THE ITERATED EQUATION OF GENERALIZED AXIALLY SYMMETRIC POTENTIAL THEORY, I

PARTICULAR SOLUTIONS

J. C. BURNS

(Received 18 March 1966, revised 27 September 1966)

1. Introduction

The iterated equation of generalized axially symmetric potential theory (GASPT) [1] is defined by the relations

$$(1) \qquad L_k^n(f) = 0, \qquad n = 1, 2, \cdots,$$

where

$$(2) \qquad L_k(f) \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{k}{y}\frac{\partial f}{\partial y}$$

and

$$L_k^n(f) = L_k[L_k^{n-1}(f)], \qquad n = 2, 3, \cdots.$$

Particular cases of this equation occur in many physical problems. In classical hydrodynamics, for example, the case $n = 1$ appears in the study of the irrotational motion of an incompressible fluid where, in two-dimensional flow, both the velocity potential ϕ and the stream function ψ satisfy Laplace's equation, $L_0(f) = 0$; and, in axially symmetric flow, ϕ and ψ satisfy the equations $L_1(\phi) = 0$, $L_{-1}(\psi) = 0$. The case $n = 2$ occurs in the study of the Stokes flow of a viscous fluid where the stream function satisfies the equation $L_k^2(\psi) = 0$ with $k = 0$ in two-dimensional flow and $k = -1$ in axially symmetric flow.

Equation (1) has been discussed by Weinstein [2] and Payne [3] who have obtained general solutions and by Weinacht [4] who has considered fundamental solutions.

In this series of papers, the properties of equation (1) will be investigated from a number of points of view. In this first paper, a variety of solutions of (1) are given in terms of solutions of the equation

$$(3) \qquad L_k(f) = 0.$$

Thus solutions of (1) are found of the forms $x^s\partial^t f_k/\partial x^t$, $r^s\partial^t f_k/\partial r^t$, $y^s f_t$ where, for any k, $f_k(x, y)$ or $f_k(r, \theta)$ are arbitrary solutions of equation (3). (r, θ are polar coordinates in the $x-y$ plane). Solutions of (1) are also derived by changes of variable: for example, a solution is found of the form

$$r^{2n-2-k-m}\frac{\partial^m f_k}{\partial r^m}\left(\frac{a^2}{r},\ \theta\right).$$

From the discussion of solutions of the form $y^s f_t$ there emerges a generalization for equation (1) of Weinstein's correspondence principle [1] which holds for equation (3).

The results given in this paper (which include some familiar results for completeness) have been selected partly with a view to their usefulness in subsequent papers. It is intended in these to present a more complete theory of general solutions of (1) of the type considered by Weinstein and Payne; a generalization of the familiar general solution of the biharmonic equation (in the present notation $L_0^2(f) = 0$) in terms of two analytic functions of the complex variable $z = x+iy$; circle theorems which have as their prototype (and include as a special case) Milne-Thomson's theorem [5], well known in classical hydrodynamics; and applications of these results to problems of physical interest, particularly in the Stokes flow of a viscous fluid.

Because this investigation was suggested by the hydrodynamical applications already mentioned, and for the sake of definiteness, the whole of this work is expressed in terms of the elliptic differential operator L_k defined in (2). As Weinstein [2] has observed, the important part of the operator is the pair of terms $\partial^2/\partial y^2+ky^{-1}\,\partial/\partial y$ and the remainder of the operator can be altered considerably without affecting many of the results. Indeed, L_k as given by (2) can be replaced by any operator of the form $X+\partial^2/\partial y^2+k/y\ \partial/\partial y$, where X is a linear operator such that $\partial/\partial y\, X(f) = X(\partial f/\partial y)$ (see [6], [3]). It will usually be clear from the context, and is in any case easily checked, when the operator L_k may be generalized in this way and in the interests of brevity no further reference will be made to such a possibility.

Functions of x, y, or of the polar coordinates r, θ, usually denoted by $f \equiv f(x, y)$ or $f \equiv f(r, \theta)$ will be assumed always to belong to the class of C^{2n} functions and the notation f_k will be used to denote a solution of (3) so that $L_k(f_k) = 0$.

2. Solutions of $L_k^n(f) = 0$ of the form $x^s\partial^t f_k/\partial x^t$

2.1 For any function f and any integers $m \geqq 0$, $n \geqq 0$, it is obvious that

$$L_k^n\left(\frac{\partial^m f}{\partial x^m}\right) = \frac{\partial^m}{\partial x^m}L_k^n(f). \tag{4}$$

2.2 For any function f and any integer $n \geqq 1$,

$$L_k^n(xf) = xL_k^n(f) + 2n\partial/\partial x L_k^{n-1}(f). \tag{5}$$

The case $n = 1$ is easily verified and the theorem is proved by induction, making use of 2.1.

