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Torsion in classifying spaces of gauge groups

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We determine when the integral homology of the classifying space of a $PU(n)$-gauge group over the sphere S^2 has torsion.

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1. Introduction

For a topological space X, we say X has torsion if its integral homology does. Let G be a compact connected Lie group. The cohomology of the connected Lie group G, its loop space ΩG and its classifying space BG has been studied by many mathematicians after the pioneering works of Hopf, Bott and Borel. The loop space ΩG has no torsion. The classifying space BG has torsion if and only if G does.

Let $P \to X$ be a principal G-bundle over a paracompact space X. Then, there is a classifying map $f\colon X \to BG$. The group of bundle automorphisms covering the identity on X is called the gauge group $\mathcal{G}(P)$. The classifying space $B\mathcal{G}(P)$ is homotopy equivalent to the path-component of the mapping space $\mathrm{Map}(X, BG)$ containing the classifying map f as in [1, 2]. If $X = S^1$, since $\pi_1(BG) = \{0\}$, the mapping space $\mathrm{Map}(S^1, BG)$ is path-connected and it has torsion if and only if G does. If $X = S^2$, since $\pi_2(BG)$ might not be zero, the mapping space $\mathrm{Map}(S^2, BG)$ may not be path-connected. The path-component that contains the trivial map is homotopy equivalent to the classifying space of the gauge group of the trivial G-bundle over S^2, and it has torsion if and only if G does. However, the situation is different for other path-components that are homotopy equivalent to classifying spaces of gauge groups of non-trivial G-bundles.

Let $SO(n)$ be the special orthogonal groups. Classification of $SO(n)$-bundles over S^2 is determined by the Stiefel–Whitney class $w_2 \in \mathbb{Z}/2 = \{0, 1\} = \pi_2(BSO(n))$. The path-component of the mapping space corresponding to the non-trivial Stiefel–Whitney class is homotopy equivalent to the classifying space of the gauge group of the non-trivial $SO(n)$-bundle over S^2. Tsukuda [5] showed that it has no torsion for $n = 3$. Minowa [3] proved that it has no torsion for $n = 3$, 4 and torsion for $n \geqslant 5$.

The special orthogonal group $SO(3)$ could be regarded as the projective unitary group $PU(2) = U(2)/S^1$. In this paper, we generalize Tsukuda's result for projective unitary groups $PU(n)$, $n \geqslant 2$ and determine when the classifying space of a $PU(n)$-gauge group over the sphere S^2 has torsion.

Throughout the rest of this paper, let n be an integer greater than or equal to 2. The second homotopy group $\pi_2(BPU(n))$ is isomorphic to the cyclic group $\mathbb{Z}/n$. We identify the cyclic group $\mathbb{Z}/n$ with its complete set of representatives $\{0, 1, \ldots, n-1\}$. Let k be an element in

$$\pi_2(BPU(n)) = \mathbb{Z}/n = \{0, 1, \ldots, n-1\}.$$

Let us denote by $\mathrm{Map}_k(S^2, BPU(n))$ the path-component of the mapping space $\mathrm{Map}(S^2, BPU(n))$ containing maps in the homotopy class k. Let p be a prime number. Unless explicitly stated, $H^*(X)$ is the mod p cohomology of the topological space X. The following is the p-local form of our result.

THEOREM 1.1. *The following holds for* $\mathrm{Map}_k(S^2, BPU(n))$.

(1) *If* $n \not\equiv 0 \mod (p)$, *it has no* p*-torsion.*

(2) *If* $n \equiv 0 \mod (p)$ *and* $k \not\equiv 0 \mod (p)$, *it has no* p*-torsion.*

(3) *If* $n \equiv 0 \mod (p)$ *and* $k \equiv 0 \mod (p)$, *it has* p*-torsion.*

As an immediate consequence of theorem 1.1, we obtain the following global form of our result.

COROLLARY 1.2. *The topological space* $\mathrm{Map}_k(S^2, BPU(n))$ *has no torsion if and only if* k *is relatively prime to* n.

In particular, for $n \geqslant 2$, the topological space $\mathrm{Map}_1(S^2, BPU(n))$ has no torsion even though the underlying Lie group $PU(n)$ has torsion.

This paper is organized as follows. In § 2, we show the existence of p-torsion in $\mathrm{Map}_k(S^2, BPU(n))$ is equivalent to the triviality of certain induced homomorphism in the mod p cohomology. Section 3 recalls the free double suspension in Takeda [**4**] and its elementary properties. Section 4 collects some elementary facts on the mod p cohomology of $BU(n)$. In § 5, we prove theorem 1.1 assuming lemma 5.6 on an $n \times n$ matrix B. In § 6, we prove lemma 5.6.

The author would like to thank Yuki Minowa for his talk on [**3**] at the Homotopy Theory Symposium at the Osaka Metropolitan University on 5 November 2023. This work was inspired by his talk.

2. Torsion

In this section, we show that the existence of p-torsion of a path-component is equivalent to the triviality of certain induced homomorphism.

