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Supersymmetric gauge theories

Local gauge invariance is a very powerful requirement. It is a symmetry principle that provides a powerful rationale for (Yang–Mills type) dynamics which, together with the ideas of spontaneous symmetry breaking, forms the basis of the Standard Model. To preserve the spectacular success of the Standard Model, it is reasonable to expect that its supersymmetric extension will incorporate the gauge principle. This then leads us to consider theories that are both supersymmetric and locally gauge invariant. In this chapter, we develop a formula analogous to Eq. (5.73) for a gauge invariant supersymmetric model with an arbitrary gauge group and any number of “matter” chiral superfields in specified representations of this group. This formula will then be our starting point for developing the Minimal Supersymmetric Standard Model or, for that matter, globally supersymmetric grand unified theories.

6.1 Gauge transformations of superfields

We saw in Chapter 4 that internal symmetry transformations must commute with the super-charge. Thus the various components of the superfields must transform in the same way under any internal symmetry transformation and, in particular, under a local gauge transformation. Hence, for a chiral scalar supermultiplet with components $(\mathcal{S}, \psi, \mathcal{F})$, we want the Lagrangian density to be invariant under the local gauge transformations,

$$\mathcal{S}_a(x) \rightarrow \left[e^{igt_A \omega_A(x)} \right]_{ab} \mathcal{S}_b(x), \quad (6.1a)$$

$$\psi_a(x) \rightarrow \left[e^{igt_A \omega_A(x)} \right]_{ab} \psi_b(x), \quad (6.1b)$$

$$\mathcal{F}_a(x) \rightarrow \left[e^{igt_A \omega_A(x)} \right]_{ab} \mathcal{F}_b(x), \quad (6.1c)$$

where we write the local transformation parameters as $\omega_A(x)$ to distinguish them from the SUSY transformation parameter α or the Majorana coordinate θ . The $\omega_A(x)$ are, of course, real functions of x , g is the gauge coupling constant, and the

t_A are matrix representations of the generators of the gauge group that satisfy the Lie algebra $[t_A, t_B] = i f_{ABC} t_C$.

What would be the corresponding transformation property of the superfield $\hat{S}(x)$? The answer would be easy if the ω_A were independent of x : then we would have simply that $\hat{S}_a \rightarrow [e^{igt_A \omega_A}]_{ab} \hat{S}_b$. However, such a form cannot be correct for a local gauge transformation because of the derivatives that appear in the expansion in Eq. (5.34) or (5.36).

Recall, however, that the expansion (5.40) of the superfield in terms of $\hat{x} = x + \frac{i}{2} \bar{\theta} \gamma_5 \gamma_\mu \theta$ has no derivatives. We would then be tempted to consider the transformations,

$$\hat{S}_a(\hat{x}, \theta) \rightarrow [e^{igt_A \omega_A(\hat{x})}]_{ab} \hat{S}_b(\hat{x}, \theta).$$

The component fields would then transform as

$$\begin{aligned} \mathcal{S}_a(\hat{x}) &\rightarrow [e^{igt_A \omega_A(\hat{x})}]_{ab} \mathcal{S}_b(\hat{x}), \\ \psi_a(\hat{x}) &\rightarrow [e^{igt_A \omega_A(\hat{x})}]_{ab} \psi_b(\hat{x}), \\ \mathcal{F}_a(\hat{x}) &\rightarrow [e^{igt_A \omega_A(\hat{x})}]_{ab} \mathcal{F}_b(\hat{x}). \end{aligned}$$

These reduce to (6.1a)–(6.1c) for $\theta = 0$. This cannot, however, be right either because after the gauge transformation, the components of \hat{S} , which was a left-chiral superfield, no longer transform as a left-chiral superfield. This is because $e^{igt_A \omega_A(\hat{x})}$ (which has only one component field ω_A) is *not* a left-chiral superfield.

To ensure that the gauge transformed left-chiral superfield remains a left-chiral superfield, we are forced to introduce a set of left-chiral scalar superfields $\hat{\Omega}_A$ with as many members as the generators of the gauge group. We then consider the superfield transformation,

$$\hat{S}_a(\hat{x}, \theta) \rightarrow [e^{igt_A \hat{\Omega}_A(\hat{x})}]_{ab} \hat{S}_b(\hat{x}, \theta), \tag{6.2}$$

which at least has the virtue that the transform of a left-chiral superfield remains a left-chiral superfield. The components then transform as

$$\mathcal{S}_a(\hat{x}) \rightarrow [e^{igt_A \hat{\Omega}_A(\hat{x})}]_{ab} \mathcal{S}_b(\hat{x}), \tag{6.3a}$$

$$\psi_a(\hat{x}) \rightarrow [e^{igt_A \hat{\Omega}_A(\hat{x})}]_{ab} \psi_b(\hat{x}), \tag{6.3b}$$

$$\mathcal{F}_a(\hat{x}) \rightarrow [e^{igt_A \hat{\Omega}_A(\hat{x})}]_{ab} \mathcal{F}_b(\hat{x}). \tag{6.3c}$$

Setting $\theta = 0$, i.e. looking at the scalar components of (6.3a)–(6.3c), we see that we almost recover (6.1a)–(6.1c), except for what looks like a complex gauge transformation parameter (since the scalar components of $\hat{\Omega}_A$ are complex functions of x). We will return to this later.

We stress here that the “parameter superfield” $\hat{\Omega}$ is not a dynamical degree of freedom. Its components are just classical functions (Grassman-valued in the fermion case) of the spacetime co-ordinates. We need to introduce such a field so that after a gauge transformation, chiral superfields transform appropriately under a supersymmetry transformation.

Now that we have been able to sensibly extend the notion of gauge transformations to chiral superfields, we proceed to examine how to couple these in a gauge invariant way. Here, and in the following, by gauge invariance we will mean invariance under this extended gauge transformation. Since all interactions of chiral superfields with one another are given by the superpotential, we begin our study with that. However, because the superpotential is simply a polynomial of chiral superfields and *does not* contain any spacetime or supercovariant derivatives, it is clear that choosing it to be invariant under global gauge transformations ensures it is also invariant under the transformations (6.2). The Lagrangian density derived from this is then also invariant.

We thus have only to worry about the Kähler potential contributions which give rise to the kinetic terms for the component fields. For renormalizable theories, the Kähler potential is given by (5.60). We see immediately that this term is not invariant under the transformation (6.2) for the chiral superfield because the gauge parameter superfield $\hat{\Omega}_A$ is intrinsically complex. We have (with matrix multiplication implied),

$$\hat{S}^\dagger \rightarrow \hat{S}^\dagger e^{-igt_A \hat{\Omega}_A^\dagger}$$

and, as a result, the Kähler potential term,

$$\hat{S}^\dagger \hat{S} \rightarrow \hat{S}^\dagger e^{-igt_A \hat{\Omega}_A^\dagger} e^{igt_B \hat{\Omega}_B} \hat{S}$$

is no longer a gauge invariant. This should not be surprising. In the usual formulation of gauge theories, kinetic terms for the scalar or fermion fields are also not gauge invariant. We have to introduce new fields (the gauge potentials) and couple these to the scalars or fermions via a gauge covariant derivative to obtain a gauge invariant Lagrangian that includes these kinetic terms.¹

Towards this end, we are led to introduce a set of gauge potential superfields $\hat{\Phi}_A$ in which the vector potentials reside. These are not chiral superfields, but are chosen to satisfy the reality conditions $\hat{\Phi}_A^\dagger = \hat{\Phi}_A$ so that their bosonic components are real while their fermionic components are Majorana. This ensures that the vector potential and the gauge field strength are real. The SUSY transformation rules for

¹ In fact, the parallel is exact since global gauge invariance of the Yukawa interactions of fermions as well as the scalar potential ensures these are also locally gauge invariant, just as the global gauge invariance of the superpotential (which leads to Yukawa interactions and the scalar potential in a supersymmetric theory) also guarantees its local gauge invariance.

its components are given by Eq. (5.29a)–(5.29g) of Chapter 5 (with the index A implied). We will see shortly that it is possible to work with the components \mathcal{S} , ψ , \mathcal{M} , and \mathcal{N} set to zero. The curl supermultiplet that contains the field strengths is then constructed from $\hat{\Phi}_A$, just as the field strengths $F_A^{\mu\nu}$ are constructed from the vector potentials.

In order to maintain local gauge invariance of the Kähler potential, we modify it to,²

$$K = \hat{\mathcal{S}}^\dagger e^{-2gt_A \hat{\Phi}_A} \hat{\mathcal{S}} \quad (6.4)$$

where it is, of course, implicit that the dimensionality of the matrix t_A depends on the representation to which the chiral superfield belongs. We then *require* that the Kähler potential remains invariant under a gauge transformation, i.e.

$$\hat{\mathcal{S}}^\dagger e^{-igt_P \hat{\Omega}_P^\dagger} e^{-2gt_A \hat{\Phi}'_A} e^{igt_Q \hat{\Omega}_Q} \hat{\mathcal{S}} = \hat{\mathcal{S}}^\dagger e^{-2gt_A \hat{\Phi}_A} \hat{\mathcal{S}}.$$

This then fixes the gauge transformation rule for the fields $\hat{\Phi}_A$ to be,

$$e^{-igt_P \hat{\Omega}_P^\dagger} e^{-2gt_A \hat{\Phi}'_A} e^{igt_Q \hat{\Omega}_Q} = e^{-2gt_A \hat{\Phi}_A}. \quad (6.5)$$

Notice that the Kähler potential is now not a polynomial in the fields since the field $\hat{\Phi}$ is exponentiated. It still has mass dimension 2, however, because as noted in the exercise following Eq. (5.60) of the last chapter, $[\hat{\Phi}] = 0$, and renormalizability is not affected.

Let us *define* a left-chiral superfield

$$gt_A \hat{W}_A \equiv -\frac{i}{8} \bar{D} D_R \left[e^{2gt_C \hat{\Phi}_C} D_L e^{-2gt_B \hat{\Phi}_B} \right] \quad (6.6)$$

where the $D_{R/L}$ are the right/left supercovariant derivatives defined in Chapter 5. Its chiral nature follows because we have already checked (see the exercise below (5.54)) that the components of D_R anticommute, so that by the “Majorana character” of D , $D_R(\bar{D} D_R) = D_R(D_R^T C D_R) = 0$. The leading component (i.e. the θ -independent term) of the superfield \hat{W}_A is a spinor, and we will call this a left-chiral *spinor* superfield (as opposed to a left-chiral scalar superfield).