2.3 THEOREM. *For any function f and any integers m, n such that $n \geqq m \geqq 1$,*

$$L_k^n(x^m f) = \mathscr{L}_{n,k}\mathscr{L}_{n-1,k} \cdots \mathscr{L}_{n-m+1,k} L_k^{n-m}(f),$$

where $\mathscr{L}_{n,k} \equiv xL_k + 2n\, \partial/\partial x$.

This is proved by induction on m, the case $m = 1$ being given by 2.2.

In particular, for $m = n-1$ and $n \geqq 2$,

$$L_k^n(x^{n-1}f) = \mathscr{L}_{n,k}\mathscr{L}_{n-1,k} \cdots \mathscr{L}_{2,k} L_k(f). \tag{6}$$

2.4 Equation (6) shows that for any function f_k and any integer t such that $0 \leqq t \leqq n-1$, $L_k^n(x^t f_k) = 0$.

Thus, if $f_{k,i}$ are a set of arbitrarily chosen solutions of (3), then

$$L_k^n[f_{k,0} + xf_{k,1} + x^2 f_{k,2} + \cdots + x^{n-1} f_{k,n-1}] = 0. \tag{7}$$

The solution of the equation $L_k(f) = 0$ given by (7) is in fact a general solution of the equation (see, for example, [3]).

2.5 Equation (4) shows that for any function f_k, $\partial^m f_k/\partial x^m$ is also a solution of (3). Combining this with (7) gives a theorem which includes all the results in 2.4 and gives a large class of solutions of (1) consisting of terms of the form $x^s \partial^t f_k/\partial x^t$.

THEOREM. *For any integers $m_i \geqq 0$ and any solutions $f_{k,i}$ of the equation $L_k(f) = 0$,*

$$L_k^n\left\{\sum_{i=0}^{n-1} x^i \frac{\partial^{m_i} f_{k,i}}{\partial x^{m_i}}\right\} = 0.$$

3. Solutions of $L_k^n(f) = 0$ of the form $r^s \partial^t f_k/\partial r^t$

In polar coordinates r, θ such that $x = r\cos\theta$, $y = r\sin\theta$,

$$L_k(f) \equiv \frac{\partial^2 f}{\partial r^2} + \frac{1+k}{r}\frac{\partial f}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2 f}{\partial \mu^2} - \frac{(1+k)\mu}{r^2}\frac{\partial f}{\partial \mu}, \tag{8}$$

where $\mu \equiv \cos\theta$.

3.1 A necessary preliminary result is a relation between $L_k^n(r^m \partial^m f/\partial r^m)$ and $L_k^n(f)$. It is shown first that for any function f and any integer $n \geqq 0$,

$$L_k^n\left(r\frac{\partial f}{\partial r}\right) = r\frac{\partial}{\partial r} L_k^n(f) + 2nL_k^n(f). \tag{9}$$

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

The case $n = 0$ is trivial and the case $n = 1$ is proved by direct substitution in the expression (8) for $L_k(f)$. The result is then proved by mathematical induction. This is the case $m = 1$ of a more general theorem which is also proved by induction.

3.2 THEOREM. *For any function f and any integers $m \geqq 1$, $n \geqq 0$,*

$$L_k^n(r^m \partial^m f/\partial r^m) = E_{n,m}E_{n,m-1} \cdots E_{n,1}L_k^n(f),$$

where

$$E_{n,m} \equiv r\partial/\partial r + 2n - m + 1.$$

The particular case $n = 0$ is familiar:

$$r^m \partial^m f/\partial r^m = (\vartheta - m + 1)(\vartheta - m + 2) \cdots \vartheta(f),$$

where $\vartheta(f) \equiv r\partial f/\partial r$. (See, for example, [7].).

Of more interest in the present context is another special case obtained by taking $f = f_k$ which gives, for any integers $m \geqq 0$, $n \geqq 1$,

$$L_k^n(r^m \partial^m f_k/\partial r^m) = 0. \tag{10}$$

3.3 Another preliminary result follows from direct substitution in (8). For any function f, and any m,

$$L_k(r^m f) = r^m L_k(f) + 2mr^{m-1}\partial f/\partial r + m(m+k)r^{m-2}f. \tag{11}$$

Equation (11) can be simplified considerably in two ways, each of which gives rise to a chain of theorems. Taking $m = 2$ gives, for any function f,

$$L_k(r^2 f) = r^2 L_k(f) + 4r\, \partial f/\partial r + 2(2+k)f; \tag{12}$$

while keeping m general and taking $f = f_k$ gives, for any function f_k and any m.

$$L_k(r^m f_k) = mr^{m-2}[2r\partial f_k/\partial r + (m+k)f_k]. \tag{13}$$

3.4 Consider first the results which follow from (12). It can be proved that, for any function f and any integer $n \geqq 1$,

$$L_k^n(r^2 f) = r^2 L_k^n(f) + 4nr\, \partial/\partial r\, L_k^{n-1}(f) + 2n(2n+k)L_k^{n-1}(f). \tag{14}$$

The case $n = 1$ is given by (12) and the proof by induction makes use of (9). (14) is the case $m = 1$ of a more general theorem which is also proved by induction.