Let us fix a fibre bundle $BU(n) \to BPU(n)$ induced by the obvious projection map $U(n) \to PU(n)$. We denote the inclusion map of its fibre by $\phi\colon BS^1 \to BU(n)$. It is a map induced by the obvious inclusion map $S^1 \to U(n)$ where S^1 consists of the scalar matrices in the unitary group $U(n)$. Consider the commutative diagram induced by the fibre bundle $BU(n) \to BPU(n)$.

$$\begin{array}{ccccc}
F_0 & \longrightarrow & \Omega^2_k BU(n) & \xrightarrow{\simeq} & \Omega^2_k BPU(n) \\
\downarrow & & \downarrow{\scriptstyle \iota_k} & & \downarrow \\
F & \xrightarrow{\varphi} & \mathrm{Map}_k(S^2, BU(n)) & \longrightarrow & \mathrm{Map}_k(S^2, BPU(n)) \\
{\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\
BU(1) & \xrightarrow{\phi} & BU(n) & \longrightarrow & BPU(n)
\end{array}$$

Both vertical maps in the bottom-right square are evaluation maps at the base point of S^2, and all maps in the bottom-right square are fibrations. Moreover, all horizontal and vertical sequences are fibre sequences. In particular, $\Omega^2_k BU(n)$ and $\Omega^2_k BPU(n)$ are fibres of evaluation maps. Since

$$\Omega^2_k BU(n) \to \Omega^2_k BPU(n)$$

is a homotopy equivalence, the fibre F_0 is contractible, and the map $F \to BS^1$ is also a homotopy equivalence.

The goal of this section is to prove the following proposition.

PROPOSITION 2.1. *The following are equivalent.*

(1) *The topological space* $\mathrm{Map}_k(S^2, BPU(n))$ *has* p*-torsion.*

(2) *The mod* p *cohomology of* $\mathrm{Map}_k(S^2, BPU(n))$ *has a non-zero odd degree element.*

(3) *The induced homomorphism* $\varphi^*\colon H^2(\mathrm{Map}_k(S^2, BU(n))) \to H^2(F)$ *is zero.*

To establish the equivalence of (1) and (2) in proposition 2.1, we use the following lemma.

LEMMA 2.2. *Let* X *be a topological space. Suppose that the integral homology groups* $H_i(X;\mathbb{Z})$ *are finitely generated abelian groups for all* i*, and the rational cohomology of* X *has no non-zero odd degree element. Then, the mod* p *cohomology* $H^*(X;\mathbb{Z}/p)$ *has a non-zero odd degree element if and only if* X *has* p*-torsion.*

Proof. First, we prove that the assumptions of lemma 2.2 imply that $H_{2j+1}(X;\mathbb{Z})$ is a finite abelian group for all j. By the universal coefficient theorem, we have an

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isomorphism

$$H^{2j+1}(X;\mathbb{Q}) \simeq \mathrm{Ext}^1(H_{2j}(X;\mathbb{Z}),\mathbb{Q}) \oplus \mathrm{Hom}(H_{2j+1}(X;\mathbb{Z}),\mathbb{Q}).$$

By the assumption that the rational cohomology of X has no non-zero odd degree element, we have

$$\mathrm{Hom}(H_{2j+1}(X;\mathbb{Z}),\mathbb{Q}) = \{0\}.$$

By the assumption that the integral homology groups $H_i(X;\mathbb{Z})$ are finitely generated, $H_{2j+1}(X;\mathbb{Z})$ is a finite abelian group.

Next, we show that if X has p-torsion, then $H^{2j+1}(X;\mathbb{Z}/p)$ is non-trivial for some j. By the universal coefficient theorem, we have an isomorphism

$$H^{2j+1}(X;\mathbb{Z}/p) \simeq \mathrm{Ext}^1(H_{2j}(X;\mathbb{Z}),\mathbb{Z}/p) \oplus \mathrm{Hom}(H_{2j+1}(X;\mathbb{Z}),\mathbb{Z}/p).$$

If X has p-torsion, $H_{2j+1}(X;\mathbb{Z})$ or $H_{2j}(X;\mathbb{Z})$ has p-torsion for some j. Therefore, $H^{2j+1}(X;\mathbb{Z}/p)$ is non-trivial.

Finally, we show that if $H^{2j+1}(X;\mathbb{Z}/p)$ is non-trivial for some j, X has p-torsion. By the universal coefficient theorem, we have an isomorphism

$$H^{2j+1}(X;\mathbb{Z}/p) \simeq \mathrm{Ext}^1(H_{2j}(X;\mathbb{Z}),\mathbb{Z}/p) \oplus \mathrm{Hom}(H_{2j+1}(X;\mathbb{Z}),\mathbb{Z}/p).$$

Suppose that

$$\mathrm{Hom}(H_{2j+1}(X;\mathbb{Z}),\mathbb{Z}/p)$$

is non-trivial. Then, since $H_{2j+1}(X;\mathbb{Z})$ is a finite abelian group, $H_{2j+1}(X;\mathbb{Z})$ has p-torsion. Suppose that

$$\mathrm{Ext}^1(H_{2j}(X;\mathbb{Z}),\mathbb{Z}/p)$$

is non-trivial. Then, since $H_{2j}(X;\mathbb{Z})$ is a finitely generated abelian group, $H_{2j}(X;\mathbb{Z})$ has p-torsion. Hence, in either case, X has p-torsion. □

Proof of proposition 2.1, (1) ⇔ (2). Let us consider the right vertical fibre sequence

$$\Omega^2_k BPU(n) \to \mathrm{Map}_k(S^2, BPU(n)) \to BPU(n)$$

and Leray–Serre spectral sequences associated with this fibre sequence. The E_2-page of the Leray–Serre spectral sequence for the integral homology consists of finitely generated abelian groups, and so are the integral homology groups of $\mathrm{Map}_k(S^2, BPU(n))$. The E_2-page of the Leray–Serre spectral sequence for the rational cohomology has no non-zero odd degree element. So the rational cohomology of $\mathrm{Map}_k(S^2, BPU(n))$ also has no non-zero odd degree element. Thus, by lemma 2.2, $\mathrm{Map}_k(S^2, BPU(n))$ has p-torsion if and only if its mod p cohomology has a non-zero odd degree element. □

Let $c_i \in H^{2i}(BU(n))$ be the mod p reduction of the i^{th} Chern class. The following proposition is what we need on the mod p cohomology of $\mathrm{Map}_k(S^2, BU(n))$ in this section. Section 5 gives a more detailed description of the generator x in terms of c_2 and the free double suspension we will define in § 3.

