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ORDINARY SINGULARITIES OF ALGEBRAIC CURVES

BY

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ABSTRACT. Let A be the local ring at a singular point p of an algebraic reduced curve. Let M (resp. $M_1, \ldots, M_h$) be the maximal ideal of A (resp. of $\bar{A}$). In this paper we want to classify ordinary singularities p with reduced tangent cone: Spec($G(A)$). We prove that $G(A)$ is reduced if and only if: p is an ordinary singularity, and the vector spaces $\mathrm{Hom}(M_i^n/M_i^{n+1}, k)$ span the vector space $\mathrm{Hom}(M^n/M^{n+1}, k)$. If the points of the projectivized tangent cone Proj($G(A)$) are in generic position then p is an ordinary singularity if and only if $G(A)$ is reduced. We give an example which shows that the preceding equivalence is not true in general.

Introduction. Let A be the local ring at a singular point p of an algebraic reduced curve C of embedding dimension $\mathrm{emdim}(A)=r+1$ and multiplicity $e(A)=s$ at p. Let M be the maximal ideal of A and $M_1, \ldots, M_h$ the maximal ideals of $\bar{A}$. Spec($G(A)$) is the tangent cone and Proj($G(A)$)$\subset \mathbb{P}^r$ the projectivized tangent cone to C at p. The scheme Proj($G(A)$) is reduced if and only if p is an ordinary singular point (see Lemma–Definition 2.1) and clearly if $G(A)$ is reduced then Proj($G(A)$) is reduced. In this paper we want to examine the converse, more precisely we want to classify ordinary singularities with reduced tangent cone. Ordinary singularities with reduced tangent cone are particularly important from many points of view. For example one can compute the conductor (see [O]), the Hilbert function (see Corollary 3.4) of A. We show that $G(A)$ is reduced if and only if: (1) p is an ordinary singularity, (2) the vector spaces $\mathrm{Hom}(M_i^n/M_i^{n+1}, k)$ span the vector space $\mathrm{Hom}(M^n/M^{n+1}, k)$, for any $n<s-1$. Moreover if $p_1, \ldots, p_s$ are the points of Proj($G(A)$), $v_n:\mathbb{P}^r \to \mathbb{P}^{(r+n/n)-1}$ is the Veronese embedding and $A_{n,s}$ is the matrix which has as columns the coordinates of the points $v_n(p_1), \ldots, v_n(p_s)$ then 2) is equivalent to $\rho(A_{n,s})=H(n)$ (for any $n<s-1$), where $\rho(A_{n,s})$ is the rank of $A_{n,s}$ and $H(n)=\dim_k(M^n/M^{n+1})$ is the Hilbert function. Using this last condition we

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prove that if the points $p_1, \ldots, p_s$ are in generic position in $\mathbb{P}^r$ (see Definition 3.1) and p is an ordinary singularity then $G(A)$ is reduced. Finally we give an example which shows that the previous equivalence is not true in general.

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Standing notation. A is the local ring at a singular closed point p of a reduced algebraic curve $C = \operatorname{Spec} R$ over an algebraically closed field k. M, $s = e(A)$ and $r+1 = \operatorname{emdim}(A)$ are respectively the maximal ideal, multiplicity and embedding dimension of the ring A. $\bar{A}$ is the normalization of A. $M_1, \ldots, M_h$ are the maximal ideals of $\bar{A}$ and $J = M_1 \cap \cdots \cap M_h$ the Jacobson radical. If B is a semilocal ring $G(B)$ is the associated graded ring with respect to its Jacobson radical.

1. **Tangents and branches at a point of a curve.** This section contains the basic notions which will be extensively used in the sequel.

There is a certain amount of vagueness in the literature concerning the use of "branch" and "tangent". Everyone is sure of these terms for plane curves. In this section we want to provide a clarification of these terms for arbitrary curves. Our definitions will be consistent with the usual plane conventions.

Definition 1.1. Let P_i be a minimal prime of the completion $\hat{A}$ (with respect to M) of A. The scheme $\operatorname{Spec}(\hat{A}/P_i)$ is a *branch* of the curve $C = \operatorname{Spec} R$ at the point p.