3.5 THEOREM. *For any function f and any integers m, n such that $n \geqq m \geqq 1$,*

$$L_k^n(r^{2m} f) = \mathscr{M}_{n,k}\mathscr{M}_{n-1,k} \cdots \mathscr{M}_{n-m+1,k}L_k^{n-m}(f),$$

where $\mathscr{M}_{n,k} \equiv r^2 L_k + 4nr\, \partial/\partial r + 2n(2n+k)$.

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

In particular, for $m = n-1$ and $n \geqq 2$,

(15) $$L_k^n(r^{2n-2}f) = \mathscr{M}_{n,k}\mathscr{M}_{n-1,k} \cdots \mathscr{M}_{2,k}L_k(f).$$

3.6 Equation (15) shows that for any f_k and any integer t such that $0 \leqq t \leqq n-1$,

(16) $$L_k^n(r^{2t}f_k) = 0.$$

Thus, if $f_{k,i}$ are a set of arbitrarily chosen solutions of (3), then

(17) $$L_k^n[f_{k,0}+r^2f_{k,1}+r^4f_{k,2}+ \cdots +r^{2n-2}f_{k,n-1}] = 0.$$

The solution of the equation $L_k^n(f) = 0$ given by (17) is in fact a general solution of the equation ([3]).

3.7 Equation (10) (with $n = 1$) shows that for any function f_k, $r^m \partial^m f_k/\partial r^m$ is also a solution of (3). Combining this with (17) gives a theorem which includes all the results in 3.6 and gives a large class of solutions of (1) consisting of terms of the form $r^s \partial^t f_k/\partial r^t$.

THEOREM. *For any integers* $m_i \geqq 0$ *and any solutions* $f_{k,i}$ *of the equation* $L_k(f) = 0$,

$$L_k^n\left\{\sum_{i=0}^{n-1} r^{2i+m_i}\frac{\partial^{m_i}f_{k,i}}{\partial r^{m_i}}\right\} = 0.$$

3.8 Equation (16) which leads to theorem 3.7 is also obtained as a special case of a theorem which follows from equation (13).

THEOREM. *If* $R_s(m, k)$ *is defined by the relations* $R_0(m, k) = 1$, $R_s(m, k) = (m+k)(m+k-2) \cdots (m+k-2s+2)$ *for* $s = 1, 2, 3, \cdots$, *then, for any function* f_k, *any* m *and any integer* $n \geqq 0$,

$$L_k^n(r^mf_k) = R_n(m, 0)r^{m-2n}\sum_{s=0}^{n} 2^{n-s}\binom{n}{s}R_s(m, k)r^{n-s}\frac{\partial^{n-s}f_k}{\partial r^{n-s}}.$$

Equation (13) gives the result when $n = 1$ and the theorem is proved by induction. If

$$U \equiv \sum_{s=0}^{n} 2^{n-s}\binom{n}{s}R_s(m, k)r^{n-s}\frac{\partial^{n-s}f_k}{\partial r^{n-s}},$$

then equation (10) (with $n = 1$) shows that $L_k(U) = 0$. From equation (13), with m replaced by $m-2n$, it now follows that if the theorem is assumed to be true for $L_k^n(r^mf_k)$, then

$$L_k^{n+1}(r^mf_k) = R_n(m, 0)(m-2n)r^{m-2n-2}[2r\partial U/\partial r+(m-2n+k)U].$$

These are the essential steps in the proof and when the differentiation on the right hand side is carried out and the resulting terms rearranged, the required expression for $L_k^{n+1}(r^m f_k)$ is found so that the inductive proof can be completed.

It will be noted that since

$$R_n(m, 0) = m(m-2) \cdots (m-2n+2),$$

equation (16) is an immediate corollary of theorem 3.8.

4. Solutions of $L_k(f) = 0$ of the form $y^s f_t$

It is convenient to introduce an operator $\mathscr{D}$ defined by the relation

$$\mathscr{D}(f) \equiv y^{-1}\partial f/\partial y,$$

and to obtain a number of relations between the operators L_k and $\mathscr{D}$.

4.1 At first the function f can be kept general. By direct calculation, it can be shown that, for any function f,

$$L_k\mathscr{D}(f) = \mathscr{D}L_{k-2}(f) \tag{18}$$

This is the case $m = 1$ of a theorem easily proved by induction: for any function f and any integer $m \geqq 0$,

$$L_k\mathscr{D}^m(f) = \mathscr{D}^m L_{k-2m}(f). \tag{19}$$

This, in turn, is the case $n = 1$ of a more general theorem also proved by induction:

THEOREM. *For any function f and any integers $m \geqq 0$, $n \geqq 0$,*

$$L_k^n\mathscr{D}^m(f) = \mathscr{D}^m L_{k-2m}^n(f).$$

4.2 A second set of relations is concerned with the operation of L_k and $\mathscr{D}$ on functions f_l. By direct calculation, it is shown (Weinstein [2]) that for any function f_l,

$$L_k(f_l) = (k-l)\mathscr{D}(f_l). \tag{20}$$

This is the case $n = 1$ of the following theorem which is proved by induction, using (18):