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Proposition 2.3. *The following hold.*

(1) $H^*(\mathrm{Map}_k(S^2, BU(n)))$ *has no non-zero odd degree element.*

(2) *As an abelian group,* $H^2(\mathrm{Map}_k(S^2, BU(n)))$ *is generated by two elements* $\pi^*(c_1)$ *and* x *such that* $\iota_k^*(x) \neq 0$.

Proof. Consider the Leray–Serre spectral sequence associated with the middle vertical fibre sequence

$$\Omega_k^2 BU(n) \to \mathrm{Map}_k(S^2, BU(n)) \to BU(n),$$

converging to the mod p cohomology of $\mathrm{Map}_k(S^2, BU(n))$. Then, the E_2-page has no non-zero odd degree element. Hence, the spectral sequence collapses at the E_2-page, and we obtain (1). Furthermore, we have

$$\begin{aligned} E_\infty^{0,2} &= H^2(\Omega_k^2 BU(n)) \simeq \mathbb{Z}/p, \\ E_\infty^{1,1} &= \{0\}, \\ E_\infty^{2,0} &= H^2(BU(n)) = \mathbb{Z}/p\{c_1\}. \end{aligned}$$

Hence, we have (2). □

Proof of proposition 2.1, (2) ⇔ (3). We consider the Leray–Serre spectral sequence associated with the middle horizontal fibre sequence

$$F \xrightarrow{\varphi} \mathrm{Map}_k(S^2, BU(n)) \longrightarrow \mathrm{Map}_k(S^2, BPU(n))$$

converging to the mod p cohomology of $\mathrm{Map}_k(S^2, BU(n))$. The mod p cohomology ring of $F \simeq BS^1$ is a polynomial ring generated by a single element u of degree 2. The E_2-page is given by

$$E_2^{*,*} = H^*(\mathrm{Map}_k(S^2, BPU(n))) \otimes H^*(F).$$

If the induced homomorphism

$$\varphi^*\colon H^2(\mathrm{Map}_k(S^2, BU(n))) \to H^2(F)$$

is non-zero, the induced homomorphism

$$\varphi^*\colon H^*(\mathrm{Map}_k(S^2, BU(n))) \to H^*(F)$$

is surjective. Then, by the Leray–Hirsh theorem, the induced homomorphism

$$H^*(\mathrm{Map}_k(S^2, BPU(n))) \to H^*(\mathrm{Map}_k(S^2, BU(n)))$$

is injective and, by proposition 2.3 (1), the mod p cohomology of $\mathrm{Map}_k(S^2, BPU(n))$ also has no non-zero odd degree element.

If the induced homomorphism

$$\varphi^*\colon H^2(\mathrm{Map}_k(S^2, BU(n))) \to H^2(F)$$

is zero, u does not survive to the E_∞-page. Hence, $d_2(u) \neq 0$ or $d_3(u) \neq 0$ must hold. Relevant subgroups of E_2-page are as follows.

$$\begin{aligned} E_2^{0,2} &= \mathbb{Z}/p\{u\}, \\ E_2^{1,1} &= \{0\}, \\ E_2^{2,1} &= \{0\}, \\ E_2^{3,0} &= H^3(\mathrm{Map}_k(S^2, BPU(n))). \end{aligned}$$

Since $d_2(u) \in E_2^{2,1} = \{0\}$, we have $d_2(u) = 0$. Therefore, $d_3(u) \neq 0$. Since $E_2^{1,1} = \{0\}$, the differential $d_2\colon E_2^{1,1} \to E_2^{3,0}$ is zero and we have $E_3^{3,0} = E_2^{3,0}$. Since

$$d_3(u) \in E_3^{3,0} \simeq H^3(\mathrm{Map}_k(S^2, BPU(n)))$$

is non-zero, the mod p cohomology of $\mathrm{Map}_k(S^2, BPU(n))$ has the non-zero odd degree element $d_3(u)$. □

3. Free double suspension

To describe the generator x of $H^2(\mathrm{Map}_k(S^2, BU(n)))$ in proposition 2.3 in more detail, we use the free double suspension

$$\sigma\colon H^*(\mathrm{Map}_k(S^2, BU(n))) \to H^{*-2}(\mathrm{Map}_k(S^2, BU(n))))$$

defined by Takeda in [4]. One may define the free double suspension over any coefficient groups. We focus on the mod p cohomology. Our definition of σ differs slightly from Takeda's $\hat{\sigma}_f^2$ in [4] but is the same homomorphism.