It is well known that there is a canonical bijection between the set of maximal ideals of $\bar{A}$ and the set of branches; namely if M_i $(1 \le i \le h)$ is a maximal ideal of $\bar{A}$, then the corresponding branch is the branch whose ideal is the kernel P_i of the lower horizontal map in the following commutative diagram:

$$\begin{array}{ccc} A & \to & \bar{A} \\ \downarrow & & \downarrow \\ \hat{A} & \to & \hat{\bar{A}}_{\hat{M}_i} \end{array} \qquad \text{“}\wedge\text{”} = \text{completion} \tag{1.2}$$

(equivalently $P_i = Q_i \cap \hat{A}$ where Q_i is the unique minimal prime ideal of $\hat{\bar{A}} = \bar{\hat{A}}$ contained in $\hat{M}$).

Further $M\bar{A} = M_1^{s_1} \cdots M_h^{s_h}$ *where* $s_i = e(\hat{A}/P_i)$ *and* $s_1 + \cdots + s_h = s = e(A)$, then

(1.3) The branch $\operatorname{Spec}(\hat{A}/P_i)$ is *nonsingular* (i.e. the ring $\hat{A}/P_i$ is regular) if and only if $s_i = 1$. The point p has all nonsingular branches if and only if $M\bar{A} = J$.

In the sequel we will call $\text{Spec}(\hat{A}/P_i)$: *the i-th branch.*

If $\text{emdim}(A)=r+1$, M has $r+1$ generators $x_0,\ldots,x_r$ such that $\bar{x}_0,\ldots,\bar{x}_r$ are a basis of the k-vector space M/M^2 and the *Zariski tangent space* $T=\text{Hom}_k(M/M^2,k)$ is canonically isomorphic (as a scheme) to the affine space $\mathbb{A}^{r+1}$. Further $G(A)=\bigoplus_{n\geq 0}(M^n/M^{n+1})=k[\bar{x}_0,\ldots,\bar{x}_r]$. Then the *affine tangent cone* $\text{Spec}(G(A))$ is a subscheme of $T=\mathbb{A}^{r+1}$ and the *projectivized tangent cone* $\text{Proj}(G(A))$ is a subscheme of $\mathbb{P}^r$ (see [M] pp. 303–309 and pp. 323–328). *From now on we set*:

$$\text{Spec}(G(A))\subset T=\mathbb{A}^{r+1}\quad\text{and}\quad\text{Proj}(G(A))\subset\mathbb{P}^r.$$

$\text{Spec}(G(A))$ consists (as a set) of a finite number of lines L_i and so $\text{Proj}(G(A))$ consists of a finite number of points p_i. The following definition extends to any curve the usual definition of tangent at a point p of a plane curve (cf. [F], p. 67).

DEFINITION 1.4. The lines L_i of the tangent cone $\text{Spec}(G(A))$ are the *tangents* to the curve C at the point p.

There is a canonical way of associating to a branch a "tangent". For this we need the notion of "blowing up". For the unexplained facts we refer the reader to [L]. We recall that, if x is an element of A s.t. $xM^n=M^{n+1}$ for all sufficiently large n, then the ring

$$B=\left\{\frac{z}{x^n}\;\middle|\; n>0,\, z\in M^n\right\}=A\left[\frac{x_0}{x},\ldots,\frac{x_r}{x}\right]\subset\bar{A}$$

is the ring obtained by blowing up the maximal ideal M in A (i.e. Spec B is the blowing up of the curve Spec A). It is well known (cf. [M], p. 319) that there is a natural isomorphism of schemes:

$$\text{Proj}(G(A))\simeq\text{Spec}(B/MB)\tag{1.5}$$

thus we have the canonical maps (recall (1.2)):

$$\{\text{branches}\}\simeq\left\{\begin{matrix}\text{closed points}\\ \text{of Spec }\bar{A}\end{matrix}\right\}\to\left\{\begin{matrix}\text{closed points of}\\ \text{blow-up of Spec }A\end{matrix}\right\}\simeq\left\{\begin{matrix}\text{closed points}\\ \text{of Proj}(G(A))\end{matrix}\right\}$$

The arrow is just the restriction of $\text{Spec }\bar{A}\to\text{Spec }B$.

Then under the previous maps the i-th branch corresponds to a point $p_i\in\text{Proj}(G(A))\subset\mathbb{P}^r$ i.e. to a line L_i of $\text{Spec}(G(A))\subset T=\mathbb{A}^{r+1}$.