Exercise *Convince yourself that none of the properties that we have derived for chiral superfields depended upon the fact that the leading component was a scalar. In other words, these properties of chiral superfields hold for \hat{W}_A also. In particular, powers of \hat{W}_A are left-chiral superfields (though not necessarily Lorentz scalars).*

² The Kähler potential (5.60) of a renormalizable theory is trivially invariant under global gauge transformations. More generally, if the Kähler potential $K(\hat{\mathcal{S}}_L, \hat{\mathcal{S}}_L^\dagger)$ is chosen to be globally gauge invariant, then $K(\hat{\mathcal{S}}_L, \hat{\mathcal{S}}_L^\dagger e^{-2gt_A \hat{\Phi}_A})$ will also be locally gauge invariant if $\hat{\Phi}_A$ transform as discussed below. This is because the product of any representation times the adjoint contains the original representation.

Since we know the corresponding transformation rule for $\hat{\Phi}_A$, we can now work out how the fields \hat{W}_A transform under a gauge transformation. We have,

$$gt_A \hat{W}_A \rightarrow -\frac{i}{8} \bar{D} D_R \left[e^{igt_P \hat{\Omega}_P} e^{2gt_C \hat{\Phi}_C} e^{-igt_{P'} \hat{\Omega}_{P'}} D_L e^{igt_{Q'} \hat{\Omega}_{Q'}} e^{-2gt_B \hat{\Phi}_B} e^{-igt_Q \hat{\Omega}_Q} \right].$$

Now, since $D_L \hat{\Omega}_{Q'}^\dagger = 0$ (because $\hat{\Omega}_{Q'}^\dagger$ is a right-chiral superfield),

$$e^{-igt_{P'} \hat{\Omega}_{P'}} D_L e^{igt_{Q'} \hat{\Omega}_{Q'}} = D_L,$$

and our gauge transformation simplifies to,

$$gt_A \hat{W}_A \rightarrow -\frac{i}{8} \bar{D} D_R \left[e^{igt_P \hat{\Omega}_P} e^{2gt_C \hat{\Phi}_C} D_L e^{-2gt_B \hat{\Phi}_B} e^{-igt_Q \hat{\Omega}_Q} \right].$$

The same type of argument allows us to move $\bar{D} D_R$ past the left-chiral superfield $e^{igt_P \hat{\Omega}_P}$, and we find that

$$gt_A \hat{W}_A \rightarrow -\frac{i}{8} e^{igt_P \hat{\Omega}_P} \left[\bar{D} D_R e^{2gt_C \hat{\Phi}_C} D_L e^{-2gt_B \hat{\Phi}_B} e^{-igt_Q \hat{\Omega}_Q} \right].$$

The D_L in the square brackets may act on either $e^{-2gt_B \hat{\Phi}_B}$ or $e^{-igt_Q \hat{\Omega}_Q}$. When it acts on the latter, the corresponding contribution to the square bracket becomes,

$$\bar{D} D_R D_L e^{-igt_Q \hat{\Omega}_Q}.$$

Since $D_R e^{-igt_Q \hat{\Omega}_Q} = 0$, we can replace $D_R D_L$ by the anticommutator, and then using (5.54), obtain $D_R D_L e^{-igt_Q \hat{\Omega}_Q} = -2i \not{\partial} C e^{-igt_Q \hat{\Omega}_Q}$. Then, since spacetime and supersymmetric covariant derivatives commute, we get

$$\begin{aligned} \bar{D} D_R D_L e^{-igt_Q \hat{\Omega}_Q} &= -2i \partial_\mu \bar{D}_L \gamma^\mu C e^{-igt_Q \hat{\Omega}_Q} = \\ 2i \partial_\mu \bar{D}_L C \gamma^{\mu T} e^{-igt_Q \hat{\Omega}_Q} &= -2i \partial_\mu (\gamma^\mu D_R e^{-igt_Q \hat{\Omega}_Q})^T = 0 \end{aligned}$$

where the expression in the brackets vanishes because $\hat{\Omega}_Q$ is left handed. Thus, we only get a contribution when D_L acts on $e^{-2gt_B \hat{\Phi}_B}$ thereby giving us our final result for the gauge transformation of \hat{W}_A ,

$$t_A \hat{W}_A \rightarrow e^{igt_P \hat{\Omega}_P} t_B \hat{W}_B e^{-igt_Q \hat{\Omega}_Q}. \quad (6.7)$$

Notice that unlike the transformation law (6.5) for the gauge potential superfields $\hat{\Phi}_A$ which entailed both $\hat{\Omega}_A$ and $\hat{\Omega}_A^\dagger$, the transformation law for $t_A \hat{W}_A$ brings in only the fields $\hat{\Omega}_A$. In fact, $t_A \hat{W}_A$ transforms as a *gauge field strength* $F_{\mu\nu}^A$ (except that the local transformation parameter is a superfield).³

³ See, for example, *Introduction to Quantum Field Theory* by M. Peskin and D. Schroeder, Perseus Press (1995), Eq. (15.36), where the field strength transforms as $t_A F_{\mu\nu A} \rightarrow e^{igt_P \alpha P} t_B F_{\mu\nu B} e^{-igt_Q \alpha Q}$.

Let us summarize the main results for local gauge transformations of superfields. Chiral superfields transform as

$$\hat{\mathcal{S}} \rightarrow e^{igt_A \hat{\Omega}_A} \hat{\mathcal{S}} \quad \text{and} \quad \hat{\mathcal{S}}^\dagger \rightarrow \hat{\mathcal{S}}^\dagger e^{-igt_A \hat{\Omega}_A^\dagger}; \quad (6.8a)$$

the gauge potential superfield transforms as

$$e^{-2gt_A \hat{\Phi}_A} \rightarrow e^{igt_P \hat{\Omega}_P^\dagger} e^{-2gt_B \hat{\Phi}_B} e^{-igt_Q \hat{\Omega}_Q}; \quad (6.8b)$$

finally, the superfields

$$gt_A \hat{W}_A = -\frac{i}{8} \bar{D} D_R \left[e^{2gt_C \hat{\Phi}_C} D_L e^{-2gt_B \hat{\Phi}_B} \right]$$

transform as

$$t_A \hat{W}_A \rightarrow e^{igt_P \hat{\Omega}_P} t_B \hat{W}_B e^{-igt_Q \hat{\Omega}_Q}. \quad (6.8c)$$

We will see below that it is the superfield \hat{W}_A that contains the field strength $F_A^{\mu\nu}$, and we will work out its other components. But first, to connect up this rather formal discussion with the usual formulation of gauge theories, let us work out the transformations (6.8b), and later (6.8c), in terms of the component fields.

6.2 The Wess–Zumino gauge

In the last chapter, we showed that under supersymmetry the $F^{\mu\nu}$, λ , and the \mathcal{D} components of the curl superfield transformed into one another, but we did not discuss the other components of this multiplet. The reason for this, as we show next, is that we can work with all but the λ , V^μ , and \mathcal{D} components of the *gauge potential superfield* set to zero. Then, because the curl superfield is derived from the gauge potential, the question of the other components does not arise.

6.2.1 Abelian gauge transformations

We begin by working out the transformations (6.8b) in component form for an Abelian theory. In this case, (6.8b) reads,

$$g \hat{\Phi}' = g \hat{\Phi} + i \frac{g}{2} (\hat{\Omega} - \hat{\Omega}^\dagger). \quad (6.9)$$

Notice that $i \frac{g}{2} (\hat{\Omega} - \hat{\Omega}^\dagger)$ is a real superfield so that $\hat{\Phi}$ remains real under a gauge transformation.

Recall that $\hat{\Omega}$ is a classical left-chiral scalar superfield. We denote its components by ω , ξ_L , and ζ . We are abusing notation here by using the symbol ω both for the (real) parameter of the local gauge transformation in (6.1a)–(6.1c) as well as for the

(complex) scalar component of $\hat{\Omega}$, but we trust that this will not cause confusion. We can expand $\hat{\Omega}$ in its canonical form,

$$\begin{aligned} \hat{\Omega} = & \omega(x) + i\sqrt{2}\bar{\theta}\xi_L(x) + i\bar{\theta}\theta_L\zeta(x) + \frac{i}{2}\bar{\theta}\gamma_5\gamma_\mu\theta\partial^\mu\omega(x) \\ & - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\bar{\theta}\not{\theta}\xi_L(x) + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\omega(x). \end{aligned} \quad (6.10)$$

Then,

$$\begin{aligned} i\frac{g}{2}(\hat{\Omega} - \hat{\Omega}^\dagger) = & i\frac{g}{2} \left\{ i\sqrt{2}\omega_I + i\sqrt{2}\bar{\theta}\xi + \frac{i}{\sqrt{2}}\bar{\theta}\theta\zeta_R + \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\zeta_I \right. \\ & \left. + \frac{i}{\sqrt{2}}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\omega_R + \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\bar{\theta}\not{\theta}\gamma_5\xi + i\frac{\sqrt{2}}{8}(\bar{\theta}\gamma_5\theta)^2\Box\omega_I \right\}, \end{aligned}$$

where, $\omega = \frac{\omega_R + i\omega_I}{\sqrt{2}}$ and $\zeta = \frac{\zeta_R + i\zeta_I}{\sqrt{2}}$. Reading off the components of (6.9) immediately tells us that under a gauge transformation, the various components of $\hat{\Phi}$ transform as,

$$S' = S - \frac{1}{\sqrt{2}}\omega_I, \quad (6.11a)$$

$$\psi' = \psi - \frac{i}{2}\gamma_5\xi, \quad (6.11b)$$

$$\mathcal{M}' = \mathcal{M} - \frac{1}{\sqrt{2}}\zeta_I, \quad (6.11c)$$

$$\mathcal{N}' = \mathcal{N} - \frac{1}{\sqrt{2}}\zeta_R, \quad (6.11d)$$

$$V^{\mu'} = V^\mu - \frac{1}{\sqrt{2}}\partial^\mu\omega_R, \quad (6.11e)$$

$$\lambda' = \lambda, \quad (6.11f)$$

$$\mathcal{D}' = \mathcal{D}. \quad (6.11g)$$

This transformation preserves the reality of the Bose fields and the Majorana nature of the Fermi fields.

The important thing to note is that even if we started with a multiplet with non-zero (S , ψ , \mathcal{M} , \mathcal{N} , V^μ , λ , \mathcal{D}), by choosing ω_I , ξ , ζ_I , and ζ_R appropriately, we can set S' , ψ' , \mathcal{M}' , and \mathcal{N}' to zero! This choice is called the *Wess–Zumino gauge*. Of course, if after setting these to zero we perform another SUSY transformation, we will re-generate these components again. Thus, the Wess–Zumino (WZ) gauge is not supersymmetric.