In this section, let X be a simply connected topological space. We denote by $*$ the base points of both S^2 and X. Let k be a homotopy class in $\pi_2(X)$ and 0 is the homotopy class in $\pi_2(X)$ containing the trivial map. Let

$$\mathrm{pr}_2\colon S^2 \times \mathrm{Map}_k(S^2, X) \to \mathrm{Map}_k(S^2, X)$$

be the obvious projection map. We use the evaluation maps

$$\mathrm{ev}\colon S^2 \times \mathrm{Map}_k(S^2, X) \to X, \quad \mathrm{ev}(s, g) = g(s),$$

and its restriction to $\mathrm{Map}_k(S^2, X) = \{*\} \times \mathrm{Map}_k(S^2, X)$,

$$\pi\colon \mathrm{Map}_k(S^2, X) \to X, \quad \pi(g) = g(*),$$

to define a homomorphism

$$\sigma\colon H^*(X) \to H^{*-2}(\mathrm{Map}_k(S^2, X)).$$

Let us fix a generator of $H^2(S^2) \simeq \mathbb{Z}/p$ and we denote it by u_2. We define σ by

$$\mathrm{ev}^*(x) - (\pi \circ \mathrm{pr}_2)^*(x) = u_2 \otimes \sigma(x).$$

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Let $\Omega^2_k X = \pi^{-1}(*)$ and denote the inclusion map by $\iota_k\colon \Omega^2_k X \to \mathrm{Map}_k(S^2,\, X)$. We define

$$\tilde{\sigma}_k\colon H^*(X) \to H^{*-2}\Omega^2_k X)$$

by $\iota_k^* \circ \sigma$. Proposition 3.1 (1) below is nothing but a particular form of proposition 2.1 in [4].

Proposition 3.1. *The homomorphism σ satisfies the following.*

(1) $\sigma(x \cdot y) = \sigma(x) \cdot \pi^*(y) + \pi^*(x) \cdot \sigma(y)$,

(2) *for a cohomology operation $\mathcal{O}$ of positive degree, $\sigma(\mathcal{O}x) = \mathcal{O}\sigma(x)$.*

Proof.

(1) Since

$$\begin{aligned}\mathrm{ev}^*(x) \cdot \mathrm{ev}^*(y) &= (u_2 \otimes \sigma(x) + 1 \otimes \pi^*(x)) \cdot (u_2 \otimes \sigma(y) + 1 \otimes \pi^*(y))\\ &= u_2 \otimes \sigma(x) \cdot 1 \otimes \pi^*(y)\\ &\quad + 1 \otimes \pi^*(x) \cdot u_2 \otimes \sigma(y) + 1 \otimes \pi^*(x) \cdot 1 \otimes \pi^*(y)\\ &= u_2 \otimes (\sigma(x) \cdot \pi^*(y) + \pi^*(x) \cdot \sigma(y)) + 1 \otimes (\pi^*(x) \cdot \pi^*(y)),\end{aligned}$$

Hence, we have

$$\mathrm{ev}^*(x \cdot y) - (\pi \circ \mathrm{pr}_2)^*(x \cdot y) = u_2 \otimes (\sigma(x) \cdot \pi^*(y) + \pi^*(x) \cdot \sigma(y)).$$

(2) is also clear from the naturality of cohomology operation.

$$\begin{aligned}\mathcal{O}(\mathrm{ev}^*(x) - (\pi \circ \mathrm{pr}_2)^*(x)) &= \mathrm{ev}^*(\mathcal{O}x) - (\pi \circ \mathrm{pr}_2)^*(\mathcal{O}x)\\ &= u_2 \otimes \sigma(\mathcal{O}x),\\ \mathcal{O}(u_2 \otimes \sigma(x)) &= u_2 \otimes \mathcal{O}\sigma(x),\end{aligned}$$

since $\mathcal{O}u_2 = 0$. Hence, we have

$$\sigma(\mathcal{O}x) = \mathcal{O}\sigma(x).$$

□

Next, we describe the relation between $\Omega^2_k X$ and $\Omega^2_0 X$. Let $X_1 \vee X_2$ be the subspace of $X_1 \times X_2$ defined by

$$X_1 \vee X_2 := \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 = * \text{ or } x_2 = *\}.$$

Let $\nu\colon S^2 \to S^2 \vee S^2$ be the pinch map collapsing the sphere's equator. We use it to define the addition on $\pi_2(X)$. Let $f\colon S^2 \to X$ be a map representing $k \in \pi_2(X)$

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and $\mathrm{c}_f\colon \Omega^2_0 X \to \{f\}$ the obvious constant map. Using f, we define

$$\mu_f\colon \Omega^2_0 X \to \Omega^2_k X$$

by

$$\begin{aligned} \mu_f(g)(s) &= f(s_1) \quad \text{if } \nu(s) = (s_1, *), \\ \mu_f(g)(s) &= g(s_2) \quad \text{if } \nu(s) = (*, s_2). \end{aligned}$$

The following lemma is a weak form of lemma 2.2 in [4]. We use it to prove proposition 5.1.

Lemma 3.2. *Let x be an element in $H^i(X)$. If $i \neq 2$, then we have*

$$\mu_f^* \circ \tilde{\sigma}_k(x) = \tilde{\sigma}_0(x).$$

Proof. We have the following commutative diagram by the definition of μ_f.