THEOREM. *For any function f_l,*

$$L_{k_1}L_{k_2}\cdots L_{k_n}(f_l) = (k_n-l)(k_{n-1}-l-2)\cdots[k_1-l-2(n-1)]\mathscr{D}^n(f_l).$$

It is immediately deduced that f_l is a solution of the equation

$$L_{k_1}L_{k_2}\cdots L_{k_n}(f_l) = 0 \tag{21}$$

provided $l = k_i-2(n-i)$ for some integer i in the range $1 \leqq i \leqq n$. Wein-

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

stein [2] has shown that a linear combination of these n solutions of (21) forms a general solution of this equation, provided that $k_i \neq k_j - 2(i-j)$ for $j < i = 2, 3, \cdots, n$.

Taking all the k_i equal to k gives

$$(22) \qquad L_k^n(f_l) = (k-l-2) \cdots [k-l-2(n-1)]\mathscr{D}^n(f_l)$$

and a family of n solutions of the equation $L_k^n(f) = 0$ given by $f_{k-2\beta}$ where $0 \leqq \beta \leqq n-1$. Weinstein's general solution of (21) shows that a linear combination of these solutions of $L_k^n(f) = 0$ forms a general solution of the equation.

Finally, it may be noted that theorem 4.1 and equation (22) can be combined to give

$$(23) \quad L_k^n \mathscr{D}^m(f_l) = (k-2m-l)(k-2m-l-2) \cdots [k-2m-l-2(n-1)]\mathscr{D}^{m+n}(f_l).$$

4.3 With the results of sections 4.1, 4.2 available, it is possible to derive results which lead to the family of solutions of the equation $L_k^n(f) = 0$ of the form $y^s f_t$. (A set of n of these solutions with $s = 0$ and $t = k-2\beta$ for $0 \leqq \beta \leqq n-1$ was found in 4.2).

A preliminary result, obtained by direct calculation, is that, for any function f, and any s,

$$(24) \qquad L_k(y^s f) = y^s L_{k+2s}(f) + s(s-1+k)y^{s-2}f.$$

Equation (24) can be simplified considerably in two ways, each of which gives rise to valuable theorems. Taking $f = f_t$ and keeping s general gives

$$(25) \qquad L_k(y^s f_t) = (k+2s-t)y^s \mathscr{D}(f_t) + s(s-1+k)y^{s-2}f_t,$$

which is obtained by the use of (20). Taking $s = 1-k$ and keeping f general gives

$$(26) \qquad L_k(y^{1-k}f) = y^{1-k}L_{2-k}(f).$$

4.4 The results which follow from equation (25) are considered first. It will be useful to have, as well, a more general form of (25) which is obtained from (24) by using (19) and (20): for any function f_t, any s, and any integer $n \geqq 0$,

$$(27) \quad L_k[y^s \mathscr{D}^n(f_t)] = (k+2s-t-2n)y^s \mathscr{D}^{n+1}(f_t) + s(s-1-k)y^{s-2}\mathscr{D}^n(f_t).$$

The main theorem provides an explicit expression for $L_k^n(y^s f_t)$.

THEOREM. *If, for integers u and v, $P_{u,v}$ and $Q_{u,v}$ are defined by the relations*

$$P_{u,v} = A_u A_{u+1} \cdots A_v \text{ for } u \leqq v, \; P_{u,v} = 1 \text{ for } u > v,$$
$$Q_{u,v} = B_u B_{u+1} \cdots B_v \text{ for } u \leqq v, \; Q_{u,v} = 1 \text{ for } u > v,$$

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

where $A_\alpha = k+2s-t-2\alpha$ *and* $B_\beta = (s-2\beta)(k-1+s-2\beta)$; *then, for any function* f_t, *any* s *and any* t, *and any integer* $n \geqq 0$,

$$L_k^n(y^s f_t) = \sum_{u=0}^{n} \binom{n}{u} P_{u,n-1} Q_{0,u-1} y^{s-2u} \mathscr{D}^{n-u}(f_t).$$

The theorem is proved by induction, the case $n = 1$ being given by equation (25). If the theorem is assumed to be true for $L_k^n(y^s f_t)$, then operating with L_k produces

$$L_k^{n+1}(y^s f_t) = \sum_{u=0}^{n} \binom{n}{u} P_{u,n-1} Q_{0,u-1} [A_{n+u} y^{s-2u} \mathscr{D}^{n+1-u}(f_t) + B_u y^{s-2t-2} \mathscr{D}^{n-u}(f_t)],$$

where the right hand side is obtained by the use of (27). After lengthy but elementary calculations, the right hand side can be rearranged to produce the required form for $L_k^{n+1}(y^s f_t)$ so that the inductive argument can be completed.