The local parameter $\omega_R(x)$ is not fixed by our choice of the WZ gauge, and the transformation corresponding to just this parameter reads,

$$V^{\mu'} = V^\mu - \frac{1}{\sqrt{2}} \partial^\mu \omega_R, \quad (6.12a)$$

$$\lambda' = \lambda, \quad (6.12b)$$

$$\mathcal{D}' = \mathcal{D}, \quad (6.12c)$$

while the other components are not affected by it. We still have the freedom to perform the gauge transformations (6.12a)–(6.12c). But these are the usual gauge transformations for an Abelian theory. The gauge field changes by a gradient, and the transformation parameter is the real part of the scalar component of the parameter superfield $\hat{\Omega}$, while the other components (which, being partners of a gauge field, must be neutral) remain invariant under the gauge transformation. In other words, the choice of the WZ gauge does not fix the gauge in the usual sense of the term.

6.2.2 Non-Abelian gauge transformations

We will now work out the transformation laws for the gauge potential superfields of a non-Abelian gauge theory. Our starting point will be Eq. 6.8b:

$$e^{-2gt_A \hat{\Phi}'_A} = e^{igt_P \hat{\Omega}_P^\dagger} e^{-2gt_B \hat{\Phi}_B} e^{-igt_Q \hat{\Omega}_Q}. \quad (6.13)$$

In this case, because the matrices t_A do not commute with one another, it is not possible to explicitly display the transformation to the WZ gauge as we did for the Abelian case above. Using the fact that a product of the exponential of three arbitrary matrices \mathbf{u} , \mathbf{v} , and \mathbf{w} can be written (using the Baker–Campbell–Hausdorff formula) as,

$$e^{\mathbf{u}} e^{\mathbf{v}} e^{\mathbf{w}} = e^{\mathbf{z}},$$

with

$$\mathbf{z} = \mathbf{u} + \mathbf{v} + \mathbf{w} + \frac{1}{2}[\mathbf{u}, \mathbf{v}] + \frac{1}{2}[\mathbf{u}, \mathbf{w}] + \frac{1}{2}[\mathbf{v}, \mathbf{w}] + \dots,$$

where the ellipsis denotes terms with nested commutators, we see that (6.13) gives us,

$$\begin{aligned} 2gt_A \hat{\Phi}'_A &= 2gt_B \hat{\Phi}_B +igt_B(\hat{\Omega}_B - \hat{\Omega}_B^\dagger) + g^2 f_{BCD} t_D (\hat{\Phi}_B \hat{\Omega}_C - \hat{\Phi}_C \hat{\Omega}_B^\dagger) \\ &\quad + \frac{ig^2}{2} f_{BCD} t_D \hat{\Omega}_B \hat{\Omega}_C^\dagger + \dots \end{aligned} \quad (6.14)$$

The nested commutators that we have ignored have even higher powers of couplings. We first observe that the first two terms of (6.14) are the same as the corresponding

equation for the Abelian case. Moreover all other terms, including the ellipsis, vanish in this case since the structure constants are zero. We now see that there is an iterative procedure for going to the WZ gauge for the non-Abelian case. To zeroth order in g , the gauge transformation that we need is identical to (6.11a)–(6.11g) discussed above. But this must be corrected to the next order in g to include terms on the second line of (6.14), and then again to include the yet higher order terms denoted by the ellipsis. The point of this argument is only to convince the reader that there is a Wess–Zumino gauge even for non-Abelian theories, where the \mathcal{S}_A , ψ_A , \mathcal{M}_A , and \mathcal{N}_A components of the field $\hat{\Phi}_A$ can be set to zero.

From now on, we will work in the WZ gauge where the gauge potential superfield can be written as,

$$\hat{\Phi}_A = \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)V_A^\mu + i\bar{\theta}\gamma_5\theta\bar{\theta}\lambda_A - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\mathcal{D}_A. \quad (6.15)$$

We must remember that the components ω_{IA} , ξ_A , and ζ_A of the parameter superfield $\hat{\Omega}_A$ are now fixed, and the only gauge freedom corresponds to transformations that depend on the parameter ω_{RA} (which, we will see, is the gauge transformation in the conventional sense). We will thus compute how the components of $\hat{\Phi}_A$ change under the gauge transformation (6.13) taking the parameter superfield with $\omega_{IA} = \xi_A = \zeta_A = 0$, and $\omega_{RA}/\sqrt{2} \equiv \alpha_A$. In other words, we take,

$$\hat{\Omega}_A = \alpha_A(x) + \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\alpha_A(x) + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\alpha_A(x) \quad (6.16)$$

and

$$\hat{\Omega}_A^\dagger = \alpha_A(x) - \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\alpha_A(x) + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\alpha_A(x). \quad (6.17)$$

This transformation clearly preserves the WZ gauge. For simplicity, we will only compute an infinitesimal gauge transformation.

In evaluating the LHS of (6.13), we need only keep terms in the expansion of the exponential to second order, since each term in (6.15) is at least quadratic in θ . Thus,

$$\begin{aligned} e^{-2gt_A\hat{\Phi}'_A} &= 1 - g\bar{\theta}\gamma_5\gamma_\mu\theta(t \cdot V^{\mu'}) - 2ig(\bar{\theta}\gamma_5\theta)\bar{\theta}(t \cdot \lambda') \\ &\quad + \frac{1}{2}(\bar{\theta}\gamma_5\theta)^2 \{gt \cdot \mathcal{D}' - (gt \cdot V^{\mu'})(gt \cdot V'_\mu)\}. \end{aligned} \quad (6.18)$$

We have used (5.24e) to cast the expression in canonical form and introduced the notation $t \cdot X$ as a shorthand for $t_A X_A$.

A straightforward substitution of (6.16), (6.17), and (6.15) into the RHS of Eq. (6.13) yields, to first order in α_A ,

$$1 - g t_A \left[\bar{\theta} \gamma_5 \gamma_\mu \theta V_A^\mu + 2i(\bar{\theta} \gamma_5 \theta) \bar{\theta} \lambda_A - \frac{1}{2}(\bar{\theta} \gamma_5 \theta)^2 \mathcal{D}_A \right] - \frac{1}{2} g^2 t_A t_B (\bar{\theta} \gamma_5 \theta)^2 V_A^\mu V_{\mu B} \\ + i g \left[(\alpha \cdot t), e^{-2g t_A \hat{\Phi}_A} \right] + \frac{g}{2} \bar{\theta} \gamma_5 \gamma_\mu \theta \left\{ \partial^\mu \alpha \cdot t, e^{-2g t_A \hat{\Phi}_A} \right\} \\ + \frac{i}{8} g (\bar{\theta} \gamma_5 \theta)^2 \left[\square \alpha \cdot t, e^{-2g t_A \hat{\Phi}_A} \right].$$

The last term of this expression vanishes because the commutator has at least two θ s yielding a term with more than four θ s. The second last term involving an anticommutator has two non-vanishing terms:

$$g \bar{\theta} \gamma_5 \gamma_\mu \theta \partial^\mu \alpha \cdot t + \frac{g^2}{2} (\bar{\theta} \gamma_5 \theta)^2 \partial^\mu \alpha_A V_{\mu B} \{t_A, t_B\}.$$

The third last term becomes

$$- i g \left[(\alpha \cdot t), g t \cdot V^\mu \bar{\theta} \gamma_5 \gamma_\mu \theta \right] + 2 g \left[(\alpha \cdot t), g \bar{\theta} t \cdot \lambda \right] \bar{\theta} \gamma_5 \theta \\ + i \frac{g}{2} \left[(\alpha \cdot t), g t \cdot \mathcal{D} \right] (\bar{\theta} \gamma_5 \theta)^2 - i \frac{g}{2} \left[(\alpha \cdot t), (g t \cdot V^\mu)(g t \cdot V_\mu) \right] (\bar{\theta} \gamma_5 \theta)^2.$$

Putting all the pieces together, the RHS of (6.13) becomes

$$1 - \bar{\theta} \gamma_5 \gamma_\mu \theta (g t \cdot V^\mu - g \partial^\mu \alpha \cdot t + i g [\alpha \cdot t, g t \cdot V^\mu]) \\ - 2i(\bar{\theta} \gamma_5 \theta) \bar{\theta} (g t \cdot \lambda + i g [\alpha \cdot t, g t \cdot \lambda]) \\ + \frac{1}{2} (\bar{\theta} \gamma_5 \theta)^2 (g \mathcal{D} \cdot t - (g t \cdot V^\mu)(g t \cdot V_\mu) + g^2 \partial^\mu \alpha_A V_{\mu B} \{t_A, t_B\} \\ + i g^2 [\alpha \cdot t, t \cdot \mathcal{D}] - i g [\alpha \cdot t, (g t \cdot V^\mu)(g t \cdot V_\mu)]). \quad (6.19)$$

Equating the coefficients of $-g \bar{\theta} \gamma_5 \gamma_\mu \theta$ in (6.19) and (6.18) leads to,

$$(t \cdot V^{\mu'}) = t \cdot V^\mu - \partial^\mu \alpha \cdot t + i g \alpha_A V_B^\mu [t_A, t_B]. \quad (6.20)$$

Comparing coefficients of $-2i g (\bar{\theta} \gamma_5 \theta) \bar{\theta}$ gives,

$$(t \cdot \lambda') = t \cdot \lambda + i g \alpha_A \lambda_B [t_A, t_B]. \quad (6.21)$$

Finally, by equating coefficients of $\frac{g}{2} (\bar{\theta} \gamma_5 \theta)^2$ we get,

$$t \cdot \mathcal{D}' - g (t \cdot V^{\mu'}) (t \cdot V'_\mu) = t \cdot \mathcal{D} + i g \alpha_A \mathcal{D}_B [t_A, t_B] - g (t \cdot V^\mu) (t \cdot V_\mu) \\ + g \partial^\mu \alpha_A V_{\mu B} \{t_A, t_B\} - i g^2 \alpha^A V_\mu^B V^{\mu C} [t_A, t_B t_C].$$

Exercise Using the relation

$$i[t_A, t_B t_C] = -f_{ABD} t_D t_C - f_{ACD} t_B t_D,$$

show that Eq. (6.20) leads to

$$\begin{aligned} -g(t \cdot V^{\mu'}) (t \cdot V'_\mu) &= -g(t \cdot V^\mu) (t \cdot V_\mu) + g V_{\mu A} \partial^\mu \alpha_B \{t_A, t_B\} \\ &\quad - ig^2 [t_A, t_B t_C] \alpha_A V_\mu^B V^{\mu C}. \end{aligned}$$

Using the result of the exercise above, it is easy to show that

$$\mathcal{D}'_C = \mathcal{D}_C - g f_{ABC} \alpha_A \mathcal{D}_B. \quad (6.22)$$

It is now easy to see from (6.20) and (6.21) that for an infinitesimal gauge transformation by a parameter α_A , the component fields of the gauge potential superfield transform as,

$$V_C^{\mu'} = V_C^\mu - \partial^\mu \alpha_C - g f_{ABC} \alpha_A V_B^\mu, \quad (6.23a)$$

$$\lambda'_C = \lambda_C - g f_{ABC} \alpha_A \lambda_B, \quad (6.23b)$$

$$\mathcal{D}'_C = \mathcal{D}_C - g f_{ABC} \alpha_A \mathcal{D}_B. \quad (6.23c)$$

The first of these is exactly what we expect for the gauge transformation of a non-Abelian gauge potential. The vector field V_C^μ does not transform covariantly in that its transformation includes the inhomogeneous $\partial^\mu \alpha_C$ piece. The fields λ_C and \mathcal{D}_C transform covariantly under the gauge transformation (i.e. the transformation is homogeneous). For the λ_C , for instance, this is just what we expect since it corresponds to fermions in the adjoint representation of the gauge group.