$$\begin{array}{ccccc} S^2 \times \Omega^2_0 X & \xrightarrow{\nu\times 1} & S^2 \times \Omega^2_0 X \vee S^2 \times \Omega^2_0 X & \xrightarrow{1\times \mathrm{c}_f \vee 1 \times 1} & S^2 \times \{f\} \vee S^2 \times \Omega^2_0 X \\ \Big\downarrow{\scriptstyle 1\times\mu_f} & & & & \Big\downarrow{\scriptstyle \mathrm{ev}\vee\mathrm{ev}} \\ S^2 \times \Omega^2_k X & & \xrightarrow{\quad\mathrm{ev}\quad} & & X, \end{array}$$

where we choose f as the base point of both $\{f\}$ and $\Omega^2_k X$, and the constant map $S^2 \to \{*\}$ as the base point of $\Omega^2_0 X$. Since the reduced mod p cohomology $\widetilde{H}^i(S^2 \times \{f\}) \simeq \widetilde{H}^i(S^2)$ is trivial for $i \neq 2$, we have isomorphisms

$$H^i(S^2 \times \{f\} \vee S^2 \times \Omega^2_0 X) \to H^i(S^2 \times \Omega^2_0 X)$$

and desired identity

$$\tilde{\sigma}_0(x) = \mu_f^* \circ \tilde{\sigma}_k(x)$$

for $x \in H^i(X)$, $i \neq 2$. □

4. Cohomology of $BU(n)$

In this section, we collect some elementary properties of the mod p cohomology ring of $BU(n)$ and the induced homomorphism

$$\phi^*\colon H^*(BU(n)) \to H^*(BS^1).$$

Let us fix a generator u of $H^2(BU(1)) = H^2(BS^1) \simeq \mathbb{Z}/p$. Let

$$\iota\colon BU(1)^n \to BU(n)$$

be the map induced by the inclusion map of the maximal torus $U(1)^n$ consisting of diagonal matrices. Let

$$B\mathrm{pr}_i\colon BU(1)^n \to BU(1) = BS^1$$

be the map induced by the projection of $U(1)^n$ to its i^{th} factor $U(1)$, defined by $(x_1, \ldots, x_n) \mapsto x_i$. We denote $B\mathrm{pr}_i^*(u) \in H^2(BU(1)^n)$ by t_i. The mod p cohomology of $BU(1)^n$ is a polynomial ring generated by $t_1, \ldots, t_n$ and the induced homomorphism

$$\iota^*\colon H^*(BU(n)) \to H^*(BU(1)^n) = \mathbb{Z}/p[t_1, \dots, t_n]$$

is injective, and its image is the set of symmetric polynomials in $t_1, \dots, t_n$. In particular, c_i is defined as the element such that $\iota^*(c_i)$ is the i^{th} elementary symmetric polynomial in $t_1, \dots, t_n$. Let us define s_i by

$$\iota^*(s_i) = \sum_{j=1}^{n} t_j^i.$$

The map $\phi\colon BS^1 \to BU(n)$ factors through

$$BS^1 \xrightarrow{\delta} BU(1)^n \xrightarrow{\iota} BU(n),$$

where δ is the map induced by the diagonal map $x \mapsto (x, \dots, x)$. Since $\delta^*(t_i) = u$ for $i = 1, \dots, n$, we have

$$\phi^*(s_i) = nu^i$$

and

$$\phi^*(c_i) = \binom{n}{i} u^i.$$

We use the following lemma 4.1 to prove proposition 5.5. The corresponding identity in symmetric polynomials is known as Newton's identity.

LEMMA 4.1. *In the mod p cohomology of $BU(n)$, for $i \geqslant 0$, we have relations*

$$s_{n+i+1} + \sum_{j=1}^{n} (-1)^j c_j s_{n+i-j+1} = 0.$$

Proof. Let us define symmetric polynomials $h_{i+2,n-1}, \dots, h_{n+i,1}$. For $\ell = i + 2, \dots, n + i$, let $h_{\ell,n+i+1-\ell}$ be the sum of monomials in the polynomial ring $\mathbb{Z}/p[t_1, \dots, t_n]$ obtained from $t_1^\ell t_2 \cdots t_{n+i+2-\ell}$ by permuting $1, \dots, n + j + 2 - \ell$ in $1, \dots, n$. Then, we have

$$\begin{aligned} \iota^*(c_1) \cdot \iota^*(s_{n+i}) &= \iota^*(s_{n+i+1}) + h_{n+i,1}, \\ \iota^*(c_j) \cdot \iota^*(s_{n+i+1-j}) &= h_{n+i+2-j,j-1} + h_{n+i+1-j,j}, \text{for } 2 \leqslant j \\ &\leqslant n-1 \text{ and } \iota^*(c_n) \cdot \iota^*(s_{i+1}) = h_{i+2,n-1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\iota^*(s_{n+i+1} + \sum_{j=1}^{n} (-1)^j c_i s_{n+i+1-j}) \\ &= \iota^*(s_{n+i+1}) - (\iota^*(s_{n+i+1}) + h_{n+i,1}) + \sum_{j=2}^{n-1} (-1)^j \left(h_{n+i+2-j,j-1} + h_{n+i+1-j,j}\right) \\ &\quad + (-1)^n h_{i+2,n-1} = 0. \end{aligned}$$

Since ι^* is injective, it completes the proof. □

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If p is an odd prime, let

$$\wp^i\colon H^j(X) \to H^{j+2i(p-1)}(X)$$

be the i^{th} Steenrod reduced power. If $p = 2$, let $\wp^1 = \mathrm{Sq}^2$ and $\wp^{2^{\ell-1}} = \mathrm{Sq}^{2^\ell}$ for $\ell \geqslant 2$, where

$$\mathrm{Sq}^i\colon H^j(X) \to H^{j+i}(X)$$

is the i^{th} Steenrod square. We define cohomology operations $\mathcal{Q}_\ell$ inductively by $\mathcal{Q}_1 = \wp^1$,

$$\mathcal{Q}_\ell = \wp^{p^{\ell-1}}\mathcal{Q}_{\ell-1} - \mathcal{Q}_{\ell-1}\wp^{p^{\ell-1}}$$

for $\ell \geqslant 2$. Cohomology operations $\mathcal{Q}_\ell$ have the following properties

(1) $\mathcal{Q}_\ell(x \cdot y) = \mathcal{Q}_\ell(x) \cdot y + x \cdot \mathcal{Q}_\ell(y)$ for x, $y \in H^*(BU(1)^n)$,

(2) $\mathcal{Q}_\ell t_i = t_i^{p^\ell}$ for t_1, …, t_n in $H^2(BU(1)^n)$.

