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MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

A meeting of the Association for Symbolic Logic was held at the Edgewater Beach Hotel, Chicago, Illinois on 29–30 April, 1965 in conjunction with the American Philosophical Association. The invited one-hour address entitled *Predicate variables in set theory* was delivered by Professor G. Hasenjaeger. A joint symposium with the American Philosophical Association was held entitled *Philosophic implications of the Gödel incompleteness theorem.* Invited symposiasts were Professors Paul Benacerraf, Hilary Putnam, and Richard Montague with Professor Alfred Tarski serving as chairman. In addition, thirteen papers were delivered and six were presented by title; the last six abstracts below were those presented by title. Professors F. B. Fitch and S. C. Kleene served as chairmen at the sessions of contributed papers.

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G. HASENJAEGER. *Predicate variables in set theory.*

Generally, predicate variables in set theory are used to indicate countable schemata of first-order axioms such as the axioms of subsets (Aussonderung) or replacement (Ersetzung). Otherwise these axioms get their full intuitive strength only from the interpretation of predicate variables in a level of "sets" exceeding the given axiomatic frame. (This strong interpretation is referred to but questioned in Lévy's ***Pacific J.*** (1960) paper.) The role of and the relation between these weak and strong interpretations will be discussed, and an obvious predicate variable version of reflexion type axioms will be presented. (*Received February 3, 1965.*)

RICHARD B. ANGELL. *Quantification without multiple occurrence-sets of variables.*

This paper presents a version of Quine's quantification theory in which no wffs a) contain the same variable in two or more quantifiers, or b) contain a variable which is both bound and free. (I.e., it is a version of quantification theory without multiple "occurrence-sets" of variables).

These restrictions on wffs are introduced by stipulation through changes in rules of formation:

F2. If ϕ is a formula, so is $\ulcorner(\alpha)\phi'\urcorner$.

F3. If ϕ and ψ are formulae, $\ulcorner(\phi{\downarrow}\psi)'\urcorner$ is a formula, where the metasyntactical expression $\ulcorner(\ldots\phi'\ldots)\urcorner$ means "any expression, χ, just like $\ulcorner(\ldots\phi\ldots)\urcorner$ except that if ϕ contains any α which has more than one occurrence-set in $\ulcorner(\ldots\phi\ldots)\urcorner$, then the occurrence-sets of α in ϕ are re-lettered, beginning with bound sets, so that no variables in ϕ have more than one occurrence-set in χ."

Introduction of these restrictions necessitate changes in definitions D1 and D5, as well as in metatheorems *101–*103. In the new version, using the accent, the results are as follows:

D1. $\ulcorner{\sim}\phi\urcorner$ for $\ulcorner(\phi{\downarrow}\phi')\urcorner$

D5. $\ulcorner(\phi \equiv \psi)\urcorner$ for $\ulcorner((\phi \supset \psi)\cdot(\psi \supset \phi)')\urcorner$

*101. $\vdash\ulcorner(\alpha)(\phi \supset \psi) \supset. ((\alpha)\phi)' \supset ((\alpha)\psi)'\urcorner$

*102. $\vdash\ulcorner(\phi \supset (\alpha)\phi')\urcorner$

*103. $\vdash\ulcorner(\alpha)\phi \supset \phi'\urcorner$

And finally, the notion of tautology in *100 must be amended slightly to permit alphabetic variants to be assigned identical truth-values in each possible case. It is then shown that the resulting system is consistent and complete with respect to Quine's original system.

The revised system permits the elimination, as redundant, of certain clauses (like, "If α is not free in ϕ", which is omitted from *102) from the metatheorems. It also makes unnecessary certain of Quine's metatheorems (E.g., *170, *171). Finally, it is shown that with suitable changes in semantic rules governing metatheoretical variables ϕ, ψ, χ, etc., the special sign '' could be dropped; and that the method here employed could be extended to other systems besides Quine's. *(Received February 4, 1965.)*