6.3 The curl superfield in the Wess–Zumino gauge

Before we can proceed with the construction of supersymmetric Lagrangians for gauge theories, we need to work out the explicit form of the curl superfield \hat{W}_A introduced in Eq. (6.6):

$$g t_A \hat{W}_A = -\frac{i}{8} \bar{D} D_R \left[e^{2g t_A \hat{\Phi}_A} D_L e^{-2g t_B \hat{\Phi}_B} \right].$$

We will work in the WZ gauge where $\hat{\Phi}_A$ is given by (6.15). The calculation is rather lengthy, so we will break it up into a number of steps, and leave it to the reader to work through the details.

Step 1: Act with $D = \partial/\partial\bar{\theta} - i\partial\theta$ on $e^{-2g t_A \hat{\Phi}_A}$ to obtain

$$\begin{aligned} D e^{-2g t_A \hat{\Phi}_A} &= -2gt \cdot V^\mu (\gamma_5 \gamma_\mu \theta) + ig \left[\bar{\theta} \theta \gamma_5 t \cdot \lambda - (\bar{\theta} \gamma_5 \theta) t \cdot \lambda - \bar{\theta} \gamma_5 \gamma_\alpha \theta \gamma^\alpha t \cdot \lambda \right] \\ &\quad + 2(\bar{\theta} \gamma_5 \theta) \gamma_5 \theta \{gt \cdot \mathcal{D} - g^2 (t \cdot V)^2\} - ig (t \cdot \partial \mathcal{Y}) \theta (\bar{\theta} \gamma_5 \theta) \\ &\quad - \frac{1}{2} g (\bar{\theta} \gamma_5 \theta)^2 \gamma_5 (t \cdot \partial \lambda), \end{aligned} \quad (6.24)$$

where we have, as usual, made use of various θ identities to cast the result in canonical form.

Step 2: Act with P_L on the above expression to obtain

$$\begin{aligned}
 D_L e^{-2gt_A \hat{\Phi}_A} &= 2gt \cdot \mathcal{Y} \theta_R - ig [2\bar{\theta} \theta_R t \cdot \lambda_L + \bar{\theta} \gamma_5 \gamma_\alpha \theta \gamma^\alpha t \cdot \lambda_R] \\
 &\quad - 2(\bar{\theta} \gamma_5 \theta) \theta_L \{gt \cdot \mathcal{D} - g^2(t \cdot V)^2\} \\
 &\quad - ig(t \cdot \not{\theta} \mathcal{Y}) \theta_L (\bar{\theta} \gamma_5 \theta) + \frac{1}{2} (\bar{\theta} \gamma_5 \theta)^2 (gt \cdot \not{\theta} \lambda_R). \tag{6.25}
 \end{aligned}$$

Step 3: Next we multiply this by $e^{2gt \cdot \hat{\Phi}}$ to find after some tedious algebra,

$$\begin{aligned}
 e^{2gt \cdot \hat{\Phi}} D_L e^{-2gt \cdot \hat{\Phi}} &= 2gt \cdot \mathcal{Y} \theta_R - 2ig \bar{\theta} \theta_R t \cdot \lambda_L - ig (\bar{\theta} \gamma_5 \gamma_\alpha \theta) \gamma^\alpha t \cdot \lambda_R \\
 &\quad - 2\bar{\theta} \gamma_5 \theta \{gt \cdot \mathcal{D} + i \frac{g}{2} (t \cdot \not{\theta} \mathcal{Y}) + \frac{1}{2} ig^2 f_{ABCTC} \mathcal{Y}_B \mathcal{Y}_A\} \theta_L \\
 &\quad + \frac{1}{2} (\bar{\theta} \gamma_5 \theta)^2 \{(gt \cdot \not{\theta} \lambda_R) + 2g^2 f_{ABC} \mathcal{Y}_{BTC} \lambda_{AR}\}. \tag{6.26}
 \end{aligned}$$

The structure constants in (6.26) come from writing the product $t_A t_B$ of Lie algebra generators as the sum of a commutator and an anticommutator. The former gives the structure constants, while the symmetry of the latter (under interchange of A and B) helps to reduce the expression to the form given above.

Step 4: Work out $\bar{D} \frac{1+\gamma_5}{2} D$ to find

$$\bar{D} \frac{1+\gamma_5}{2} D = -\frac{\partial}{\partial \theta_a} P_{Rab} \frac{\partial}{\partial \bar{\theta}_b} + \bar{\theta} \theta_L \square - 2i(P_R \not{\theta} \theta)_c \frac{\partial}{\partial \theta_c}. \tag{6.27}$$

We can now work out the action of $\frac{\partial}{\partial \theta_a} P_{Rab} \frac{\partial}{\partial \bar{\theta}_b}$ on various terms in (6.26) involving θ . Using (5.26a)–(5.26e), we obtain:

$$\frac{\partial}{\partial \theta_a} P_{Rab} \frac{\partial}{\partial \bar{\theta}_b} \bar{\theta} P_R \theta = 4, \tag{6.28a}$$

$$\frac{\partial}{\partial \theta_a} P_{Rab} \frac{\partial}{\partial \bar{\theta}_b} (\bar{\theta} \gamma_5 \gamma_\alpha \theta) = 0, \tag{6.28b}$$

$$\frac{\partial}{\partial \theta_a} P_{Rab} \frac{\partial}{\partial \bar{\theta}_b} (\bar{\theta} \gamma_5 \theta) \theta_{Lc} = 4\theta_{Lc}, \quad \text{and} \tag{6.28c}$$

$$\frac{\partial}{\partial \theta_a} P_{Rab} \frac{\partial}{\partial \bar{\theta}_b} (\bar{\theta} \gamma_5 \theta)^2 = -8\bar{\theta} \theta_L. \tag{6.28d}$$

We thus obtain the action of $\bar{D}\frac{1+\gamma_5}{2}D$ on (6.26) to find

$$\begin{aligned} & 8igt \cdot \lambda_L + 8\{gt \cdot \mathcal{D} + i\frac{g}{2}(t \cdot \not{\partial} \mathcal{Y}) + i\frac{g^2}{2}f_{ABCTC} \mathcal{Y}_B \mathcal{Y}_A\}\theta_L \\ & + 4\bar{\theta}_L\{(gt \cdot \not{\partial}\lambda_R) + 2g^2f_{ABC} \mathcal{Y}_B t_C \lambda_{AR}\} + 2g\bar{\theta}_L(t \cdot \square \mathcal{Y})\theta_R \\ & + ig(\bar{\theta}\gamma_5\theta)^2(t \cdot \square\lambda_L) - 4ig\partial_\mu(t \cdot \mathcal{Y})\gamma^\mu\theta_L - 8g\bar{\theta}\not{\partial}\theta_L(t \cdot \lambda_L) + 4g\not{\partial}t \cdot \lambda_R\bar{\theta}_L \\ & + 8i\bar{\theta}\not{\partial}\theta_L\{gt \cdot \mathcal{D} + i\frac{g}{2}(t \cdot \not{\partial} \mathcal{Y}) + \frac{i}{2}g^2f_{ABCTC} \mathcal{Y}_B \mathcal{Y}_A\}\theta_L. \end{aligned} \quad (6.29)$$

Step 5: To complete our calculation, we will exploit the fact that \hat{W}_A is a left-chiral superfield; hence, its dependence on $\bar{\theta}_L$ and θ_R can arise only through \hat{x} . To obtain \hat{W}_A , we can thus pick off the terms involving only $\bar{\theta}_R$ and θ_L from Eq. (6.29), including of course the θ independent term, and then simply change the argument in the component fields from x to \hat{x} . These terms are,

$$\begin{aligned} & 8igt \cdot \lambda_L + 8\{gt \cdot \mathcal{D} + i\frac{g}{2}(t \cdot \not{\partial} \mathcal{Y}) + i\frac{g^2}{2}f_{ABCTC} \mathcal{Y}_B \mathcal{Y}_A\}\theta_L \\ & + 4\bar{\theta}_L\{(gt \cdot \not{\partial}\lambda_R) + 2g^2f_{ABC} \mathcal{Y}_B t_C \lambda_{AR}\} - 4ig\partial_\mu t \cdot \mathcal{Y}\gamma^\mu\theta_L + 4g\bar{\theta}_L\not{\partial}t \cdot \lambda_R \\ & = 8igt \cdot \lambda_L + 4ig\gamma^\mu\gamma^\nu[(\partial_\mu V_{\nu A} - \partial_\nu V_{\mu A})t_A + gf_{ABC}V_{\mu B}V_{\nu A}t_C]\theta_L \\ & + 8g\bar{\theta}_L t_C [\not{\partial}\delta_{AC} + gf_{ABC} \mathcal{Y}_B]\lambda_{RA} + 8gt \cdot \mathcal{D}\theta_L. \end{aligned} \quad (6.30)$$

Since $F_{\mu\nu A} = \partial_\mu V_{\nu A} - \partial_\nu V_{\mu A} - gf_{ABC}V_{\mu B}V_{\nu C}$, the term in the first square brackets above is just $t_A F_{\mu\nu A}$. Also, recall that the gauge group structure constants furnish a representation – the adjoint representation of the gauge group: $[t_C^{\text{adj}}]_{AB} = -if_{CAB}$. Using this, the second set of square brackets above yields

$$[\not{\partial}\delta_{AC} + gf_{ABC} \mathcal{Y}_B]\lambda_{RA} = [\not{\partial}\delta_{CA} + ig(t_B^{\text{adj}})_{CA} \mathcal{Y}_B]\lambda_{RA},$$

which is the gauge covariant derivative acting on the field λ_A that always belongs to the adjoint representation of the gauge group.