We give now another equivalent way of associating a tangent to the ith branch (i.e. to M_i) (this second point of view is consistent with the one of [Sh], p. 123, for plane curves).

Let $M/M^2\to M_i/M_i^2$ be the natural homomorphism $(1\leq i\leq h)$. It induces the homomorphism $T_i\to T$ of the corresponding tangent spaces $T_i=\text{Hom}_k(M_i/M_i^2,k)$, $T=\text{Hom}_k(M/M^2,k)$.

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LEMMA 1.6. *The homomorphism $T_i \to T$ is injective if and only if the i-th branch is nonsingular.*

Proof. $T_i \to T$ is injective if and only if $M/M^2 \to M_i/M_i^2$ is surjective i.e. it is not null (in fact $\dim_k(M_i/M_i^2) = 1$). But $M/M^2 \to M_i/M_i^2$ is null if and only if $M \subset M_i^2$ i.e. $M\bar{A} = M_1^{s_1} \cdots M_i^{s_i} \cdots M_h^{s_h} \subset M_i^2$. But this is equivalent to $s_i > 1$, i.e. the ith branch is singular (see (1.3)).

Then if the ith branch is nonsingular we identify T_i to a subspace of T.

PROPOSITION 1.7. *The following equality of subspaces of T holds: $T_i = L_i$.*

Proof. Let $(a_{i0}, \ldots, a_{ir}) \in \mathbb{P}^r$ be the point of $\operatorname{Proj}(G(A))$ corresponding to the line L_i of $\operatorname{Spec}(G(A))$. With a suitable choice of coordinates we can assume that $a_{i0} \neq 0$. As a subspace of $T = \operatorname{Hom}(M/M^2, k)$, L_i has basis the homomorphism $\phi(\bar{x}_0) = a_{i0}, \ldots, \phi(\bar{x}_r) = a_{ir}$. To show that $L_i = T_i$ one has to prove that the natural homomorphism $M/M^2 \to M_i/M_i^2$ is given by $\bar{x}_j \to a_{ij}t_i$ for a suitable basis of the one dimensional vector space M_i/M_i^2; but, if $a_{i0} \neq 0$, $\sqrt{\bar{x}_0 G(A)}$ is the homogeneous maximal ideal of $G(A)$, then $x_0M^n = M^{n+1}$ for all sufficiently large n. Hence the ring B obtained by blowing up M is

$$B = A\left[\frac{x_1}{x_0}, \ldots, \frac{x_r}{x_0}\right]$$

and under the isomorphism $\operatorname{Proj}(G(A)) \simeq \operatorname{Spec}(B/MB)$ (cf. (1.5)) the point $(a_{i0}, \ldots, a_{ir})$ corresponds to the unique maximal ideal N_i of B which contains the ideal

$$\left(\frac{x_1}{x_0} - \frac{a_{i1}}{a_{i0}}, \ldots, \frac{x_r}{x_0} - \frac{a_{ir}}{a_{i0}}\right).$$

Further $x_0 \in N_i^2$ then

$$a_{i0}x_j - a_{ij}x_0 = a_{i0}x_0\left(\frac{x_j}{x_0} - \frac{a_{ij}}{a_{i0}}\right) \in N_i^2 \subset M_i^2.$$

Now the result is clear.

DEFINITION 1.8. The line $L_i = T_i$ is *the tangent to the* ith *branch* P_i.

REMARK. There is a third equivalent way of associating a tangent to a branch. Let $\hat{T}_i = \operatorname{Hom}_k(\hat{M}_i/\hat{M}_i^2, k)$ and $\hat{T} = \operatorname{Hom}_k(\hat{M}/\hat{M}^2, k)$ be the Zariski tangent spaces to the ith branch and to Spec $\hat{A}$. We have $\hat{T} = T$ and so if the ith branch is nonsingular $\hat{T}_i$ can be naturally identified to a subspace of T. Clearly $\hat{T}_i = T_i = L_i$.

2. **Ordinary singular points. Points with reduced tangent cone.** The following result–definition extends to any curve the usual definitions of ordinary singularity of a plane curve (cf. [F], p. 105, [H], ch. I, ex. 5.14, [Sh], p. 123).