WILLIAM H. HANSON. *Syntactical systems that reflect the logical-factual distinction.*
We show that Carnap's well-known semantical distinction between L-true (logically true) and F-true (factually true) sentences can be mirrored in syntactical (i.e., logistic) systems. Such a system based on the propositional calculus (pc) can be constructed by adding an enumerably infinite list of propositional constants, t_1, f_1, t_2, f_2, ..., to the usual primitive symbols of pc. Wffs are then defined as usual. For the axioms of the new system take any complete set for pc (call these *L-axioms*) and t_1, $\sim f_1$, t_2, $\sim f_2$, ... (call these *F-axioms*). Take as rules those rules needed for the pc axioms that have been chosen. *Theorems* are then defined as usual, *L-theorems* as those theorems that can be proved without using any F-axioms, and *F-theorems* as those theorems that are not L-theorems.

Semantically, the constants t_i (f_i) are thought of as atomic, factually true (false) sentences, which are distinct from each other in meaning in the sense that no conjunction made up exclusively of t_i's, f_i's and their negations is self-contradictory (assuming, of course, that no constant appears both negated and unnegated in the same conjunction). We define a wff to be *true* if and only if it takes the value T for all possible assignments of truth-values to its variables for which the constants t_i and f_i are assigned T and F, respectively, *L-true* if and only if it takes the value T for all possible assignments of truth-values to its variables and constants, and *F-true* if and only if it is true but not L-true.

Theorem. A wff is a theorem (L-theorem, F-theorem) if and only if it is true (L-true, F-true).

The syntactical procedure has been extended to both S5 and the first-order functional calculus. The Theorem continues to hold, for appropriate extensions of the semantical concepts involved, in both cases. The semantical concepts of the first-order-extension are obtained from the usual semantics of the first-order functional calculus by definitions analogous to those of the preceding paragraph. It can be shown that the F-theorems of the first-order-extension are not recursively enumerable. The semantical concepts of the S5-extension are based on a modification of Kripke's methods for modal logic. *(Received January 28, 1965.)*

A. A. MULLIN. *On the complexity of algorithms.*
Recently, B. M. Kloss [The definition of complexity of algorithms, ***Doklady Akad. Nauk. SSSR,*** vol. 157, No. 1 (1964), pp. 38–40] considered the question of the complexity of effective processes. He has given (Theorem 2, ***op. cit.***) a necessary and sufficient condition for an effectively calculable function to satisfy the condition that for no principal enumeration the complexity of all partial recursive functions is bounded above, *viz.*, that it assumes each of its values only a finite number of times. This note provides a concrete basis for extending his results by giving two elementary, but basic, infinite recursive families of p.r. functions which satisfy the conditions of Theorem 2, ***op. cit.,*** and which are closely related to K. Gödel's method for numbering the WFFs of A^2 in his Incompleteness Theorems. Let N be the set of natural numbers. The first infinite class is $\{\psi^i : i \epsilon N\}$ whose members can be ordered without repetition as defined in the author's note, ***Bull. Amer. Math. Soc.*** 69 (1963), pp. 446–447. The other infinite recursive class is $\{(\psi^*)^i : i \epsilon N\}$, an additive version of the previous

example, as defined in the author's paper, ***Zeitschr. f. math. Logik Grundlagen*** 10 (1964), pp. 199–201; with "powers" of functions defined by composition. (*Received December 13, 1964.*)

JAAKKO HINTIKKA. *Confirmation no longer paradoxical.*

In a paper read at the second International Congress of Logic, Methodology and Philosophy of Science in 1964 I have outlined a new approach to quantitative inductive logic. The basic idea of this approach is to use consistent constituents (strongest consistent statements with given parameters and with a fixed maximal length for their nested sequences of quantifiers) in the way Carnap uses structure-descriptions in assigning *a priori* probabilities to state-descriptions in *The logical foundations of probability*. In this new inductive logic, the general implication (*) $(x)(x$ is a raven $\supset x$ is black) is confirmed only by black ravens provided that it is assumed that the number of ravens in the universe of discourse is fixed (the same in all state-descriptions). It is argued that this numerical assumption is tantamount to the intuitive assumption that we can always decide whether a given member of the domain is a raven or not (independently whether we can decide whether it is black or not). If the converse assumption is also made, the generalization (*) is confirmed by black ravens more than by non-black non-ravens to the extent to which the former are rarer in the whole universe than the latter; it is disconfirmed by black non-ravens to the extent to which ravens are rarer at large than among all black objects. (*Received January 27, 1965.*)