4.5 Theorem 4.4 can be used to find all solutions of the equation $L_k^n(f) = 0$ of the form $y^s f_t$.

THEOREM. $L_k^n(y^s f_t) = 0$ *for all functions* f_t *if, and only if,* $s = 2\alpha$, $t = k+2\alpha-2\beta$ *or* $s = 1-k+2\beta$, $t = 2-k+2\beta-2\alpha$, *where* α, β *are non-negative integers such that* $0 \leqq \alpha+\beta \leqq n-1$.

Theorem 4.4 shows that for all integers v such that $0 \leqq v \leqq n-1$,

$$L_k^n(y^s f_t) = \left\{ \sum_{u=0}^{v} + \sum_{u=v+1}^{n} \right\} \left\{ \binom{n}{u} P_{u,n-1} Q_{0,u-1} y^{s-2u} \mathscr{D}^{n-u}(f_t) \right\}. \tag{28}$$

From the definitions of $P_{u,v}$ and $Q_{u,v}$ it can be seen that the highest common factor of the coefficients $P_{u,n-1}Q_{0,u-1}$ in the first and second sums on the right hand side of (28) are respectively $P_{v,n-1}$ and $Q_{0,v}$. Hence $L_k^n(y^s f_t) = 0$ for all functions f_t if, and only if, s and t are such that $P_{v,n-1} = 0$ and $Q_{0,v} = 0$ for some v in $0 \leqq v \leqq n-1$.

LEMMA. *The criterion* (A) $P_{v,n-1} = Q_{0,v} = 0$ *for some* v *in* $0 \leqq v \leqq n-1$ *is equivalent to the criterion* (B) $P_{v,n-1} = B_v = 0$ *for some* v *in* $0 \leqq v \leqq n-1$.

The proof depends on two results which follow immediately from the definitions of $P_{u,v}$, $Q_{u,v}$:

(i) $$Q_{0,v} = B_0 B_1 \cdots B_v;$$

(ii) $$\text{if} \quad 0 \leqq w \leqq v, \; P_{w,n-1} = P_{w,v-1} P_{v,n-1}.$$

If (A) holds, it follows from (ii) that $P_{w,n-1} = 0$ for all w in $0 \leqq w \leqq v$ and from (i) that $B_w = 0$ for some w in $0 \leqq w \leqq v$. Hence (B) holds. The converse is obvious from (i). (This concise form of proof is due to Dr. M. F. Newman.)

The lemma leads to the statement that $L_k^n(y^s f_t) = 0$ for all functions f_t if, and only if, s and t are such that, for some v in $0 \leqq v \leqq n-1$, $P_{v,n-1} = 0$ and $B_v = 0$, i.e.

$$(s-2v)(s+k-1-2v) = 0 \quad \text{and} \quad \prod_{\gamma=v}^{n-1} (k+2s-t-2\gamma) = 0.$$

This requires that $s = 2v$ and $t = k+4v-2\gamma$ or $s = 1-k+2v$ and $t = 2-k+4v-2\gamma$, where v and γ are integers such that $0 \leqq v \leqq \gamma \leqq n-1$. Rearrangement of this last statement with appropriate changes of notation gives the result of the theorem.

The theorem shows that all solutions of the equation $L_k^n(f) = 0$ of the form $y^s f_t$ fall into two families, each containing $\frac{1}{2}n(n+1)$ members:

$$a_{\alpha\beta} = y^{2\alpha} f_{k+2\alpha-2\beta} \quad \text{and} \quad A_{\alpha\beta} = y^{1-k+2\beta} f_{2-k+\beta-2\alpha}.$$

These solutions will be considered in detail in a later paper where the main aim will be to construct general solutions of the equation in the form of linear combinations of n terms chosen from these two families.

4.6 The consequences of equation (26) are now considered. Taking f to be any f_{2-k} in (26) gives

$$L_k(y^{1-k} f_{2-k}) = 0,$$

which shows that, for any f_{2-k}, $y^{1-k} f_{2-k}$ can be expressed in the form f_k, a result which will be denoted by

$$y^{1-k} f_{2-k} \to f_k. \tag{29}$$

Writing $2-k$ for k in (29) shows that, for any function f_k,

$$f_k \to y^{1-k} f_{2-k} \tag{30}$$

The symbol $\leftrightarrow$ is now introduced to express (29) and (30) in the single statement:

$$f_k \leftrightarrow y^{1-k} f_{2-k}. \tag{31}$$

(Two functions related as in (31) will be said to be *equivalent*.)

Equation (31) is well-known as Weinstein's correspondence principle [1].

Among other deductions from (26) is one that will be useful; for any function f,

$$L_k(y^{1-k} f) = y^{1-k}[L_{-k}(f) + 2\mathscr{D}(f)]. \tag{32}$$

4.7 Equation (26) is the case $n = 1$ of a more general result which can be proved by induction:

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

THEOREM. *For any function f, and any integer $n \geqq 1$,*

(33) $$L_k^n(y^{1-k}f) = y^{1-k}L_{2-k}^n(f).$$

This theorem leads to generalized forms of (31) and (32) which have previously been given by Weinacht [4].