With these properties, we have the following lemma 4.2. We will use it to prove proposition 5.2.

Lemma 4.2. *In the mod p cohomology of $BU(n)$, for $\ell \geqslant 1$, we have*

$$\mathcal{Q}_\ell c_2 = s_1 s_{p^\ell} - s_{p^\ell+1}.$$

Proof. On the one hand, since

$$\iota^*(c_2) = \sum_{1\leqslant i<j\leqslant n} t_i t_j,$$

by direct calculation, we have

$$\iota^*(\mathcal{Q}_\ell(c_2)) = \sum_{1\leqslant i<j\leqslant n} (t_i^{p^\ell} t_j + t_i t_j^{p^\ell}).$$

On the other hand, we have

$$\iota^*(s_{p^\ell} s_1 - s_{p^\ell+1}) = \left(\sum_{i=1}^n t_i^{p^\ell}\right)\left(\sum_{j=1}^n t_j\right) - \sum_{i=1}^n t_i^{p^\ell+1} = \sum_{1\leqslant i<j\leqslant n} (t_i^{p^\ell} t_j + t_i t_j^{p^\ell}).$$

Hence, we obtain the desired identity. □

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5. Proof of theorem 1.1

In this section, we consider the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & \mathrm{Map}_k(S^2, BU(n)) \\ \simeq\downarrow & & \downarrow\pi \\ BS^1 & \xrightarrow{\phi} & BU(n) \end{array}$$

We begin with the following refinement of proposition 2.3 (2).

Proposition 5.1. *As an abelian group,* $H^2(\mathrm{Map}_k(S^2, BU(n)))$ *is generated by* $\pi^*(c_1)$ *and* $\sigma(c_2)$.

Proof. Let $\lambda\colon BSU(n) \to BU(n)$ and $\lambda'\colon \Omega^2 BSU(n) \to \Omega^2_0 BU(n)$ be maps induced by the inclusion map $SU(n) \to U(n)$. We have the following commutative diagram by lemma 2.2 and the naturality of cohomology suspension.

$$\begin{array}{ccc} H^2(\Omega^2 BSU(n)) & \xleftarrow{\tilde{\sigma}} & H^4(BSU(n)) \\ \lambda'^*\uparrow & & \uparrow\lambda^* \\ H^2(\Omega^2_0 BU(n)) & \xleftarrow{\tilde{\sigma}_0} & H^4(BU(n)) \\ \mu_f^*\uparrow & & \uparrow = \\ H^2(\Omega^2_k BU(n)) & \xleftarrow{\tilde{\sigma}_k} & H^4(BU(n)) \end{array}$$

The top horizontal homomorphism $\tilde{\sigma}$ is the composition of cohomology suspensions

$$H^4(BSU(n)) \to H^3(\Omega BSU(n)) \to H^2(\Omega^2 BSU(n))$$

and it is an isomorphism. Since $H^4(BSU(n)) \simeq \mathbb{Z}/p$ is generated by $\lambda^*(c_2)$, we have

$$\lambda'^* \circ \mu_f^* \circ \tilde{\sigma}_k(c_2) = \tilde{\sigma} \circ \lambda^*(c_2) \neq 0.$$

Therefore, we obtain

$$\tilde{\sigma}_k(c_2) = \iota_k^* \circ \sigma(c_2) \neq 0.$$

By proposition 2.3 (2), $\pi^*(c_1)$ and $\sigma(c_2)$ generate $H^2(\mathrm{Map}_k(S^2, BU(n)))$. □

Let $u \in H^2(F) = H^2(BS^1) \simeq \mathbb{Z}/p$ be the generator fixed in § 4. Let us define α_i, $\beta \in \mathbb{Z}/p$ by

$$\alpha_i u^i = \varphi^* \circ \sigma(s_{i+1}),$$
$$\beta u = \varphi^* \circ \sigma(c_2).$$

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Proposition 5.2. *If $n \equiv 0 \mod (p)$, we have $\beta = -\alpha_{p^\ell}$ for $\ell \geqslant 1$.*

Proof. On the one hand, by the definition of β, we have

$$\varphi^* \circ \sigma(c_2) = \beta u.$$

Applying $\mathcal{Q}_\ell$, we have

$$\varphi^* \circ \sigma(\mathcal{Q}_\ell c_2) = (\beta u)^{p^\ell} = \beta u^{p^\ell}.$$

On the other hand, by lemma 4.2, in the mod p cohomology of $BU(n)$, we have the relation

$$\mathcal{Q}_\ell c_2 = s_1 s_{p^\ell} - s_{p^\ell+1}.$$

Applying $\varphi^* \circ \sigma$, we have

$$\begin{aligned}\varphi^* \circ \sigma(\mathcal{Q}_\ell c_2) &= \varphi^* \circ \sigma(s_1) \cdot \phi^*(s_{p^\ell}) + \phi^*(s_1) \cdot \varphi^* \circ \sigma(s_{p^\ell}) - \varphi^* \circ \sigma(s_{p^\ell+1}) \\ &= n\alpha_1 u^{p^\ell} + n\alpha_{p^{\ell-1}} u^{p^\ell} - \alpha_{p^\ell} u^{p^\ell} \\ &= -\alpha_{p^\ell} u^{p^\ell}.\end{aligned}$$

Hence, we have $\beta = -\alpha_{p^\ell}$. □

Summing up propositions 5.1 and 5.2, we have the following proposition 5.3. It reduces the proof of theorem 1.1 to the computation of α_p.

