The Focal Circles of Circular Cubics.

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1. The following method of establishing the existence and properties of the Focal Circles of a Circular Cubic is, as far as I know, new, and it has the advantage of dispensing, almost entirely, *explicitly* with analysis, while many of the properties can be proved without using the complicated method of generating the curve given in Salmon. The results which I have arrived at in connexion with the Nodal Cubic and Cuspidal Cubic are not given in Salmon.

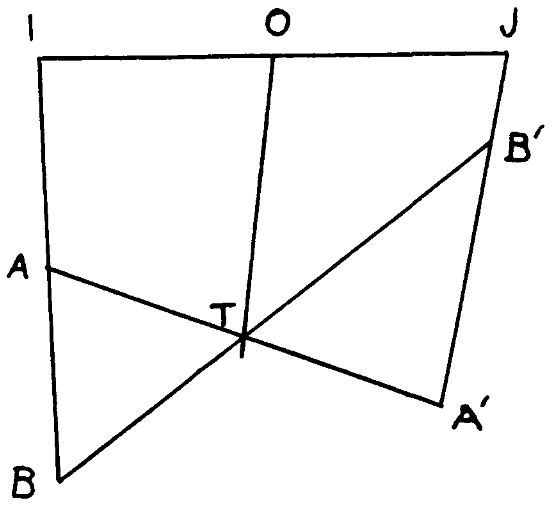

Fig. 1.

2. Let I and J (Fig. 1) be the Circular Points at Infinity, and let the third point of intersection of the Line at Infinity with the Cubic be O.

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Let OT be one of the four tangents from O, T being the point of contact.

Let IAB be any chord through I cutting the Cubic at A and B.

Let AT and BT meet the curve again at A′ and B′ respectively.

Then two Cubics through the nine points

$$I, O, J, A, T, A', B, T, B'$$

are (1) the Cubic itself,

and (2) the three lines IOJ, ATA′, BTB′.

Thus the third Cubic through eight of these points, viz., the lines IAB, OTT, A′B′, must also pass through the ninth point J.

Hence A′B′ passes through J.

Thus a one-to-one algebraic correspondence exists between the two chords IAB and JA′B′ connected as above.

3. Let IAB and JA′B′ meet in P.

Then by the theorem of Article 2, P describes a conic passing through I and J, *i.e.* a circle (which we shall call F).

4. *The Circle F passes through a group of four foci.*

Let A and B coincide.

Then plainly A′ and B′ also coincide.

In this case the intersection of the two corresponding chords through I and J become the intersection of tangents, and their point of intersection is thus a focus. Hence the circle F passes through this focus. Also, since four tangents can be drawn from I to the Cubic, their intersections with the four corresponding tangents from J are foci of the Cubic, and all lie on the circle F.

5. *The circle F passes through the points of contact of the four tangents drawn from T.*

Consider where two corresponding rays through I and J meet on the Cubic.

Either A and A′ or B and B′ become coincident.

Hence if the focal-circle F cut the Cubic in K, the tangent at K passes through T.

6. *T is the centre of the circle F.*

Let A (Fig. 2) be a point very near to T on the Cubic.

Join A to T, meeting the Cubic again in A′, a point close to O (since AT is in the limit the tangent at T). Hence we see that corresponding to the chord IT is the chord JI, in the limit.

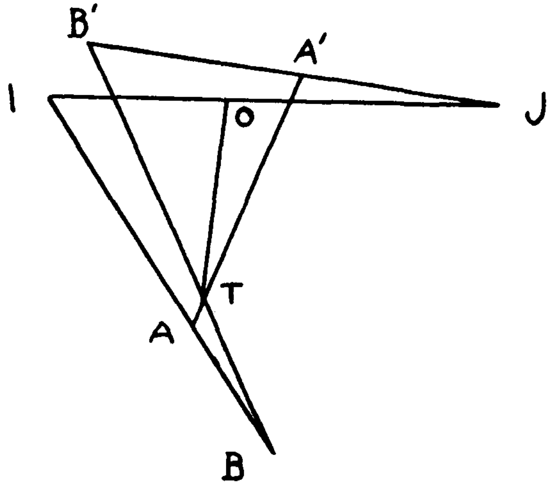

Fig. 2.