Thus, the $\bar{\theta}_L$ and θ_R independent part of $\bar{D}D_R\{e^{2gt \cdot \hat{\Phi}}D_L e^{-2gt \cdot \hat{\Phi}}\}$ is:

$$\begin{aligned} & 8igt_A \lambda_{AL} + 4ig\gamma^\mu\gamma^\nu t_A F_{\mu\nu A}\theta_L \\ & + 8g\bar{\theta}_L t_A (\not{\mathcal{D}}\lambda_R)_A + 8gt_A \mathcal{D}_A \theta_L. \end{aligned}$$

Comparing this with our definition (6.6) of $gt_A \hat{W}_A$, and then replacing the argument x with \hat{x} we find that, in the WZ gauge,

$$\hat{W}_A(\hat{x}, \theta) = \lambda_{LA}(\hat{x}) + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu A}(\hat{x})\theta_L - i\bar{\theta}_L(\not{\mathcal{D}}\lambda_R)_A - i\mathcal{D}_A(\hat{x})\theta_L, \quad (6.31)$$

where $\not{\mathcal{D}}_{AC} = \not{\partial}\delta_{AC} + ig(t_B^{\text{adj}})_{AC} \mathcal{Y}_B$ is the gauge covariant derivative in the adjoint representation. The interested reader can explicitly check that, aside from the

factor $8igt_A$, expanding the component fields about $\hat{x} = x$ indeed reproduces the expression in (6.29).

We note the following:

- We have already remarked that \hat{W}_A is a left-chiral *spinor* superfield. We see explicitly that the θ independent term in \hat{W}_A is a spinor. In addition to its gauge index A , it carries a spinor index which we have suppressed.
- \hat{W}_A has components λ_A , $F_{\mu\nu A}$, and \mathcal{D}_A ; the spinor λ_A and the scalar \mathcal{D}_A are the same components that are in the gauge potential superfield $\hat{\Phi}_A$, but instead of the vector potential, the third component of \hat{W}_A is the field strength.
- The product $\hat{W}_A \hat{W}_A$ is gauge invariant but not Lorentz invariant. Since $\hat{W}_A^c = C \overline{\hat{W}_A}^T$ transforms as the adjoint representation also, but is a right-chiral superfield, the combination

$$\overline{\hat{W}_A^c} \hat{W}_A$$

is a gauge-invariant, Lorentz-invariant bilinear in \hat{W}_A , and is a product of only left-chiral superfields. Its F -term is, therefore, a candidate for the Lagrangian density.

Exercise Show that

$$\begin{aligned} \overline{\hat{W}_A^c} = & \bar{\lambda}_{RA} + \frac{1}{2} F_{\mu\nu A} \bar{\theta}_R \gamma^\nu \gamma^\mu \\ & - i \bar{\theta}_L \left[-\bar{\lambda}_{LA} \overleftarrow{\not{\partial}} - g f_{CBA} \bar{\lambda}_{LC} \not{V}_B \right] - i \mathcal{D}_A \bar{\theta}_R. \end{aligned} \tag{6.32}$$

6.4 Construction of gauge kinetic terms

We have just seen that the F -term of $\overline{\hat{W}_A^c} \hat{W}_A$ is a candidate for a supersymmetric action. Moreover, inspection of (6.31) and (6.32) shows that this term contains a contribution proportional to $F_A^{\mu\nu} F_{\mu\nu A}$ so that this term potentially contains the gauge kinetic term. Before computing it, however, let us do some dimensional analysis to see the constraints imposed by renormalizability.

The dimensionality of the superfield \hat{W}_A can be worked out as

$$[\hat{W}_A] = [\bar{D} D D] = \left[\left(\frac{\partial}{\partial \theta} \right)^3 \right] = \frac{3}{2}.$$

Since renormalizability requires that the (composite) superfield whose F -term is proportional to the Lagrangian density have mass dimension ≤ 3 , this function can at most be quadratic in \hat{W}_A . We are thus left with just $\overline{\hat{W}_A^c} \hat{W}_A$ as the most general

Lorentz and gauge invariant bilinear.⁴ Notice also that since \hat{W}_A carries with it a spinor index, it can only enter via even powers, assuming that the other fields in the theory are just chiral scalar superfields. We do not, therefore, have to worry about products of \hat{W}_A s and \hat{S}_{Li} in a renormalizable theory.

We are thus led to compute the $\bar{\theta}\theta_L$ term of $\overline{\hat{W}_A^c} \hat{W}_A$. We can do so by simply using the form (6.31) for \hat{W}_A and setting $\hat{x} = x$ because, since $\hat{x} - x$ is already bilinear in θ , any other contribution to the $\bar{\theta}\theta_L$ term can only come from the θ independent term of this product. But this contribution is proportional to $\bar{\theta}\gamma_\mu\theta$ and not $\bar{\theta}\theta_L$ that we are looking for, and so does not contribute. We thus have,

$$\begin{aligned} & \overline{\hat{W}_A^c} \hat{W}_A \Big|_{\bar{\theta}\theta_L \text{ term}} \\ &= \left[\bar{\lambda}_{RA} + \frac{1}{2} F_{\mu\nu A} \bar{\theta}_R \gamma^\nu \gamma^\mu + i \bar{\theta}\theta_L \left(\bar{\lambda}_{LA} \overleftarrow{\not{\partial}} + g f_{CBA} \bar{\lambda}_{CL} \not{Y}_B \right) - i \mathcal{D}_A \bar{\theta}_R \right] \\ & \times \left[\lambda_{LA} + \frac{1}{2} \gamma^{\mu'} \gamma^{\nu'} F_{\mu'\nu' A} \theta_L - i \bar{\theta}\theta_L \left(\not{\partial} \lambda_{RA} + g f_{C'B'A} \not{Y}'_B \lambda_{RC'} \right) - i \mathcal{D}_A \theta_L \right] \Big|_{\bar{\theta}\theta_L \text{ term}} . \end{aligned}$$

The sources of $\bar{\theta}\theta_L$ terms are,

1. $\left(-i \bar{\lambda}_{RA} (\not{\partial} \lambda_{RA} + g f_{CBA} \not{Y}_B \lambda_{RC}) + i (\bar{\lambda}_{LA} \overleftarrow{\not{\partial}} + g f_{CBA} \bar{\lambda}_{CL} \not{Y}_B) \lambda_{LA} \right) \bar{\theta}\theta_L$
2. $\frac{1}{4} F_{\mu\nu A} F_{\mu'\nu' A} \bar{\theta}_R \gamma^\nu \gamma^\mu \gamma^{\mu'} \gamma^{\nu'} \theta_L,$
3. $-\mathcal{D}_A \mathcal{D}_A \bar{\theta}\theta_L,$ and
4. $-\frac{i}{2} F_{\mu\nu A} \bar{\theta}_R \gamma^\nu \gamma^\mu \mathcal{D}_A \theta_L - \frac{i}{2} F_{\mu\nu A} \bar{\theta}_R \gamma^\nu \gamma^\mu \mathcal{D}_A \theta_L.$

In the first term above, we can integrate by parts and shift the derivative acting on $\bar{\lambda}_A$ to one acting on λ_A , at the cost of a sign. Then, up to a surface term that we will not display, the derivative terms together yield the usual kinetic term for the spinor field λ aside from a factor of -2 .

The second term can be simplified by noting that,

$$\begin{aligned} \bar{\theta}_R \gamma^\nu \gamma^\mu \gamma^{\mu'} \gamma^{\nu'} \theta_L &= \frac{1}{4} Tr \left[\gamma^\nu \gamma^\mu \gamma^{\mu'} \gamma^{\nu'} P_L \{ \bar{\theta}\theta \mathbf{I} + \bar{\theta} \gamma_5 \theta \cdot \gamma_5 - \bar{\theta} \gamma_5 \gamma_\rho \theta \cdot \gamma_5 \gamma^\rho \} \right] \\ &= \frac{1}{2} (\bar{\theta}\theta_L) Tr \left[\gamma^\nu \gamma^\mu \gamma^{\mu'} \gamma^{\nu'} P_L \right] \end{aligned}$$

where we have used (5.21). This then simplifies to

$$\left(\frac{1}{2} F_{\mu\nu A} F_A^{\mu\nu} + \frac{i}{4} \epsilon^{\nu\mu\mu'\nu'} F_{A\mu\nu} F_{A\mu'\nu'} \right) \bar{\theta}\theta_L. \tag{6.33}$$

⁴ It is clear from (6.31) that $\gamma_5 \hat{W}_A = -\hat{W}_A$ so that $\overline{\hat{W}_A^c} \gamma_5 \hat{W}_A$ is not an independent term.

The first term in (6.33) is the usual gauge kinetic term apart from a factor -2 , while the second term can be re-written as

$$\begin{aligned} & \frac{1}{4} \epsilon^{\nu\mu\mu'\nu'} F_{A\mu\nu} F_{A\mu'\nu'} \\ &= -\epsilon^{\mu\nu\mu'\nu'} \left(\partial_\mu A_{A\nu} - \frac{g}{2} f_{ABC} A_{B\mu} A_{C\nu} \right) \left(\partial_{\mu'} A_{A\nu'} - \frac{g}{2} f_{AB'C'} A_{B\mu'} A_{C\nu'} \right) \\ &= -\epsilon^{\mu\nu\mu'\nu'} \partial_\mu \left(A_{A\nu} \partial_{\mu'} A_{A\nu'} - \frac{g}{3} f_{ABC} A_{A\nu} A_{B'\mu'} A_{C'\nu'} \right) \\ &\quad - \epsilon^{\mu\nu\mu'\nu'} \frac{g^2}{4} f_{ABC} f_{AB'C'} A_{B\mu} A_{C\nu} A_{B'\mu'} A_{C'\nu'}. \end{aligned}$$

In the last step the $1/3$ enters because it does not matter upon which of the three gauge potentials the derivative acts – they all give the same contribution. We will leave it to the reader (see exercise below) to check that the last line of the expression above vanishes, so that the second term of (6.33) turns out to be a total derivative and makes no contribution to the equations of motion.

Finally, the last term in our list contracts a symmetric and antisymmetric tensor, and so identically vanishes.

Exercise Verify that

$$\epsilon_{\mu\nu\mu'\nu'} \frac{g^2}{4} f_{ABC} f_{AB'C'} A_B^\mu A_C^\nu A_{B'}^{\mu'} A_{C'}^{\nu'} = 0.$$

Hints: one way to verify this is to note that we may write,

$$\begin{aligned} t_P f_{PBC} A_B^\mu A_C^\nu &= [t_B, t_C] A_B^\mu A_C^\nu \equiv [\mathbf{A}^\mu, \mathbf{A}^\nu] \\ t_Q f_{Q B'C'} A_{B'}^{\mu'} A_{C'}^{\nu'} &= [t_{B'}, t_{C'}] A_{B'}^{\mu'} A_{C'}^{\nu'} \equiv [\mathbf{A}^{\mu'}, \mathbf{A}^{\nu'}], \end{aligned}$$

where we have introduced matrices, $\mathbf{A}^\mu \equiv A_B^\mu t_B$ etc. Then since $\text{Tr}(t_P t_Q) \propto \delta_{PQ}$, the term in question becomes $\epsilon_{\mu\nu\mu'\nu'} \text{Tr}(\mathbf{A}^\mu \mathbf{A}^\nu \mathbf{A}^{\mu'} \mathbf{A}^{\nu'})$ (aside from a multiplicative constant), and so vanishes because of the cyclic property of the trace.