4.8 Denote an arbitrary solution of the equation $L_k^n(f) = 0$ by $f_k^{(n)}$. (It will, however, be convenient to continue to denote solutions of $L_k(f) = 0$ by f_k.)

THEOREM. *Generalized Weinstein correspondence principle.*

(34) $$f_k^{(n)} \leftrightarrow y^{1-k}f_{2-k}^{(n)}.$$

This result is derived from (33) exactly as (31) was derived from (26).

This generalization of Weinstein's correspondence principle can be used to solve problems in the Stokes flow of a viscous fluid in very much the same way that the simpler principle (31) can be used in inviscid flow [1]. This, and other applications of the principle, will be discussed in later papers.

Another result which will be useful later is a generalization of (32) and is easily proved by induction: for any function f and any integer $n \geqq 1$,

(35) $$L_k^n(y^{1-k}f) = y^{1-k}[L_{-k}^n(f)+2n\mathscr{D}L_{-k}^{n-1}(f)].$$

5. Solutions of $L_k^n(f) = 0$ obtained by changes of variable

New solutions of $L_k^n(f) = 0$ can be obtained by changes of variable and the changes of the independent variables to be considered are those which result from reflection in the axis $x = 0$ of the $x-y$ plane or inversion in a circle with centre at the origin.

5.1 Reflection in the y-axis is easily disposed of as the operator L_k is even in x. Thus, every solution $f(x, y)$ of the equation $L_k^n(f) = 0$ gives rise to another solution $f(-x, y)$.

5.2 The case of inversion in a circle of radius a is of much more interest. Introduce new coordinates ξ, η related to x, y so that the point (ξ, η) is the inverse in the circle $r = a$ of the point (x, y). Thus, if r, θ are polar coordinates in the $x-y$ plane, then polar coordinates in the $\xi-\eta$ plane are ρ, θ where $\rho r = a^2$. Let the operator in the $\xi-\eta$ plane which corresponds to L_k in the $x-y$ plane be Λ_k i.e.

$$\Lambda_k \equiv \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \frac{k}{\eta}\frac{\partial}{\partial\eta}.$$

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

If $f(r, \theta)$ is any function, define $\boldsymbol{f}(\rho, \theta)$ such that $\boldsymbol{f}(\rho, \theta) \equiv f(a^2/\rho, \theta)$; thus $\boldsymbol{f}(\rho, \theta)$ and $f(r, \theta)$ have the same value at corresponding points.

It is easily proved that, for any function $f(r, \theta)$ and the corresponding function $\boldsymbol{f}(\rho, \theta)$,

$$\Lambda_k(\boldsymbol{f}) = \left(\frac{r}{a}\right)^4 \left[L_k(f) - \frac{2k}{r}\frac{\partial f}{\partial r}\right]. \tag{36}$$

5.3 Combining (36) and (11) shows that for any function $f(r, \theta)$ and the corresponding function $\boldsymbol{f}(\rho, \theta)$,

$$\Lambda_k\left[\left(\frac{\rho}{a}\right)^{-m}\boldsymbol{f}\right] = \left(\frac{r}{a}\right)^{m+4}\left[L_k(f) + \frac{2(m-k)}{r}\frac{\partial f}{\partial r} + \frac{m(m-k)}{r^2}f\right]. \tag{37}$$

Equation (37) can be simplified either by taking $m = k$ to give

$$\Lambda_k\left[\left(\frac{\rho}{a}\right)^{-k}\boldsymbol{f}\right] = \left(\frac{r}{a}\right)^{k+4} L_k(f); \tag{38}$$

or by taking $f = f_k$ (with corresponding function $\boldsymbol{f}_k$) to give

$$\Lambda_k\left[\left(\frac{\rho}{a}\right)^{-m}\boldsymbol{f}_k\right] = \left(\frac{r}{a}\right)^{m+4}\left[\frac{2(m-k)}{r}\frac{\partial f_k}{\partial r} + \frac{m(m-k)}{r^2}f_k\right]. \tag{39}$$

Each of these equations (38) and (39) gives rise to new results.