JOHN M. VICKERS. *Definability of theoretical concepts in elementary theories.*

Tarski shows that a necessary and sufficient condition for the term t of the theory T to be definable by means of the other terms of T on the basis of T is that the sentence

$$(x)(x = t \equiv \mathrm{T}(x))$$

should be derivable from T, where x does not occur in T and $T(x)$ is the result of replacing t throughout T by x. Here definability of t means that there is a systematic procedure for replacing any sentence S of T by a sentence S′ of T which does not include t and such that the equivalence of S and S′ is a consequence of T.

In this paper attention is restricted for the most part to questions of definability in elementary theories with equivalence. Tarski's result is extended to provide a necessary and sufficient condition for a predicate P (of degree n) of the theory T to be definable by means of the other constants of T on the basis of T, namely that if Q is a predicate letter of degree n not occurring in T then

$$(x_1, \ldots, x_n)(Qx_1, \ldots, x_n \equiv Px_1, \ldots, x_n) \equiv T(Q)$$

is derivable from T, where $T(Q)$ results from replacing P throughout T by Q.

These necessary and sufficient syntactical conditions for definability are shown to be equivalent to the following semantical conditions: Definability of the term t to the condition that every two normal models of T which agree up to t also agree for t; definability of the predicate P to the condition that every two models of T on the same base which agree up to P also agree for P.

The conditions under which a theory has an extension in which a given one of its terms or predicates can be defined are investigated. The relevance of definability to the meanings of Ramsey sentences are investigated. (*Received February 1, 1965.*)

H. HIŻ. *Ontological definitions in augmented protothetics.*

Assuming protothetics and an axiom which introduces a grammatical category A different from any category in protothetics and the grammatical category B of a functor that forms a sentence with one argument of the category A, e.g.,

1. $\forall\varphi\forall a\ulcorner\supset(\varphi\{a\}\ \varphi\{a\})\urcorner$, where the brackets are of the shapes not occurring in protothetics, one can prove any of the so-called ontological definitions (for the category B) as well as various definitions by identity. One can illustrate the procedure by deducing the ontological definition of class complementation n (4 below).

Df. o. $\forall a\forall b\ulcorner\equiv(o\{ab\}\forall\varphi\ulcorner\equiv(\varphi\{a\}\ \varphi\{b\})\urcorner)\urcorner$

Df. e. $\forall\varphi\forall\psi\ulcorner\equiv(e\{\varphi\psi\}\&(\forall a\ulcorner\supset(\varphi\{a\}\ \psi\{a\})\urcorner\exists a\ulcorner\varphi\{a\}\urcorner\forall a\forall b\ulcorner\supset(\&(\varphi\{a\}\ \varphi\{b\})o\{ab\})\urcorner))\urcorner$

Df. n. $\forall\varphi\forall a\ulcorner\equiv(n\{\varphi\}\{a\}\sim(\varphi\{a\}))\urcorner$

By substituting ψ/φ in Df e and applying the result to Df e,

2. $\forall\varphi\forall\psi\ulcorner\equiv(e\{\varphi\psi\}\&(\forall a\ulcorner\supset(\varphi\{a\}\ \psi\{a\})\urcorner e\{\varphi\varphi\}))\urcorner$

In 2 substitute $\psi/n\{\psi\}$,

3. $\forall\varphi\forall\psi\ulcorner\equiv(e\{\varphi n\{\psi\}\}\&(\forall a\ulcorner\supset(\varphi\{a\}\ n\{\psi\}\{a\})\urcorner e\{\varphi\varphi\}))\urcorner$

Using Df n to 3,

4. $\forall\varphi\forall\psi\ulcorner\equiv(e\{\varphi n\{\psi\}\}\&(\forall a\ulcorner\supset(\varphi\{a\}\sim(\psi\{a\}))\urcorner e\{\varphi\varphi\}))\urcorner$