We say that p is an s-fold point if $s = e(A)$.

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LEMMA–DEFINITION 2.1. *Let p be an s-fold point. Then the following conditions are equivalent. A point p which satisfies these conditions is said to be an* ordinary point.

(1) *the branches at p are nonsingular and their tangents are distinct,*

(2) $\operatorname{Proj}(G(A))$ *is reduced,*

(3) *there are s tangents at p,*

(4) $\operatorname{Proj}(G(A))$ *consists of s points.*

Proof. Let $\operatorname{Spec} B$ be the blowing up of $\operatorname{Spec} A$. Thus $\operatorname{Spec}(B/MB) \simeq \operatorname{Proj}(G(A))$ (cf. (1.5)). Therefore $\dim_k(B/MB) \geq$ number of tangents at p; and equality holds if and only if B/MB is reduced. Since MB is a principal ideal ([L], prop. 1.1): $\dim_k(B/MB) = s$; and $B = \bar{B} = \bar{A}$ if B/MB is reduced. Thus: number of tangents $= s \Leftrightarrow B/MB$ is reduced $\Leftrightarrow B = \bar{A}$ and $M\bar{A} = M_1 \cap \cdots \cap M_s = J$. Recalling (1.3) and Proposition 1.7 the equivalence of (1), (2), (3), (4) is now clear.

REMARK. It is well known that $G(A)$ reduced implies $\operatorname{Proj}(G(A))$ reduced (see [H], ch. II, ex. 2.3). Then a point with reduced tangent cone is an ordinary point.

Let t_i be a basis of the vector space M_i/M_i^2 $(1 \leq i \leq h)$ and $\bar{x}_0, \ldots, \bar{x}_r$ be a basis of M/M^2 (emdim $A = r+1$); If all the branches at the point p are nonsingular then the $h = s$ (cf. (1.3)) and the natural homomorphism $d_{1i}: M/M^2 \to M_i/M_i^2$ is surjective (Lemma 1.6). Thus $d_{1i}(\bar{x}_j) = a_{ij}t_i$ $(0 \leq j \leq r,\ a_{ij} \in k)$ with $a_{ij} \neq 0$ for at least a j. Then $p_i = (a_{i0}, \ldots, a_{ir})$ is a point of $\mathbb{P}^r$.

COROLLARY 2.2. *Let* $p_1 = (a_{10}, \ldots, a_{1r}), \ldots, p_s = (a_{s0}, \ldots, a_{sr}) \in \mathbb{P}^r$. *If* $\operatorname{Proj}(G(A))$ *is reduced then the points p_i are distinct and* $\operatorname{Proj}(G(A)) = \{p_1, \ldots, p_s\}$.

Proof. The points p_i are the points of $\operatorname{Proj}(G(A))$ (see the proof of Proposition 1.7). Further $\operatorname{Proj}(G(A))$ consists of s points (Lemma 2.1), so the p_i are necessarily distinct.

Let now $d_{ni}: M^n/M^{n+1} \to M_i^n/M_i^{n+1}$ be *the natural homomorphism.* It induces *the homomorphism* $\delta_{in}: \operatorname{Hom}(M_i^n/M_i^{n+1}, k) \to \operatorname{Hom}(M^n/M^{n+1}, k)$ of the dual spaces.

LEMMA 2.3. *If the i-th branch is nonsingular then δ_{in} is injective for any $n > 0$.*

Proof. If δ_{in} is not injective for an $n > 0$ then the homomorphism $d_{ni}: M^n/M^{n+1} \to M_i^n/M_i^{n+1}$ is not surjective i.e. it is null; hence $M^n \subset M_i^{n+1}$. But if the ith branch is nonsingular $(M\bar{A})^n = M_1^{ns_1} \cdots M_i^n \cdots M_h^{ns_h}$ (cf. (1.3)) and $(M\bar{A}) \subset M_i^{n+1}$ implies $M_i^n \subset M_i^{n+1}$. Contradiction.

Then if the ith branch is nonsingular we can identify the vector space $\mathrm{Hom}(M_i^n/M_i^{n+1}, k)$ to a subspace of the vector space $\mathrm{Hom}(M^n/M^{n+1}, k)$. *From now on we set*:

$T_i^n = \mathrm{Hom}(M_i^n/M_i^{n+1}, k)$, $T^n = \mathrm{Hom}(M^n/M^{n+1}, k)$ (*if* $n>0$ and $1 \le i \le h$), $T_i^1 = T_i$, $T^1 = T$.