5.4 Equation (38) is the simplest case of the theorem:

THEOREM. *For any function $f(r, \theta)$ and the corresponding function $\boldsymbol{f}(\rho, \theta)$, and for any integer $n \geqq 0$,*

$$\Lambda_k^n\left[\left(\frac{\rho}{a}\right)^{-k-2+2n}\boldsymbol{f}\right] = \left(\frac{r}{a}\right)^{k+2+2n} L_k^n(f).$$

Equation (38) gives the case $n = 1$ and the theorem is proved by induction. Equation (37) is used to show that

$$\begin{aligned} &\Lambda_k^{n+1}\left[\left(\frac{\rho}{a}\right)^{-k+2n}\boldsymbol{f}\right] \\ &= \Lambda_k^n\left\{\left(\frac{r}{a}\right)^{k+2-2n}\left[\left(\frac{r}{a}\right)^2 L_k(f) - \frac{4nr}{a^2}\frac{\partial f}{\partial r} - \frac{2n(k-2n)}{a^2}f\right]\right\}. \end{aligned} \tag{40}$$

If the theorem as stated is assumed to be true, so that

$$\Lambda_k^n\left[\left(\frac{r}{a}\right)^{k+2-2n} f\right] = \left(\frac{r}{a}\right)^{k+2+2n} L_k^n(f)$$

for any function f, then (40) becomes

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

$$\Lambda_k^{n+1}\left[\left(\frac{\rho}{a}\right)^{-k+2n} f\right] \tag{41}$$
$$= \left(\frac{r}{a}\right)^{k+2+2n} L_k^n \left\{\left(\frac{r}{a}\right)^2 L_k(f) - \frac{4nr}{a^2}\frac{\partial f}{\partial r} - \frac{2n(k-2n)}{a^2} f\right\}.$$

The right hand side of (41) is evaluated with the help of equations (14) and (9) and gives the required expression so that the induction can be completed.

5.5 Theorem 5.4 can be used to deduce new solutions of the equation $L_k^n(f) = 0$.

THEOREM. *If $f_k^{(n)}(r, \theta)$ is a solution of $L_k^n(f) = 0$, then so is*

$$\left(\frac{r}{a}\right)^{-k-2+2n} f_k^{(n)}\left(\frac{a^2}{r}, \theta\right).$$

From theorem 5.4, it is clear that since $L_k^n[f_k^{(n)}(r, \theta)] = 0$,

$$\Lambda_k^n\left[\left(\frac{\rho}{a}\right)^{-k-2+2n} f_k^{(n)}\left(\frac{a^2}{\rho}, \theta\right)\right] = 0. \tag{42}$$

Replacing ρ by r thoughout equation (42) gives the required result.

In particular, if $f_k(r, \theta)$ is any solution of $L_k(f) = 0$, then

$$L_k^n\left[\left(\frac{r}{a}\right)^{-k-2+2n} f_k\left(\frac{a^2}{r}, \theta\right)\right] = 0. \tag{43}$$

(See [8]).

5.6 THEOREM. *For any function $f_k(r, \theta)$, and any integer $m \geqq 0$,*

$$L_k\left[r^{-k-m}\frac{\partial^m f_k}{\partial r^m}\left(\frac{a^2}{r}, \theta\right)\right] = 0.$$

Equation (43) (with $n = 1$) gives the case $m = 0$ and the theorem is proved by induction using equation (10) with $m = n = 1$ and the identity

$$r\frac{\partial}{\partial r}\left[r^{-k-m}\frac{\partial^m f}{\partial r^m}\left(\frac{a^2}{r}, \theta\right)\right]$$
$$= -(k+m)r^{-k-m}\frac{\partial^m f}{\partial r^m}\left(\frac{a^2}{r}, \theta\right) - a^2 r^{-k-m-1}\frac{\partial^{m+1} f}{\partial r^{m+1}}\left(\frac{a^2}{r}, \theta\right).$$

This theorem produces a whole family of new solutions of the equation $L_k(f) = 0$ when one solution $f_k(r, \theta)$ is known.

5.7 Taken in conjunction with equation (16), theorem 5.6 gives a large class of solutions of the equation $L_k^n(f) = 0$ which may be compared with those given by theorem 3.7.

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

THEOREM. *For any integers $m_i \geqq 0$ and any solutions $f_{k,i}(r, \theta)$ of the equation $L_k(f) = 0$,*

$$L_k^n \left\{ \sum_{i=0}^{n-1} r^{2i-k-m_i} \frac{\partial^{m_i} f_{k,i}}{\partial r^{m_i}} \left(\frac{a^2}{r}, \theta \right) \right\} = 0.$$

5.8 Equation (39) with m replaced by $k-m$ and r and ρ interchanged everywhere gives

$$\begin{aligned} &L_k \left[\left(\frac{r}{a} \right)^{m-k} f_k \left(\frac{a^2}{r}, \theta \right) \right] \\ (44) \quad &= \frac{m}{a} \left(\frac{r}{a} \right)^{m-2} \left[-2 \left(\frac{r}{a} \right)^{-k-1} \frac{\partial f_k}{\partial r} \left(\frac{a^2}{r}, \theta \right) + \frac{m-k}{a} \left(\frac{r}{a} \right)^{-k} f_k \left(\frac{a^2}{r}, \theta \right) \right]. \end{aligned}$$

Equation (44) closely resembles equation (13) and this suggests the possibility of obtaining an expression for $L_k^n[(r/a)^{m-k} f_k(a^2/r, \theta)]$ similar to that obtained in theorem 3.8 for $L_k^n[r^m f_k(r, \theta)]$. Indeed, since from theorem 5.5 it is known that $(r/a)^{-k} f_k(a^2/r, \theta)$ is a solution of $L_k(f) = 0$, it should be possible to replace $f_k(r, \theta)$ in theorem 3.8 by $(r/a)^{-k} f_k(a^2/r, \theta)$ and so derive the required expression. However, it appears to be easier to prove the result directly.