In a similar manner we introduce the definition

Df. =. $\forall\varphi\forall\psi\ulcorner\equiv(=\{\varphi\psi\}\forall a\ulcorner\equiv(\varphi\{a\}\ \psi\{a\})\urcorner)\urcorner$

and derive

5. $\forall\varphi\forall\psi\ulcorner\equiv(=\{\varphi n\{\psi\}\}\forall a\ulcorner\equiv(\varphi\{a\}\sim(\psi\{a\}))\urcorner)\urcorner$

All definitions here are in accordance with the rule of definitions in protothetics augmented by the categories of 1. By the rules of protothetics with similar augmentation one can prove for e the formula which was used as the only axiom of Leśniewski's ontology. (*Received February 2, 1965.*)

RICHMOND H. THOMASON. *An approach to infinitary propositional calculus.*

We will consider three cutfree systems $\mathbf{LK}_\infty$, $\mathbf{LJ}_\infty$, and $\mathbf{LS4}_\infty$ of infinitary propositional calculus, corresponding respectively to classical two-valued logic, intuitionistic logic, and the modal logic **S4**. Wffs of these systems are characterized as follows:

i) p_i is wf for all natural numbers i,

ii) if A and B are wf, so are $\sim A$, $A\supset B$, and $\Box A$,

iii) if Γ is an (at most denumerable) set of wffs, then $\wedge\Gamma$ and $\vee\Gamma$ are wf.

(This could be replaced, if desired, by a definition using transfinite induction up to ω_1.) Where Γ and Δ are denumerable sets of wffs, $\Gamma\vdash\Delta$ is a (wf) *sequent*.

Necessity-wffs and *possibility-wffs* are characterized as follows:

i) any wff having the form $\Box A$ [$\sim\Box A$] is a necessity-wff [possibility-wff];

ii) if Γ is a denumerable set of necessity-wffs [possibility-wffs], then $\wedge\Gamma$ is a necessity-wff [possibility-wff].

Besides the usual structural rules and logical rules for the connectives $\sim$ and $\supset$, the infinitary systems have the following primitive rules:

$$\frac{\Gamma;\ A\vdash\Delta}{\Gamma;\ \wedge(A;\ \Theta)\vdash\Delta} \qquad \frac{\Gamma\vdash\Delta;\ A_0,\ \ \Gamma\vdash\Delta;\ A_1,\ \ldots}{\Gamma\vdash\Delta;\ \wedge\{A_0,\ A_1,\ \ldots\}}$$

$$\frac{\Gamma;\ A_0\vdash\Delta,\ \ \Gamma;\ A_1\vdash\Delta,\ \ldots}{\Gamma;\ \vee\{A_0,\ A_1,\ \ldots\}\vdash\Delta} \qquad \frac{\Gamma\vdash\Delta;\ A}{\Gamma\vdash\Delta;\ \vee(A;\ \Theta)}$$

where in $\mathbf{LJ}_\infty$ only unit-sets or the empty set may appear on the right.

$\mathbf{LS4}_\infty$ has in addition the primitive rules:

$$\frac{\Theta\vdash\Xi;\ A}{\Theta\vdash\Xi;\ \Box A} \qquad \frac{\Gamma;\ A\vdash\Delta}{\Gamma;\ \Box A\vdash\Delta}$$

where every wff in Θ is a necessity-wff and every wff in Ξ a possibility-wff.

By means of transfinite induction, an elimination theorem can be established for each of these three systems. (*Received February 1, 1965.*)

ANTHONY C. SCOVILLE. *The cardinal number of the continuum.*

The present paper is a short extract from an extensive work in progress which undertakes to provide a comprehensive exploration for a solution to the philosophical problem of The One and The Many. Here we shall confine ourselves to the attempt to determine the number of elements contained by any continuous series (ordered set). While it is the author's belief that the analysis to follow by no means provides the fundamental solution to Cantor's "Continuum Problem", a deeper treatment is necessarily too broad for adequate presentation in the present short report as it would involve a far-reaching recasting of the axioms of set theory.

Nevertheless, following the general pattern laid down by Cantor, Huntington, Gödel, and Kamke with the addition of one postulate, it is possible to demonstrate that

$$N_c = \aleph_1 = \aleph_0^{\aleph_0}. \tag{1}$$

The further postulate which is necessary is that all the members of a set must be unique in the sense that every element must be potentially enumerable as a member of a partial subset

$$S_n = (a, b, c \ldots) \tag{2}$$

of the totality of all sets S_c composing a continuous series.