Thus from the theory of homographic correspondence, IT is the tangent to F at I.

Similarly JT is the tangent to F at J.

Hence T is the centre of the circle F.

7. We thus have :—

In a Circular Cubic there are four tangents parallel to the real asymptote.

Corresponding to each of these four tangents, there is a set of four foci (there being sixteen foci in all).

The directrices of all the four foci in a set pass through the same point, viz., the point of contact of the corresponding tangent parallel to the asymptote.

The four foci in a set lie on a circle (called a Focal Circle) whose centre is the point of concurrency of their directrices.

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A Focal Circle cuts the Cubic in the four points of contact of the four tangents drawn from the point of concurrency of the four directrices.

8. *Case of the Nodal Cubic.*

In this case only two tangents can be drawn from O to the Nodal Cubic (see Fig. 1).

Hence in the case of the Nodal Cubic there are only two Focal Circles.

9. *The two Focal Circles of a Nodal Cubic pass each through the Node.*

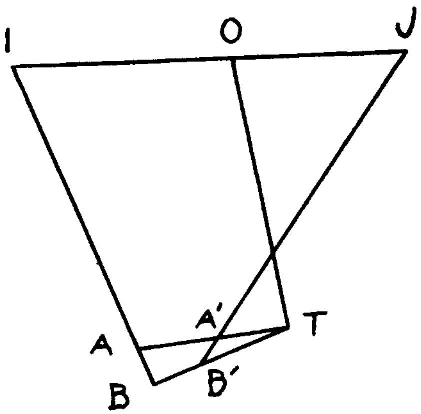

Fig. 3.

Let IAB (Fig. 3) be a chord drawn very near the Node cutting the curve in A and B.

Then plainly A′ is very near A
and B′ is very near B,

while by the theorem of Art. 2, A′B′ passes through J. Thus in the limit A, B, A′, B′, all coincide in the Node, and since two corresponding chords intersect in the Node, the Node lies on the Focal Circle. Similarly it lies on the other Focal Circle.

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10. *The tangents at the Node to the two Focal Circles are the perpendicular bisectors of the angles between the Nodal Tangents to the Cubic.*

Let N be the Node, and let the tangents from O touch the Cubic at T and T′, (*i.e.* the centres of the Focal Circles).

Then since the tangents at T and T′ meet in O, a point on the curve, we know that NT and NT′ harmonically separate the Nodal Tangents at N to the Cubic.

Also the lines joining the extremities of all chords through O to N form an involution having NT and NT′ as double rays. ∴ NT and NT′ harmonically separate NI and NJ, *i.e.* are perpendicular. Hence the theorem follows immediately.

11. *The Cuspidal Cubic has only one focus.*

This is plain since only one tangent can be drawn from I to meet the corresponding tangent from J.

12. The following theorem will also be plain from what precedes:—

If a circle be described having T as centre and passing through the Focus, it also passes through the point of contact of the tangent from T and touches the Cuspidal Tangent at the Cusp.

13. The following two theorems are evident from Articles 2 and 4, and are best placed side by side.

Let X and Y be two points on the Cubic.

Let P_1, P_2, P_3, P_4 be the points of contact of the tangents from X.

Let Q_1, Q_2, Q_3, Q_4 be the points of contact of the tangents from Y.

Then

$$\text{(i)} \quad X[P_1P_2P_3P_4] = Y[Q_1Q_2Q_3Q_4]$$

$$\text{(ii)} \quad X[Q_1Q_2Q_3Q_4] = Y[P_1P_2P_3P_4].$$

The first is Salmon's well-known theorem. The second, though very simple, is unknown to several authorities on geometry whom I have consulted.