Collecting all terms from the computation of the coefficient of $\bar{\theta} \theta_L$ in $\widehat{W}_A^c \widehat{W}_A$ and inserting an additional $-1/2$ to put the gauge kinetic terms in canonical form, we obtain a supersymmetric and gauge-invariant Lagrangian density \mathcal{L}_{GK} for the gauge field kinetic terms,

$$\mathcal{L}_{\text{GK}} = \frac{i}{2} \bar{\lambda}_A \not{D}_{AC} \lambda_C - \frac{1}{4} F_{\mu\nu A} F_A^{\mu\nu} + \frac{1}{2} \mathcal{D}_A \mathcal{D}_A, \tag{6.34}$$

where, as before,

$$\begin{aligned} F_{\mu\nu A} &= \partial_\mu V_{\nu A} - \partial_\nu V_{\mu A} - g f_{ABC} V_{\mu B} V_{\nu C} \quad \text{and} \\ (\not{D} \lambda)_A &= \not{\partial} \lambda_A + i g (t_B^{\text{adj}} \not{Y}_B)_{AC} \lambda_C. \end{aligned}$$

We see that we have the usual gauge kinetic term for the Yang–Mills field. We also have a gauge invariant kinetic term for the massless spin $\frac{1}{2}$ fields λ_A in the adjoint representation of the gauge group. The spin zero fields \mathcal{D}_A enter without any derivatives so these will turn out to be auxiliary fields that satisfy *algebraic* equations of motion. The quanta of the theory whose Lagrangian density is \mathcal{L}_{GK} would thus be massless vector bosons together with a set of massless spin $\frac{1}{2}$ fermions, both in the adjoint representation of the gauge group. These fermions are termed *gauginos*.

Exercise *In our derivation of the Lagrangian density (6.34), we dropped surface terms. Show that these can be written as,*

$$\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}F_A^{\mu\nu}F_A^{\rho\sigma} - \frac{i}{2}\partial_\mu(\overline{\lambda}_A\gamma^\mu\gamma_5\lambda_A).$$

These terms do not contribute to the field equations. Even so, they might be relevant in a non-Abelian gauge theory when instanton effects are important. In Abelian gauge theories, they have no effect.

We remark that the superfield $\overline{\hat{W}}_A^c\hat{W}_A$ is intrinsically complex, and it was only fortuitous that the surface terms that we ignored were anti-Hermitian. This would not be a problem, since, as in the case of the Lagrangian density for chiral superfields that we obtained in the last chapter, we would simply have added the Hermitian conjugate. The point, however, is that (since the Hermitian and anti-Hermitian parts are separately supersymmetric and gauge invariant) a supersymmetric gauge theory may include terms proportional to the surface terms shown in the preceding exercise. In non-Abelian gauge theories, the corresponding constant of proportionality is conventionally written as θ (not to be confused with the Grassmann number θ that we have been using).

Finally we note that it is only in the WZ gauge that we can set the scalar components of $\hat{\Phi}_A$ to zero. This is what gave us only a finite number of terms in the expansion (6.6) of \hat{W}_A , and not an infinite series. The latter would have resulted in a non-polynomial Lagrangian where renormalizability would not have been at all clear.

6.5 Coupling chiral scalar to gauge superfields

We have already seen that the gauge interactions of chiral superfields enter via the Kähler potential (6.4),

$$\hat{\mathcal{S}}_L^\dagger e^{-2gt_A\hat{\Phi}_A}\hat{\mathcal{S}}_L. \quad (6.35)$$

There is one such term for every chiral scalar superfield that we introduce. The $(\bar{\theta}\gamma_5\theta)^2$ component is a candidate Lagrangian density. In the previous chapter we

had seen that, for $g = 0$, these gave rise to the kinetic energy terms for the scalar and fermion components of the chiral supermultiplet.

To work out this quartic term in θ , we substitute the expansions for $\hat{\mathcal{S}}_L^\dagger$ and $\hat{\mathcal{S}}_L$ from Eqs. (5.34) and (5.37) together with the unprimed version of (6.18) for the exponential of $\hat{\Phi}$ in the WZ gauge. There are then four sources of θ^4 terms in (6.35). First, we have the quartic terms from just $\hat{\mathcal{S}}_L^\dagger \hat{\mathcal{S}}_L$ multiplying the 1 from the exponential, which was exactly what we had worked out in Chapter 5. These terms are,

$$\hat{\mathcal{S}}_L^\dagger \hat{\mathcal{S}}_L \Big|_{\theta^4} = -\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2 \left\{ \frac{i}{2}\bar{\psi}\not{\theta}\psi + \mathcal{F}^\dagger\mathcal{F} + (\partial_\mu\mathcal{S})^\dagger(\partial^\mu\mathcal{S}) \right\}. \tag{6.36a}$$

Next we have contributions from,

$$-\hat{\mathcal{S}}_{La}^\dagger g(t \cdot V^\mu)_{ab} \hat{\mathcal{S}}_{Lb} (\bar{\theta}\gamma_5\gamma_\mu\theta), \tag{6.36b}$$

where the chiral superfields contribute two factors of θ . Then we have another contribution from,

$$-2ig(\bar{\theta}\gamma_5\theta)\hat{\mathcal{S}}_{La}^\dagger \bar{\theta}(t \cdot \lambda)_{ab} \hat{\mathcal{S}}_{Lb} \tag{6.36c}$$

with one θ from the chiral superfields, and finally, we have a contribution from,

$$\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2 \hat{\mathcal{S}}_{La}^\dagger \{gt \cdot \mathcal{D} - g^2(t \cdot V)^2\}_{ab} \hat{\mathcal{S}}_{Lb}. \tag{6.36d}$$

Non-zero terms from (6.36b) can only come when we have either θ_L and a $\bar{\theta}_L$ or θ_R and a $\bar{\theta}_R$ from the chiral superfields. These contributions are:

$$\begin{aligned} 2\bar{\psi}_a\theta_R[-gt \cdot V^\mu]_{ab}\bar{\theta}\psi_{Lb}(\bar{\theta}\gamma_5\gamma_\mu\theta) - \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial_\mu\mathcal{S}_a^\dagger[-gt \cdot V^\nu]_{ab}\mathcal{S}_b(\bar{\theta}\gamma_5\gamma_\nu\theta) \\ + \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\mathcal{S}_a^\dagger[-gt \cdot V^\nu]_{ab}\partial_\mu\mathcal{S}_b(\bar{\theta}\gamma_5\gamma_\nu\theta). \end{aligned}$$

Using (5.21) on the first of these terms, and (5.24d) on the remaining terms to cast this in the canonical form, we obtain,

$$\frac{1}{2}g\bar{\psi}(t \cdot \mathcal{Y})\psi_L(\bar{\theta}\gamma_5\theta)^2 - \frac{i}{2}(\bar{\theta}\gamma_5\theta)^2\partial_\mu\mathcal{S}^\dagger[gt \cdot V^\mu]\mathcal{S} + \frac{i}{2}(\bar{\theta}\gamma_5\theta)^2\mathcal{S}^\dagger[gt \cdot V^\mu]\partial_\mu\mathcal{S}.$$

Note that in the first term here we can rewrite (for reasons that will become clear shortly) the fermion bilinear using,

$$\begin{aligned} -\bar{\psi}_a(t \cdot \mathcal{Y})_{ab} \frac{1-\gamma_5}{2} \psi_b &= \bar{\psi}_b(t \cdot \mathcal{Y})_{ab} \frac{1+\gamma_5}{2} \psi_a \\ &= \frac{1}{2} [-\bar{\psi}(t \cdot \mathcal{Y})\psi_L + \bar{\psi}(t^* \cdot \mathcal{Y})\psi_R]. \end{aligned}$$

The terms from (6.36c) are

$$-2ig(\bar{\theta}\gamma_5\theta)(-i\sqrt{2}\bar{\psi}_a\theta_R)\bar{\theta}(t \cdot \lambda)_{ab}\mathcal{S}_b - 2ig(\bar{\theta}\gamma_5\theta)\mathcal{S}_a^\dagger\bar{\theta}(t \cdot \lambda)_{ab}(-i\sqrt{2}\bar{\theta}\psi_{Lb}).$$

Again, casting these in canonical form using the relations for bilinears of Majorana spinors together with (5.21), we obtain

$$\frac{g}{\sqrt{2}}(\bar{\theta}\gamma_5\theta)^2(t\cdot\lambda)_{ab}\psi_{Ra}\mathcal{S}_b + \frac{g}{\sqrt{2}}(\bar{\theta}\gamma_5\theta)^2\mathcal{S}_a^\dagger(t_A)_{ab}\bar{\lambda}_A\psi_{Lb}.$$

We take the Lagrangian density to be the coefficient of $-\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2$ in (6.35) which, as we saw in the last chapter, gives the correctly normalized kinetic terms for the scalar and fermion components of $\hat{\mathcal{S}}$.

Collecting all the terms from Eq. (6.36a)–(6.36d), we find that the contribution to the Lagrangian density from $\hat{\mathcal{S}}^\dagger e^{-2gt_A\hat{\Phi}_A}\hat{\mathcal{S}}$ is,

$$\begin{aligned} \mathcal{L}_{\text{gauge}} = & \frac{i}{2}\bar{\psi}\not{\partial}\psi + (\partial_\mu\mathcal{S})^\dagger(\partial^\mu\mathcal{S}) + \mathcal{F}^\dagger\mathcal{F} \\ & + i(\partial_\mu\mathcal{S})^\dagger g(t\cdot V^\mu)\mathcal{S} - i\mathcal{S}^\dagger g(t\cdot V^\mu)\partial_\mu\mathcal{S} - \mathcal{S}^\dagger [gt\cdot\mathcal{D} - g^2(t\cdot V)^2]\mathcal{S} \\ & + \frac{1}{2}[-g\bar{\psi}(t\cdot\mathcal{Y})\psi_L + g\bar{\psi}(t^*\cdot\mathcal{Y})\psi_R] \\ & - \left(\sqrt{2}g\mathcal{S}^\dagger t_A\bar{\lambda}_A \frac{1-\gamma_5}{2}\psi + \text{h.c.} \right). \end{aligned} \quad (6.37)$$

We can now cast the interactions of the scalar and fermion components of the chiral superfields with gauge bosons in the familiar form using gauge covariant derivatives introduced in Chapter 1. The covariant derivatives on \mathcal{S} are,

$$D_\mu\mathcal{S} = \partial_\mu\mathcal{S} + ig t\cdot V_\mu\mathcal{S} \quad (6.38a)$$

$$(D_\mu\mathcal{S})^\dagger = (\partial_\mu\mathcal{S})^\dagger - ig\mathcal{S}^\dagger t\cdot V_\mu. \quad (6.38b)$$

For the action of the covariant derivative on the Majorana spinor ψ , we must be careful because (3.3) shows that its left- and right-handed components are complex conjugates of one another.⁵ Thus, if ψ_L transforms according to a representation given by t_A , then ψ_R transforms according to the conjugate representation whose generators are given by $-t_A^*$.