Proposition 5.3. *The following are equivalent.*

(1) $\varphi^* \colon H^2(\mathrm{Map}_k(S^2, BU(n))) \to H^2(F)$ *is zero,*

(2) $\phi^*(c_1) = 0$ *and* $\beta = 0$,

(3) $\phi^*(c_1) = 0$ *and* $\alpha_p = 0$.

Proof. Since $H^2(\mathrm{Map}_k(S^2, BU(n)))$ is generated by $\pi^*(c_1)$ and $\sigma(c_2)$, (1) and (2) are equivalent. Under the assumption that $\phi^*(c_1) = 0$, we have $n \equiv 0 \mod (p)$. Then, by proposition 5.2, we have

$$\beta = -\alpha_p.$$

Hence, (2) and (3) are equivalent. □

By computing α_p, we complete the proof of theorem 1.1.

Proposition 5.4. *We have $\alpha_0 = k$.*

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Proof. Let $f\colon S^2 \to BU(n) \in \mathrm{Map}_k(S^2, BU(n))$. By definition, we have

$$f^*(c_1) = ku_2.$$

Let

$$\mathrm{i}_f\colon S^2 \to S^2 \times \mathrm{Map}_k(S^2, BU(n))$$

be a map defined by $t \mapsto (t, f)$. Then, we have

$$f = \mathrm{ev} \circ \mathrm{i}_f$$

and

$$\pi \circ \mathrm{pr}_2 \circ \mathrm{i}_f$$

is a constant map $S^2 \to \{f(*)\}$. It implies that

$$\mathrm{i}_f^*(\mathrm{ev}^*(c_1) - (\pi \circ \mathrm{pr}_2)^*(c_1)) = f^*(c_1) = ku_2.$$

When we restrict i_f^* to $H^2((S^2, *) \times \mathrm{Map}_k(S^2, BU(n)))$, it is injective. So, we have

$$\mathrm{ev}^*(c_1) - (\pi \circ \mathrm{pr}_2)^*(c_1) = ku_2 \otimes 1.$$

Hence, by the definition of σ, we have $\sigma(c_1) = k$. □

Proposition 5.5. *If $n \equiv 0 \mod (p)$, we have $\alpha_p = k$.*

We use the following lemma 5.6 to prove proposition 5.5. We will prove it in the next section. Let B be an $n \times n$ matrix whose (i, j)-entry is given by integers

$$b_{1,j} = (-1)^{j+1}\binom{n}{j}$$

for $1 \leqslant j \leqslant n$ and $b_{i,j} = 1$ if $i = j + 1$, $b_{i,j} = 0$ if $i \neq j + 1$ for $2 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$.

Lemma 5.6. *When we regard the matrix B as an element in $SL_n(\mathbb{Z}/p)$, the order of B is a power of p.*

Proof of proposition 5.5. By lemma 4.1, in $H^*(BU(n))$, we have

$$s_{n+i+1} + \sum_{j=1}^{n}(-1)^j c_j s_{n+i+1-j} = 0$$

for $i \geqslant 0$. Applying $\varphi^* \circ \sigma$, we have

$$\alpha_{n+i}u^{n+i} + \sum_{j=1}^{n}(-1)^j \phi^*(c_j) \cdot \alpha_{n+i-j}u^{n+i-j} + \sum_{j=1}^{n}(-1)^j \varphi^* \circ \sigma(c_j) \cdot \phi^*(s_{n+i-j+1}) = 0.$$

Since $\phi^*(s_{n+i-j+1}) = 0$, we obtain

$$\alpha_{n+i}u^{n+i} + \sum_{j=1}^{n}(-1)^j \phi^*(c_j) \cdot \alpha_{n+i-j}u^{n+i-j} = 0.$$

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Furthermore, since $\phi^*(c_j) = \binom{n}{j} u^j$, we have

$$\alpha_{n+i} + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \alpha_{n+i-j} = 0.$$

Thus, we have

$$\begin{aligned} \alpha_{n+i} &= \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \alpha_{n+i-j}, \\ \alpha_{n-1+i} &= \alpha_{n-1+i}, \\ &\vdots \\ \alpha_{1+i} &= \alpha_{1+i}. \end{aligned}$$

Therefore, put these identities together in matrix form, using the $n \times n$ matrix B that we just defined, we have

$$\begin{pmatrix} \alpha_{n+i} \\ \vdots \\ \alpha_{1+i} \end{pmatrix} = B \begin{pmatrix} \alpha_{n-1+i} \\ \vdots \\ \alpha_i \end{pmatrix} = \cdots = B^{i+1} \begin{pmatrix} \alpha_{n-1} \\ \vdots \\ \alpha_0 \end{pmatrix},$$

for $i \geqslant 0$. By lemma 5.6, the order of B as an element of $SL_n(\mathbb{Z}/p)$ is a power of p. Hence, for some positive integer ℓ, we have

$$\alpha_{p^\ell} = \alpha_0 = k.$$

By proposition 5.2, we have $\alpha_{p^\ell} = -\beta = \alpha_p$. Therefore, we obtain $\alpha_p = k$. □

Proposition 5.7 below is immediate from proposition 5.5 and it completes the proof of theorem 1.1.

