want to express their sincere thanks to the referee for his or her valuable remarks and suggestions, which made this paper more readable.

Funding statement. This survey is supported by the National Natural Science Foundation of China (Grant No. 11971034) and the Natural Science Foundation of Anhui Universities (Grant Nos. 2022AH051370 and 2023AH052234).

Competing interest. The authors declare no competing interest.

References

[1] Alsmeyer, G. & Rösler, U. (1996). The bisexual Galton-Watson process with promiscuous mating: Extinction probabilities in the super-critical case. *Ann. Appl. Probab.* 6(3): 922–939.
[2] Daley, D. (1968). Extinction conditions for certain bisexual Galton-Watson processes. *Z Wahr.* 9(4): 315–322.
[3] González, M. & Molina, M. (1996). On the limit behaviour of a superadditive bisexual Galton-Watson branching process. *J. Appl. Probab.* 33(4): 960–967.
[4] González, M. & Molina, M. (1997). On the L^2 convergence of a superadditive bisexual Galton-Watson branching process. *J. Appl. Probab.* 34(3): 575–582.
[5] González, M., Molina, M., & Mota, M. (2001). Nonparametric estimation of the offspring distribution and the mean vector for a bisexual Galton-Watson branching process. *Commun. Stat-Theor. M.* 30(4): 497–516.
[6] González, M., Molina, M., & Mota, M. (2001). On the limit behaviour of a supercritical bisexual Galton-Watson branching process with immigration of mating units. *Stoch. Anal. Appl.* 19(6): 933–943.
[7] Gonzalez, M., Molina, M., & Mota, M. (2002). Bisexual Galton-Watson branching process with immigration of females and males: Asymptotic behaviour. *Markov Processes and Related Fields* 21(4): 229–254.
[8] Li, Y.Q., Hu, Y.L., & Zhang, Y. (2015). Limiting behaviors of bisexual branching processes in random environments. *Sci. China. Ser.* 45(5): 611–622.
[9] Li, Y.Q., Xiao, S., & Peng, Z.H. (2020). Moment convergence criteria for bisexual branching process in random environments. *Acta Math. Appl. Sin.* 43(4): 639–653.
[10] Ma, S.X. (2006). Bisexual Galton-Watson branching processes in random environments. *Acta Math. Appl. Sin. Engl. Ser.* 22(3): 419–428.
[11] Molina, M. & Mota, M. (2004). Limiting behavior for superadditive bisexual Galton-Watson processes in varying environments. *Sociedad de Estadlstica e Investigacion Operativa.* 13(2): 481–499.
[12] Ren, M. & Wang, J.J. (2020). Limiting behaviors of the bisexual branching processes with migration in random environments. *J. Jilin. Inst. Chem. Technol.* 37(11): 85–90.
[13] Ren, M., Wang, J.J., & Wang, Y.P. (2021). Probability generating functions and extinction conditions for the bisexual branching processes affected by infectivity of virus in random environments. *Wuhan Univ. (Nat. Sci. Ed.)* 67(3): 263–269.
[14] Song, M.Z. & Shao, J. (2020). The limit properties of the conditional mean growth rate for a kind of bisexual branching process. *J. Eng. Math.* 37(3): 314–324.
[15] Song, M.Z. & Wu, Y.F. (2015). Limiting behavior of the bisexual Galton-Watson branching process with random control functions in random environments. *Math. Appl.* 28(1): 33–40.
[16] Song, M.Z., Wu, Y.F., & Hu, S.P. (2015). The limited properties of the bisexual branching process with population-size-dependent immigration in random environments. *J. Zhejiang. Univ-Sci.* 42(3): 282–285.

Cite this article: Ren M. and Zhang G. (2025). The bisexual branching processes affected by viral infectivity and with random control functions in random environments. *Probability in the Engineering and Informational Sciences* 1–18. https://doi.org/10.1017/S0269964824000287