An aside on conjugate representations Consider a field ϕ that transforms under some representation of a group, and let t_A be a matrix representation of the corresponding generators. Then if $\phi \rightarrow e^{i\alpha_A t_A}\phi$, $\phi^* \rightarrow e^{-i\alpha_A t_A^*}\phi^* = e^{i\alpha_A(-t_A)^*}\phi^*$. In other words, the conjugate field transforms with generators $(-t_A)^*$.

It is easy to see that these satisfy the same Lie algebra $[t_A, t_B] = if_{ABC}t_C$ as the generators t_A . Since the structure constants can be chosen to be real, we have

⁵ Only if we insist that the left- and right-handed components transform the same way, can we conclude that the fermion must belong to a real representation. For the case of the $U(1)$ group, this will mean that the charge of the fermion is zero. But we stress that it is possible to represent each chirality of a charged particle by a Majorana field. Then one of the chiral components of this Majorana field corresponds to the field of the particle, while the other corresponds to the antiparticle field.

$[t_A^*, t_B^*] = -if_{ABC}t_C^*$, or $[-t_A^*, -t_B^*] = if_{ABC}(-t_C^*)$. Thus, if t_A is a set of representation matrices, then $-t_A^*$ is equally good. If the set of $d \times d$ matrices t_A furnish a representation denoted by $\underline{\mathbf{d}}$, the matrices $-t_A^*$ provide another equally good representation of the same dimensionality. This is known as the conjugate representation and is denoted by $\underline{\mathbf{d}}^*$.

The gauge covariant derivative on a Majorana fermion ψ whose left-chiral component transforms via a representation furnished by t_A is thus given by,

$$D_\mu \psi = \partial_\mu \psi + ig(t \cdot V_\mu)\psi_L - ig(t^* \cdot V_\mu)\psi_R. \tag{6.38c}$$

We can then write the Lagrangian (6.37) as,

$$\begin{aligned} \mathcal{L}_{\text{gauge}} = & \frac{i}{2} \bar{\psi} \not{D} \psi + (D_\mu S)^\dagger (D^\mu S) + \mathcal{F}^\dagger \mathcal{F} \\ & - g S^\dagger t \cdot DS + \left(-\sqrt{2} g S^\dagger t_A \bar{\lambda}_A \frac{1-\gamma_5}{2} \psi + \text{h.c.} \right). \end{aligned} \tag{6.39}$$

6.5.1 Fayet–Iliopoulos D-term

We have seen that the D -term of any superfield is a candidate for a Lagrangian. The D -term of a product of chiral superfields, being a total derivative, is not interesting. However, the D -term of the gauge potential multiplet $\hat{\Phi}_A$ is not a derivative of anything. It is independent of the terms that we have considered so far. As we saw in (6.23c), it is however gauge covariant and not gauge invariant, unless of course $f_{ABC} = 0$, i.e. when the gauge group is Abelian. We can thus include

$$\mathcal{L}_{\text{FI}} = \xi_p D_p \tag{6.40}$$

in the Lagrangian density, where p runs over each $U(1)$ factor of the gauge group, where ξ_p are coupling constants with mass dimension $[\xi_p] = 2$.

It is easy to see that the D -term of higher powers of $\hat{\Phi}$ is not gauge invariant.

6.6 A master Lagrangian for SUSY gauge theories

We now collect the various contributions to the Lagrangian density of a renormalizable supersymmetric gauge theory that we have obtained into a single master formula which will serve as the starting point for SUSY model building. Our Lagrangian density consists of,

$$\mathcal{L} = \mathcal{L}_{\text{GK}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_F + \mathcal{L}_{\text{FI}}, \tag{6.41}$$

where \mathcal{L}_{GK} , $\mathcal{L}_{\text{gauge}}$, and \mathcal{L}_{FI} have been constructed in this chapter, and \mathcal{L}_F is as given in Eq. (5.70) of Chapter 5. \mathcal{L}_{GK} and $\mathcal{L}_{\text{gauge}}$ have been explicitly constructed

to be gauge invariant. Since the superpotential \hat{f} is a sum of products of superfields without any (spacetime or supercovariant) derivatives, \mathcal{L}_F will be just a product of fields with no derivatives. Thus, the condition for \mathcal{L}_F to be locally gauge invariant is that it simply be globally gauge invariant. This is guaranteed if the superpotential is globally gauge invariant.

The complete Lagrangian for renormalizable, supersymmetric gauge theories is

$$\begin{aligned} \mathcal{L} = & \sum_i (D_\mu \mathcal{S}_i)^\dagger (D^\mu \mathcal{S}_i) + \frac{i}{2} \sum_i \bar{\psi}_i \not{D} \psi_i + \sum_i \mathcal{F}_i^\dagger \mathcal{F}_i \\ & + \frac{i}{2} \sum_{A,B} \bar{\lambda}_A \not{D}_{AB} \lambda_B - \frac{1}{4} \sum_A F_{\mu\nu A} F_A^{\mu\nu} + \frac{1}{2} \sum_A \mathcal{D}_A \mathcal{D}_A \\ & + \left(-\sqrt{2}g \sum_i \mathcal{S}_i^\dagger t \cdot \bar{\lambda} \frac{1-\gamma_5}{2} \psi_i + \text{h.c.} \right) - g \sum_{i,A} \mathcal{S}_i^\dagger (t_A \mathcal{D}_A) \mathcal{S}_i \\ & - \sum_p \xi_p \mathcal{D}_p + \sum_i \left\{ -i \left(\frac{\partial \hat{f}}{\partial \mathcal{S}_i} \right)_{\hat{s}=S} \mathcal{F}_i + i \left(\frac{\partial \hat{f}}{\partial \mathcal{S}_i} \right)_{\hat{s}=S}^\dagger \mathcal{F}_i^\dagger \right\} \\ & - \frac{1}{2} \sum_{i,j} \bar{\psi}_i \left[\left(\frac{\partial^2 \hat{f}}{\partial \mathcal{S}_i \partial \mathcal{S}_j} \right)_{\hat{s}=S} \frac{1-\gamma_5}{2} + \left(\frac{\partial^2 \hat{f}}{\partial \mathcal{S}_i \partial \mathcal{S}_j} \right)_{\hat{s}=S}^\dagger \frac{1+\gamma_5}{2} \right] \psi_j, \end{aligned} \quad (6.42)$$

where i, j denote the matter field types, A is the gauge group index, and p runs over all the $U(1)$ factors of the gauge group.

To obtain our final formula, we may eliminate the auxiliary fields \mathcal{F}_i and \mathcal{D}_A via their equations of motion, which are purely algebraic:

$$\mathcal{F}_i = -i \left(\frac{\partial \hat{f}}{\partial \mathcal{S}_i} \right)_{\hat{s}=S}^\dagger \quad \text{and} \quad \mathcal{F}_i^\dagger = i \left(\frac{\partial \hat{f}}{\partial \mathcal{S}_i} \right)_{\hat{s}=S} \quad (6.43a)$$

$$\mathcal{D}_A = g \sum_i \mathcal{S}_i^\dagger t_A \mathcal{S}_i + \xi_A. \quad (6.43b)$$

Substituting into Eq. (6.42), we arrive at the master formula for supersymmetric gauge theories:

$$\begin{aligned} \mathcal{L} = & \sum_i (D_\mu \mathcal{S}_i)^\dagger (D^\mu \mathcal{S}_i) + \frac{i}{2} \sum_i \bar{\psi}_i \not{D} \psi_i + \sum_{\alpha,A} \left[\frac{i}{2} \bar{\lambda}_{\alpha A} (\not{D} \lambda)_{\alpha A} - \frac{1}{4} F_{\mu\nu \alpha A} F_{\alpha A}^{\mu\nu} \right] \\ & - \sqrt{2} \sum_{i,\alpha,A} \left(\mathcal{S}_i^\dagger g_{\alpha} t_{\alpha A} \bar{\lambda}_{\alpha A} \frac{1-\gamma_5}{2} \psi_i + \text{h.c.} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{\alpha, A} \left[\sum_i \mathcal{S}_i^\dagger g_\alpha t_{\alpha A} \mathcal{S}_i + \xi_{\alpha A} \right]^2 - \sum_i \left| \frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i} \right|_{\hat{\mathcal{S}}=S}^2 \quad (6.44) \\
& -\frac{1}{2} \sum_{i, j} \bar{\psi}_i \left[\left(\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j} \right)_{\hat{\mathcal{S}}=S} \frac{1 - \gamma_5}{2} + \left(\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j} \right)_{\hat{\mathcal{S}}=S}^\dagger \frac{1 + \gamma_5}{2} \right] \psi_j,
\end{aligned}$$

where the covariant derivatives are given by,

$$D_\mu \mathcal{S} = \partial_\mu \mathcal{S} + i \sum_{\alpha, A} g_\alpha t_{\alpha A} V_{\mu \alpha A} \mathcal{S}, \quad (6.45a)$$

$$D_\mu \psi = \partial_\mu \psi + i \sum_{\alpha, A} g_\alpha (t_{\alpha A} V_{\mu \alpha A}) \psi_L - i \sum_{\alpha, A} g_\alpha (t_{\alpha A}^* V_{\mu \alpha A}) \psi_R, \quad (6.45b)$$

$$(\mathcal{D}\lambda)_{\alpha A} = \mathcal{D}\lambda_{\alpha A} + i g_\alpha \left(t_{\alpha B}^{\text{adj}} \mathcal{V}_{\alpha B} \right)_{AC} \lambda_{\alpha C}, \quad (6.45c)$$

$$F_{\mu\nu\alpha A} = \partial_\mu V_{\nu\alpha A} - \partial_\nu V_{\mu\alpha A} - g_\alpha f_{\alpha ABC} V_{\mu\alpha B} V_{\nu\alpha C}. \quad (6.45d)$$

The index α that suddenly appears in (6.44) is simply to allow for several gauge couplings that would be present if the gauge group is not simple.

Exercise Observe that unlike ordinary derivatives, the covariant derivatives defined above do not commute. Show that their commutator is given by,

$$[D_\mu, D_\nu] = i \sum_{\alpha, A} g_\alpha t_{\alpha A} F_{\mu\nu\alpha A}. \quad (6.46)$$

We will return to this result when we consider the covariant derivative in general relativity.

We note the following features of our master Lagrangian density (6.44).