THEOREM. *If $S_s(m, k, n)$ is defined by the relations $S_0(m, k, n) = 1$, $S_s(m, k, n) = (m-k-2n+2s)(m-k-2n+2s-2) \cdots (m-k-2n+2)a^{-s}$ for $s = 1, 2, 3, \cdots$ then, for any function $f_k(r, \theta)$ and any integer $n \geqq 0$,*

$$\begin{aligned} &L_k^n \left[\left(\frac{r}{a} \right)^{m-k} f_k \left(\frac{a^2}{r}, \theta \right) \right] \\ &= S_n(m, 0, n) \left(\frac{r}{a} \right)^{m-2n} \sum_{s=0}^{n} (-2)^{n-s} \binom{n}{s} S_s(m, k, n) \left(\frac{r}{a} \right)^{-k-n+s} \frac{\partial^{n-s} f_k}{\partial r^{n-s}} \left(\frac{a^2}{r}, \theta \right). \end{aligned}$$

Equation (44) gives the result when $n = 1$ and the theorem is proved by induction. If

$$V = \sum_{s=0}^{n} (-2)^{n-s} \binom{n}{s} S_s(m, k, n) \left(\frac{r}{a} \right)^{-k-n+s} \frac{\partial^{n-s} f_k}{\partial r^{n-s}} \left(\frac{a^2}{r}, \theta \right),$$

then theorem 5.6 shows that $L_k(V) = 0$. From equation (13), with m replaced by $m-2n$, it follows that if the theorem is assumed to be true for the operator L_k^n, then

$$\begin{aligned} &L_k^{n+1} \left[\left(\frac{r}{a} \right)^{m-k} f_k \left(\frac{a^2}{r}, \theta \right) \right] \\ &= S_n(m, 0, n) \frac{m-2n}{a^2} \left(\frac{r}{a} \right)^{m-2n-2} \left[2r \frac{\partial V}{\partial r} + (m-2n+k) V \right]. \end{aligned}$$

https://doi.org/10.1017/S1446788700004110 Published online by Cambridge University Press

These are the essential steps in the proof and when the differentiation on the right hand side is carried out and the resulting terms rearranged, the required expression for $L_k^{n+1}[(r/a)^{m-k} f_k(a^2/r, \theta)]$ is found so that the inductive proof can be completed.

It will be noted that since

$$S_n(m, 0, n) = m(m-2) \cdots (m-2n+2)a^{-n},$$

it follows that, for any integer t such that $0 \leqq t \leqq n-1$,

$$L_k^n[(r/a)^{2t-k} f_k(a^2/r, \theta)] = 0,$$

a result which is included in theorem 5.7 and can be used to provide an alternative proof of that theorem.

5.9 Particular cases of several of the theorems of section 5 have been given by Butler [9], Collins [10] and other authors in establishing circle theorems for particular equations of the type $L_k^n(f) = 0$. The general results found here will be used in a later paper to obtain general circle theorems.

References

[1] A. Weinstein, 'Generalized axially symmetric potential theory', *Bull. Amer. Math. Soc.* 59 (1953), 20—28.

[2] A. Weinstein, 'On a class of partial differential equations of even orders', *Ann. Mat., Pura Appl.* 39 (1955), 245—254.

[3] L. E. Payne, 'Representation formulas for solution of a class of partial differential equations', *J. Math. and Phys.* 38 (1959), 145—149.

[4] R. J. Weinacht, *Contributions to Differential Equations III* (1964), 43—55.

[5] L. M. Milne-Thomson, 'Hydrodynamical Images', *Proc. Cambridge Philos. Soc.* 36 (1940), 246—247; or *Theoretical hydrodynamics* (Macmillan, 4th ed. 1960), 154.

[6] A. Weinstein, 'Sur une classe d'équations aux derivées partielles singulières', *Coll. Intern. du Centre Natl. de la Recherche Scientifique, La théorie des équations aux derivées partielles* (1956), 179—186.

[7] A. R. Forsyth, *A treatise on differential equations* (Macmillan, 6th ed. 1943), 62.

[8] A. Huber, 'Some results on generalised axially symmetric potentials', *Proc. Conf. Part. Diff. Eq., Univ. of Maryland* (1955), 147—155.

[9] S. J. F. Butler, 'A note on Stokes's stream function for motion with a spherical boundary', *Proc. Cambridge Philos. Soc.* 49 (1953), 169—174.

[10] W. D. Collins, 'A note on Stokes's stream function for the slow steady motion of a viscous fluid before plane and spherical boundaries', *Mathematika* 1 (1954), 125—130.

The Australian National University
Canberra, A.C.T.