Let n be a positive integer, $N = \binom{n+r}{r}$ and $v_n : \mathbb{P}^r \to \mathbb{P}^{N-1}$ be a Veronese (or n-ple) mapping: $v_n(x_0, \ldots, x_r) = (f_0(x_0, \ldots, x_r), \ldots, f_N(x_0, \ldots, x_r))$, where $f_j(x_0, \ldots, x_r)(1 \le j \le N)$ are all the monomials of degree n. Let $S = \{p_1, \ldots, p_s\} \subset \mathbb{P}^r$ be the set of Corollary 2.2. Consider a subset of S which for simplicity we call $p_1, \ldots, p_u$ and let $A_{n,u}$ be the matrix which has columns the coordinates of the points $v_n(p_1), \ldots, v_n(p_u)$. $\rho(A_{n,u})$ denotes the rank of $A_{n,u}$ (this does not depend on the coordinates chosen for the point p_i).

PROPOSITION 2.4. Let p be an ordinary s-fold point and let $V = T_1^n + \cdots + T_u^n \subset T^n$. Then $\dim V = \rho(A_{n,u})$. Further, if $n \ge s-1$, $T_1^n + \cdots + T_s^n = T^n$ and $\dim T^n = s$.

Proof. Let $\{t_i^n\}$ $(t_i \in M_i)$ be a basis of M_i^n/M_i^{n+1} and $\tau_i \in T_i^n$ be the homomorphism defined by $\tau_i(t_i^n) = 1$. If $f = f(\bar{x}_0, \ldots, \bar{x}_r) = \sum_{j=1}^N b_j f_j(\bar{x}_0, \ldots, \bar{x}_r) \in M^n/M^{n+1}$ (f_j monomials of degree n, $b_j \in k$), then $\tau_i d_{ni}(f) = \tau_i(f(p_i)t_i^n) = f(p_i) = \sum_j b_j f_j(p_i)$. To prove the first equality it is enough to show that the vectors $\tau_1 d_{n1}, \ldots, \tau_m d_{mn}$ are linearly independent in T if and only if the points $v_n(p_1), \ldots, v_n(p_m)$ are linearly independent in $\mathbb{P}^{N-1}$, for every $m \le n$. But $\sum_{i=1}^m c_i \tau_i d_{ni} = 0$ is equivalent to $\sum_{i,j} c_i b_j f_j(p_i) = 0$ for any b_j i.e. to $\sum_{i=1}^m c_i v_n(p_i) = 0$. The second part of the proposition is a consequence of the following facts. (1) If $V(n, s)$ is the linear system of all hypersurfaces of degree n containing S then $\dim V(n, s) = N - \rho(A_{n,s}) - 1$ (as linear subvariety of $\mathbb{P}^{N-1}$). Further, if $n \ge s-1$, $\dim V(n, s) = N - s - 1$ (same proof of [F], p. 110 for plane curves), hence $\rho(A_{n,s}) = s$, 2) $\dim T^n = \dim(M^n/M^{n+1}) \le s$, for any $n > 0$ (cf. [L] theorem 1.9).

PROPOSITION 2.5. p is an ordinary point if and only if the homomorphism $T_i^n \to T^n$ are injective and $T_i^n \ne T_j^n$ for every $i \ne j$ and $n > 0$.

Proof. Owing to Lemma 1.6 and 2.1 we have only to prove that $T_i^n \ne T_j^n$ for $i \ne j$. But if $T_i^n = T_j^n$ then the matrix $A_{n,2}$ of the points $v_2(p_i)$, $v_2(p_j)$ has rank 1 (see Proposition 2.4) so $v_2(p_i) = v_2(p_j)$ and then $p_i = p_j$, i.e. $T_i = T_j$. Hence p is not ordinary.

Now we need the following result (which can be proved in more general hypotheses). The natural homomorphism $M^n/M^{n+1} \to J^n/J^{n+1}$, $n \ge 0$ induces a homomorphism of graded rings $G(A) \to G(\bar{A}) = \bigoplus_{n \ge 0} (J^n/J^{n+1})$.