The solution (1) may be generalized to sets of higher power than $\aleph_1$. However, the engendered series do not possess unique members. They are pure operational constructs which need transformation to (1) before we can ascertain the extensive existence of the elements of such a series. (*Received January 6, 1965.*)

W. W. TAIT. *Cut elimination in infinite propositional logic.*

Propositional formulae (p.f.) are built up from atoms p, q, etc. using negation $\neg A$ and infinite disjunctions $\bigvee_{\xi<\alpha} A_\xi$ where α is an ordinal. $\mathfrak{M}$, $\mathfrak{M}_1$, etc. denote finite sets of p.f. and $\mathfrak{M} \cup A$ denotes $\mathfrak{M} \cup \{A\}$. The rules of inference, for deriving finite sets of p.f., are: 1^0. $\mathfrak{M} \cup p \cup \neg p$ 2^0. $\mathfrak{M} \cup A \vdash \mathfrak{M} \cup \neg\neg A$ 3^0. $\mathfrak{M} \cup A_\eta \vdash \mathfrak{M} \cup \bigvee_{\xi<\alpha} A_\xi$ (for any $\eta < \alpha$) 4^0. $\mathfrak{M} \cup \neg A_\eta$ for all $\eta < \alpha \vdash \mathfrak{M} \cup \neg\bigvee_{\xi<\alpha} A_\xi$, and the cut rule 5^0. $\mathfrak{M} \cup A$, $\mathfrak{M}' \cup \neg A \vdash \mathfrak{M} \cup \mathfrak{M}'$. A is called the *cut formula* (c.f.) in 5^0. These rules are valid if we interpret $\mathfrak{M}$ as a disjunction of its p.f.; and also they are complete. $\rho A \leqq \beta$ means that there is an assignment of ordinals B^* to the subformulae B of A with $p^* = 0$, $(\neg B)^* = B^* + 1$, $B_\eta^* < (\bigvee_{\xi<\alpha} B_\xi)^*$ for all $\eta < \alpha$ and $A^* \leqq \beta$. If D is a derivation, $\delta D \leqq \gamma$ means that $\rho\neg A \leqq \gamma$ for every c.f. A in D. $|D| \leqq \delta$ means that we can assign ordinals $\leqq \delta$ to the steps in D (in tree form) so that the ordinal of the conclusion of each inference exceeds the ordinals of the premises. Let $\chi_\alpha^0 = 2^\alpha$, and for $\gamma > 0$, χ_α^γ is the α^{th} simultaneous solution of $\chi_\beta^{\gamma'} = \beta$ for all $\gamma' < \alpha$.

Theorem. If D is a derivation of $\mathfrak{M}$ and $\delta D \leqq \beta + \omega^\gamma$, then there is a derivation D′ of $\mathfrak{M}$ with $\delta D' \leqq \beta$ and $|D'| \leqq \chi_{|D|}^\gamma$.

The only proof-theoretic result needed for this is

Lemma. If D and D′ are derivations of $\mathfrak{M} \cup A$ and $\mathfrak{M}' \cup \neg A$ resp., with δD, $\delta D' \leqq \beta$ and $\rho A \leqq \beta$, then there is a derivation D″ of $\mathfrak{M} \cup \mathfrak{M}'$ with $\delta D'' \leqq \beta$ and $|D''| \leqq |D| \circ |D'|$, where $\alpha \circ \beta$ is the *natural sum* of α and β. [Cf. Schütte, *Beweistheorie*].

If we consider only derivations D with $\delta D < \omega^\gamma$ and $|D| < \alpha$ and which can be represented by primitive recursive (p.r.) functions (so that we are considering only countable p.f. and derivations) and if we represent the ordinals in a suitable p.r. well ordering, then the proof of the above results can be carried out in p.r. arithmetic with definition of functions by recursion up to any $\beta < \chi_\alpha^\gamma$.