PROPOSITION 5.7. *The following holds.*

(1) *If* $n \not\equiv 0 \mod (p)$, *then* $\phi^*(c_1) \neq 0$,

(2) *If* $n \equiv 0 \mod (p)$ *and* $k \not\equiv 0 \mod (p)$, *then* $\alpha_p \neq 0$,

(3) *If* $n \equiv 0 \mod (p)$ *and* $k \equiv 0 \mod (p)$, *then* $\phi^*(c_1) = 0$ *and* $\alpha_p = 0$.

6. Proof of lemma 5.6

In this section, we deal with unimodular $n \times n$ matrices. Unless otherwise clear from the context, matrix entries are integers. What we do in what follows is to find the transpose of the Jordan matrix similar to the matrix B in § 5.

PROPOSITION 6.1. *There is a unimodular* $n \times n$ *matrix* A *such that* $A^{-1}BA = D$ *where* (i, j)*-entry* $d_{i,j}$ *of* D *is* $d_{i,j} = 1$ *if* $i = j$ *or* $i = j + 1$ *and* $d_{i,j} = 0$ *if otherwise.*

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We prove this proposition by giving such a matrix A explicitly. Before we do it, we complete the proof of lemma 5.6.

Proof of lemma 5.6. By proposition 6.1, we have

$$B = ADA^{-1}.$$

The matrix D belongs to the subgroup U_n of $SL_n(\mathbb{Z}/p)$ consisting of lower triangular matrices whose diagonal entries are 1. The subgroup U_n is a p-group. Therefore, the order of D is a power of p. Hence, the order of B is also the power of p. □

Now, we prove proposition 6.1 by defining A explicitly.

Proof of proposition 6.1. Let A be the $n \times n$ unimodular upper triangular matrix whose $(i,\, j)$-entry is given by

$$a_{i,j} = \binom{n-i}{n-j}.$$

We show that $(i,\, j)$-entries of BA and AD are equal to $\binom{n-i+1}{n-j}$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$.

Recall that B is the $n \times n$ unimodular matrix whose $(i,\, j)$-entry is given as follows: For $i = 1$, $1 \leqslant j \leqslant n$, the $(1,\, j)$-entry of B is given by

$$b_{1,j} = (-1)^{j+1}\binom{n}{j},$$

and, for $2 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, the $(i,\, j)$-entry of B is given by

$$\begin{aligned} b_{i,j} &= 1 \quad \text{if } i = j+1, \\ b_{i,j} &= 0 \quad \text{otherwise.} \end{aligned}$$

(1) For $1 \leqslant j \leqslant n$, the $(1,\, j)$-entry of BA is given by

$$\begin{aligned} \sum_{\ell=1}^{n} b_{1,\ell}a_{\ell,j} &= \sum_{\ell=1}^{j} b_{1,\ell}a_{\ell,j} \\ &= \sum_{\ell=1}^{j}(-1)^{\ell+1}\binom{n}{\ell}\cdot\binom{n-\ell}{n-j} \\ &= \sum_{\ell=1}^{j}(-1)^{\ell+1}\frac{n!}{(n-\ell)!\ell!}\cdot\frac{(n-\ell)!}{(n-j)!(j-\ell)!} \\ &= \sum_{\ell=1}^{j}(-1)^{\ell+1}\frac{n!}{(n-j)!j!}\cdot\frac{j!}{(j-\ell)!\ell!} \end{aligned}$$

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$$= \sum_{\ell=1}^{j} (-1)^{\ell+1} \binom{n}{n-j} \binom{j}{\ell}$$

$$= \binom{n}{n-j} \left(\sum_{\ell=1}^{j} (-1)^{\ell+1} \binom{j}{\ell} \right)$$

$$= \binom{n}{n-j}$$

$$= \binom{n-1+1}{n-j}$$

For $2 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, the (i, j)-entry of BA is given by

$$\begin{aligned} \sum_{\ell=1}^{n} b_{i,\ell} a_{\ell,j} &= b_{i,i-1} a_{i-1,j} \\ &= a_{i-1,j} \\ &= \binom{n-i+1}{n-j}. \end{aligned}$$

(2) For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, the (i, j)-entry of AD is given by

$$\begin{aligned} \sum_{\ell=1}^{n} a_{i,\ell} d_{\ell,j} &= a_{i,j} d_{j,j} + a_{i,j+1} d_{j+1,j} \\ &= a_{i,j} + a_{i,j+1} \\ &= \binom{n-i}{n-j} + \binom{n-i}{n-j-1} \\ &= \binom{n-i+1}{n-j}. \end{aligned}$$

It completes the proof of proposition 6.1 □

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References

1 M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. R. Soc. London Ser. A* **308** (1983), 523–615.

2 D. H. Gottlieb. Applications of bundle map theory. *Trans. Am. Math. Soc.* **171** (1972), 23–50.

3 Y. Minowa. *On the cohomology of the classifying spaces of $SO(n)$-gauge groups over S^2.* ArXiv:2304.08702v1 (2023).

4 M. Takeda. Cohomology of the classifying spaces of $U(n)$-gauge groups over the 2-sphere. *Homol. Homotopy Appl.* **23** (2021), 17–24.

5 S. Tsukuda. On the cohomology of the classifying space of a certain gauge group. *Proc. R. Soc. Edinburgh Sect. A* **127** (1997), 407–409.

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