PROPOSITION 2.6. The ring $G(A)$ is reduced if and only if the homomorphism $G(A) \to G(\bar{A})$ is injective.

Proof. (Cf. [O], proposition 2.2).

Let $H(n)=\dim_k(M^n/M^{n+1})=\dim T^n$ be the Hilbert function of the maximal ideal M of A. Clearly $\dim(T_1^n+\cdots+T_h^n)\le H(n)$. Further if p is an ordinary point and $n\ge s-1$ then the vector spaces T_i^n $(1\le i\le s)$ span T^n (Proposition 2.4).

THEOREM 2.7. *The following conditions are equivalent:*

(1) $G(A)$ *is reduced,*

(2) p *is an ordinary singular point and the vector spaces* T_i^n $(1\le i\le s)$ *span* T^n *for every* $n<s-1$.

(3) $\operatorname{Proj}(G(A))=\{p_1,\ldots,p_s\}$ *is reduced and* $\rho(A_{n,s})=H(n)$ *for every* $n<s-1$.

Proof. $G(A)$ is reduced if and only if the homomorphisms $M^n/M^{n+1}\to J^n/J^{n+1}$ are injective (cf. Proposition 2.6) for every $n>0$ i.e. the induced homomorphisms $d_n:\operatorname{Hom}(J^n/J^{n+1},k)\to\operatorname{Hom}(M^n/M^{n+1},k)=T^n$ are surjective. But $\operatorname{Hom}(J^n/J^{n+1},k)=\bigoplus_{i=1}^s\operatorname{Hom}(M_i^n/M_i^{n+1},k)$ and so $\operatorname{Im} d_n=T_1^n+\cdots+T_s^n$. Now recalling Proposition 2.4, Lemma 2.1 and Remark, the result is clear.

3. **Points in generic position. Applications and examples.** The following definition is taken from [O] in which one can find a rather complete discussion of the notion of generic position.

We recall that if $p_1,\ldots,p_m$ are points of $\mathbb{P}^r$, $N=\binom{n+r}{r}$ and $v_n:\mathbb{P}^r\to P^{N-1}$ is a Veronese embedding then $\rho(A_{n,m})$ is the rank of the matrix which has as columns the coordinates of the points $v_n(p_1),\ldots,v_n(p_m)$.

DEFINITION 3.1. The points $p_1,\ldots,p_m$ are *in generic position* if $\rho(A_{n,m})=\operatorname{Min}\{N,m\}$ (i.e. $\rho(A_{n,m})$ is the greatest possible), for any $n>0$.

The following lemma provides a method for deciding if m points are in generic position.

LEMMA 3.2. *Let d be the least degree of a hypersurface containing $p_1,\ldots,p_m$ i.e. $d=\operatorname{Min}\{n\mid\rho(A_{n,m})<N\}$, then $p_1,\ldots,p_m$ are in generic position if and only if $v_d(p_1),\ldots,v_d(p_m)$ are linearly independent in $\mathbb{P}^{N-1}$.*

Proof. (Cf. [O], lemma 3.2).

EXAMPLES. (1). One or two points of $\mathbb{P}^r$ are always in generic position, (2) any set of points of $\mathbb{P}^1$ is always in generic position, (3) $r+1$ points of $\mathbb{P}^r$ $(r>1)$ are in generic position if and only if they do not lie on a hyperplane, (4) six points of $\mathbb{P}^2$ are in generic position if and only if they do not lie on a conic, (5) ten points of $\mathbb{P}^3$ are in generic position if and only if they do not lie on a quadric.

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Now it is easy to prove the following:

THEOREM 3.3. *If* Proj$(G(A))$ *is reduced and consists of points in generic position in* $\mathbb{P}^r$ *then* $G(A)$ *is reduced.*

Proof. $\rho(A_{n,s}) \le H(n) = \dim(T^n)$ for any $n > 0$ (Cf. proposition 2.4). If $s \le N$ then $\rho(A_{n,s}) = s = e(A)$; but $H(n) \le e(A)$ (cf. [L], theorem 1.9) then $\rho(A_{n,s}) = H(n)$. If $N < s$ then $\rho(A_{n,s}) = N$ and $H(n) = \dim_k (M^n/M^{n+1}) \le N$ so $\rho(A_{n,s}) = H(n)$. Then the result follows from Theorem 2.7).