1. The first line is the usual gauge-invariant kinetic energies for the components of the chiral and gauge superfields. The derivatives that appear are gauge-covariant derivatives appropriate to the particular representation in which the field belongs. For example, if we are talking about SUSY QCD, for quark fields in the first line of Eq. (6.44) the covariant derivative contains triplet $SU(3)_C$ matrices: i.e. $D_\mu = \partial_\mu + i g_s \frac{\lambda_A}{2} V_A^\mu$, whereas the covariant derivative acting on the gauginos in the following line will contain octet matrices. These terms completely determine how *all* particles couple to gauge bosons. As in any gauge theory, this coupling is fixed by the minimal coupling prescription.
2. The next line describes the interactions of gauginos with the scalar and fermion components of chiral superfields. We will see later that matter particles as well as Higgs bosons together with their superpartners belong to chiral scalar supermultiplets. Thus, this term describes how gauginos couple matter fermions

to their superpartners, or Higgs bosons to their superpartners. Notice that these interactions are completely determined by the gauge couplings. Here $t_{\alpha A}$ is the appropriate dimensional matrix representation of the group generators for the α th factor of the gauge group, while g_α is the corresponding gauge coupling constant (one for each factor of the gauge group). Matrix multiplication is implied.

3. The third line describes the scalar potential. This has two distinct contributions. The first term on this line is determined solely by the gauge interactions and has its origin in the auxiliary field \mathcal{D}_A . This term is referred to as the D -term contribution to the scalar potential. The second term comes from the superpotential \hat{f} . We saw in the last chapter that this term arises when the auxiliary fields \mathcal{F}_i are eliminated from the Lagrangian density. This set of terms is, therefore, referred to as F -term contributions to the scalar potential.
4. Finally, the last line of Eq. (6.44) describes the non-gauge, superpotential interactions of matter and Higgs fields as well as fermion mass terms. Since this line describes the interaction of fermion pairs with scalars, the Yukawa interactions of the SM can arise from this term. In other words, all the Yukawa couplings are contained in the superpotential.
5. We note here that in a supersymmetric theory, the scalar potential contains no new couplings other than the gauge couplings and the “Yukawa couplings” and fermion mass terms already present in the superpotential. This is the result of supersymmetry which relates the masses as well as couplings of fermions and bosons within a supermultiplet. Additional terms in the scalar potential are possible if supersymmetry is softly broken.

We conclude this chapter by presenting a recipe for the construction of renormalizable supersymmetric gauge theories.

- (a) Choose a gauge group and the representations for the various supermultiplets, taking care to ensure that the theory is free of chiral anomalies. Matter fermions and Higgs bosons form parts of chiral scalar supermultiplets, $\hat{\mathcal{S}}_{Li}$, while gauge bosons reside in the real gauge supermultiplet $\hat{\Phi}_A$. Keep in mind that we will need a chiral scalar superfield for every chiral component of matter fermions that we want to introduce.
- (b) Choose a superpotential function which is a globally gauge-invariant polynomial (of degree ≤ 3 for renormalizable interactions) of the various left-chiral superfields.
- (c) The interactions of all particles with gauge bosons are given by the usual “minimal coupling” prescription.
- (d) Couple the gauginos to matter via the gauge interactions given in the second line of (6.44).

- (e) Write down the additional self-interactions of the scalar matter fields as given by the third line of (6.44).
- (f) Write down the non-gauge interactions of matter fields coming from the superpotential. The form of these is given by the last two terms of (6.44).

This theory is, of course, exactly supersymmetric. The final step for obtaining realistic models is to incorporate supersymmetry breaking. This forms the subject of the following chapter.

Exercise Construct the Lagrangian density for supersymmetric quantum electrodynamics using (6.44) and the recipe just mentioned.

Remember that you will need to introduce **two** left-chiral scalar supermultiplets in order to obtain a massive Dirac electron. The left-handed part of the Majorana fermion field in the first multiplet will annihilate the left-handed (Dirac) electron, while the corresponding component in the second multiplet will annihilate a left-handed positron. By the Majorana property, the right-handed part of this fermion will annihilate right-handed electrons. The Dirac electron field is then the sum of the left-handed part of the first and the right-handed part of the second. There are, therefore, two scalar partners (one for each chiral component) of the Dirac electron.

Show that the interaction of the photon with the Dirac electron is exactly as you would expect in QED, while the corresponding couplings to the scalar electrons are as in scalar QED. Work out the couplings of the photino (the SUSY partner of the photon) to the electron and the scalar electron.

Before concluding, we remark that the action for supersymmetric gauge theories can also be written as an integral over superspace. We have,

$$S = -\frac{1}{4} \int d^4x d^4\theta \left[\hat{\mathcal{S}}^\dagger e^{-2g t_A \hat{\Phi}_A} \hat{\mathcal{S}} + 2\xi_p \hat{\Phi}_p \right] - \frac{1}{2} \left[\int d^4x d^2\theta_L \hat{f}(\hat{\mathcal{S}}) + \text{h.c.} \right] - \frac{1}{4} \int d^4x d^2\theta_L \widehat{W}_A^c \widehat{W}_A, \quad (6.47)$$

where the ξ_p are dimensionful couplings for Fayet–Iliopoulos terms, one for each $U(1)$ factor of the gauge group.

It is, perhaps, worth emphasizing here that supersymmetry is also restrictive in a sense that we have not yet encountered because we have been dealing with renormalizable theories. In this case, supersymmetry mandated the existence of superpartners with well-defined interactions, but (aside from the holomorphy requirement on the superpotential), did not restrict the spacetime structure of the interactions. However, not all interactions that we might imagine in ordinary field theory can be incorporated in a supersymmetric theory. This is exemplified in

the exercise below where we assert that an arbitrary “Pauli magnetic moment” of fermions is forbidden in a $U(1)$ gauge theory.

Exercise Show that supersymmetry precludes the introduction of the “Pauli term” (even if it is generalized to include transitional magnetic moments), $\overline{\psi}_1 \sigma_{\mu\nu} \psi_2 F^{\mu\nu} + \text{h.c.}$, in a globally supersymmetric Abelian gauge theory.

One way to proceed is as follows. Since the Pauli term is a dimension 5 operator, in a supersymmetric theory it must arise either from a dimension 4 term in the superpotential, or from a dimension 3 term in the Kähler potential. Moreover, since this term is (anti-)linear in ψ_1 , ψ_2 , and $F_{\mu\nu}$, it must originate in a superfield term that includes (at least) one power of $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$ (or in the case of the Kähler potential, possibly $\hat{\mathcal{S}}_1^\dagger$ or $\hat{\mathcal{S}}_2^\dagger$), the left-chiral superfields whose spinor components are ψ_1 and ψ_2 , and one power of the left-chiral spinor curl superfield \hat{W} exhibited in (6.31) whose θ component is the gauge field strength $F_{\mu\nu}$. But the mass dimension $[\hat{\mathcal{S}}_1] = [\hat{\mathcal{S}}_2] = 1$, and $[\hat{W}] = 3/2$, so that $[\hat{\mathcal{S}}_1 \hat{\mathcal{S}}_2 \hat{W}] = 7/2 > 3$, showing that this term cannot originate in a dimension 3 superfield operator in the Kähler potential.

Finally, note that though $\hat{\mathcal{S}}_1 \hat{\mathcal{S}}_2 \hat{W}$ is a (possibly) gauge invariant left-chiral superfield, it is not Lorentz invariant because it is a spinor under Lorentz transformations: in order to be able to include it in the superpotential, we have to contract the spinor index on \hat{W} . We do so by letting a supersymmetric covariant derivative (remember that this also has a spinor index) act on any **one** of the superfields in the product: this then results in a dimension 4 superfield product as required. We have, however, already seen that the supercovariant derivative acting on a left-chiral superfield does not leave it as a left-chiral superfield, so that terms that include such a supercovariant derivative are not allowed in the superpotential. We thus conclude that the “Pauli term” is absent if supersymmetry is unbroken.⁶

Notice that our argument relies only upon dimensional counting and hence applies equally to electric as well as magnetic dipole moments. Also, its validity is independent of whether these dipole moments are diagonal (for Dirac fermions) or transitional.

We thus conclude that in supersymmetric models, anomalous magnetic moments or radiative transitions of elementary fermions (contained in chiral supermultiplets) are possible only if supersymmetry is broken. In other words, contributions from supersymmetric partners in the loops exactly cancel SM contributions if supersymmetry is unbroken. Measurements of anomalous magnetic moments of SM fermions or radiative decays of heavy quarks or leptons potentially provide information about supersymmetry breaking. We will return to this in Chapter 9.

⁶ This was first noted by S. Ferrara and E. Remiddi, *Phys. Lett.* **B53**, 347 (1974).

6.7 The non-renormalization theorem

Supersymmetric theories have better ultra-violet behavior than their non-supersymmetric counterparts. We have already seen an illustration of this in our examination of the one-loop corrections in the Wess–Zumino model, where it was shown that quadratically divergent loop integrals all cancelled. It is now understood that this apparently miraculous cancellation of quadratic divergences is a general consequence of the *SUSY non-renormalization theorem* which states that to any order in perturbation theory, any loop correction can be written as a D -term, i.e. one particle irreducible loop corrections do not generate F -terms. In particular, there are no loop corrections to the superpotential.

This was first established by using supergraph methods,⁷ a perturbative technique that maintains manifest supersymmetry throughout the calculation in the same way that Feynman diagram techniques keep the Lorentz covariance manifest.⁸ A more direct proof of this theorem was given by Seiberg who recognized that the holomorphy of the superpotential (which is a direct consequence of supersymmetry) suffices to establish that there are no perturbative loop corrections to the superpotential, as long as the regularization procedure preserves supersymmetry and gauge invariance.⁹

D -terms in the action of a supersymmetric theory lead to the kinetic energy terms for the components of chiral superfields, so that corrections to these lead to so-called “wave function renormalization”. Since loop corrections do not change the superpotential, superpotential masses and couplings are renormalized only because of the wave function renormalization; i.e. *supersymmetry precludes additional renormalization of the mass terms in the superpotential*. The reader familiar with the basics of renormalization in quantum field theory will immediately recognize that the wave function renormalization is at most logarithmically divergent in the cut-off, thereby establishing that supersymmetric theories are free of quadratic divergences to all orders in perturbation theory. This is important because the existence of quadratic divergences played the central role in persuading us that there must be new physics at the TeV scale. It is the non-renormalization theorem that assures us that TeV scale superpartners can stabilize the electroweak symmetry breaking sector of the supersymmetric extension of the SM in the sense discussed in Chapter 2.

⁷ M. T. Grisaru, W. Siegel and M. Roček, *Nucl. Phys.* **B159**, 429 (1979).

⁸ Supergraph methods were introduced by A. Salam and J. Strathdee, *Phys. Rev.* **D11**, 1521 (1975) and developed by other authors. See e.g. J. Honerkamp *et al.*, *Nucl. Phys.* **B95**, 397 (1975) and S. Ferrara, *Nucl. Phys.* **B93**, 261 (1975).

⁹ N. Seiberg, *Phys. Lett.* **B318**, 469 (1993).