The previous results give precise computations of the Hilbert function for curves with reduced tangent cone. These computations seem to have interested many authors (cf. [S], ch. 2, §3):

COROLLARY 3.4. *If* $G(A)$ *is reduced then* $H(n) = \rho(A_{n,s})$ *for any* $n > 0$. *If in addition the points of* Proj$(G(A))$ *are in generic position then* $H(n) = \text{Min}\left\{\binom{n+r}{r}, e(A)\right\}$, *hence* $H(n) = e(A)$ *if* $\binom{n+r}{r} \ge e(A)$.

Proof. The statement is an immediate consequence of Theorem 2.7 and of Definition 3.1.

REMARK 3.5. It is well known that if emdim$(A) = H(1) = e(A)$ or $H(1) = 2$ (for example if C is a plane curve) then $G(A)$ is a Cohen Macaulay ring (cf. [S], ch. 2, proposition 3.4, theorem 3.10 or [D]). So in these cases Proj$(G(A))$ reduced implies $G(A)$ reduced.

The following example shows that the condition "the vector spaces T_i^n span the vector space T^n" is necessary in Theorem 2.7 i.e. in general Proj$(G(A))$ reduced does not imply $G(A)$ reduced.

EXAMPLE 1. Let $\bar{R} = \prod_{i=1}^4 k[t_i]$, and consider the ring $R = k[u_1, u_2, u_3] \subset \bar{R}$ where $u_1 = (t_1, 0, t_3, -t_4)$, $u_2 = (0, t_2, t_3, t_4)$, $u_3 = (0, 0, t_3^2, 0)$ i.e. $R = k[X, Y, Z]/((Y, Z) \cap (X, Z) \cap (X+Y, Z) \cap (X-Y, Z-X^2))$. Then the origin is an ordinary singularity of Spec R. Now let $N = (u_1, u_2, u_3)$ and $A = R_N$. We want to prove that $H(1) = 3$ i.e. the vectors $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in N/N^2$ are linearly independent. N^2 is generated by $u_1^2 = (t_1^2, 0, t_3^2, t_4^2)$, $u_2^2 = (0, t_2^2, t_3^2, t_4^2)$, $u_3^2 = (0, 0, t_3^4, 0)$, $u_1u_2 = (0, 0, t_3^2, -t_4^2)$, $u_1u_3 = (0, 0, t_3^3, 0)$. Then if $c_1u_1 + c_2u_2 + c_3u_3 \in N^2$ we have $c_1t_1 \in (t_1^2)$, $c_2t_2 \in (t_2^2)$ hence $c_1 = c_2 = 0$ so $u_3 \in N^2$ i.e. $u_3 = (0, 0, t_3^2, 0) = f_1u_1^2 + f_2u_2^2 + f_3u_3^2 + f_4u_1u_2 + f_5u_1u_3$ $(f_1, f_2, f_3 \in k[u_1, u_2, u_3])$. But it is easy to see that this implies $t_3^2 \in (t_3^3)$. Contradiction. So $H(1) = 3$. Now the points of Proj R are collinear so the corresponding spaces T_i, $1 \le i \le 3$, do not span the tangent space $T = \mathbb{A}^3$ to Spec A at p. Then $G(A)$ is not reduced (see Theorem 2.7).

The converse of Theorem 3.3 is obviously false. In fact let A be the local ring of n lines passing through the origin. Then $G(A)$ is reduced but

Proj($G(A)$) may consist of points not in generic position. Here is another example:

EXAMPLE 2. Let $A=R_N$ be the local ring at the point $p \leftrightarrow (t^s-1, t(t^s-1), t^2(t^s-1))$ of $R=k[t^s-1, t(t^s-1), t^2(t^s-1)]$ (car $k=0$). Then it is easy to see that p is an ordinary singularity and the points $p_1, \ldots, p_s$ of Proj($G(A)$) lie on a conic hence if $s>5$ they are not in generic position ($\rho(A_{2,s})=5$). Furthermore it is easy to check that $J^n \cap M^{n-1}=M^n$ for any $n>0$, so $G(A) \to G(\bar{A})$ is injective and $G(A)$ is reduced by Proposition 2.6.

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