Let Γ_0 be the least fixed point of the function $\varphi(\alpha) = \chi_0^\alpha$. The above theorem can be applied to prove that if induction up to α is proved in some system in the autonomous hierarchy of ramified analysis, hyperarithmetical analysis or ramified set theory, then $\alpha < \Gamma_0$, and thus unifies the proofs of these results due to Schütte and Feferman. (*Received March 5, 1965.*)

FREDERIC B. FITCH. *Tree proofs in modal logic.*

Tree-proof procedures are given for the systems M, S4, S5, and B (the Brouwersche system), and for deontic systems DM, DS4, DS5, and DB. (DM is like M but lacks $\Box A \supset A$, while retaining $\Box A \supset \Diamond A$, where $\Diamond A$ is $\sim\Box\sim A$. Add $\Box A \supset \Box\Box A$ to DM to give DS4, and add $\Diamond A \supset \Box\Diamond A$ to DM to give DB, and add both of them to DM to give DS5.) The method combines features of S. A. Kripke's method of semantical analysis [***Zeitschrift für mathematische Logik und Grundlagen der Mathematik*** 9 (1963) 67–96] and features of the Anderson-Belnap tree-proof method [***J. Symbolic Logic*** 24 (1959) 320–321]. It differs from Kripke by introducing symbols for universes into the object language, thus avoiding semantical tableaux, but making it necessary to show that this extension of the object language is a conservative extension. If α symbolizes a universe, then $\alpha_1, \alpha_2, \ldots,$ symbolize universes possible relative to α. If A is an ordinary sentence of modal logic, then αA is a *secondary* sentence, asserting that A is true in α, while A is a *primary* sentence. Truth-functional combinations of secondary sentences are secondary sentences. Tree-proof axioms are primary sentences having A and $\sim A$ as disjunctive parts, or secondary sentences having αA and $\alpha\sim A$ as disjunctive parts, where A is atomic. Some examples of tree-proof rules of inference for DM: $(\alpha_1 A \vee \alpha\Diamond A) \rightarrow \alpha\Diamond A$; $\alpha_1 A \rightarrow \alpha\Box A$; $(\alpha A \vee \alpha B) \rightarrow \alpha(A \vee B)$; $\alpha\sim A \rightarrow \sim\alpha A$; $\alpha\Box\sim A \rightarrow \alpha\sim\Diamond A$; $\alpha\Diamond\sim A \rightarrow \alpha\sim\Box A$; if $A \rightarrow B$, then $(C \vee A) \rightarrow (C \vee B)$ and $(A \vee C) \rightarrow (B \vee C)$. The tree-proof method can be extended to deal with quantifiers and identity. The various systems considered are correlated in obvious ways with formal properties of the relation between universes. (*Received February 3, 1965.*)

W. E. SINGLETARY. *A note on the finite axiomatization of partial propositional calculi.*

Definition; A partial propositional calculus is a system having $\supset$, $\sim$, [,], and an infinite list of propositional variables $p_1, q_1, r_1, p_2, q_2, r_2, \ldots$ as primitive symbols. Its well-formed formulas are (1) a propositional variable standing alone, (2) $[A \supset B]$, where A and B are well-formed formulas, and (3) $\sim A$, where A is a well-formed formula. Its axioms are a recursive (possibly infinite) set of tautologies, and its two rules of inference are modus ponens and substitution.

Theorem. For each recursively enumerable degree of unsolvability D there exists a class of partial propositional calculi $\{P\}_D$ such that the problem to determine of an arbitrary member P of $\{P\}_D$ whether or not P is finitely axiomatizable is of degree D.

To each positive integer n recursively assign a well-formed formula as follows: W_1 is $[p_1 \supset p_1]$ and W_{n+1} is $[p_1 \supset W_n]$. Let S be a set of ordered pairs of positive integers having the properties (1) there is a recursive procedure to determine of any ordered pair (m, n) whether or not (m, n) is in S, (2) the problem to determine of an arbitrary n whether or not there is an m such that (m, n) is in S is of degree D, and (3) the set of second members of the ordered pairs of S is infinite. Let L represent the single Łukasiewicz axiom for the complete propositional calculus. Then P_S is specified by the following axioms (1) $\sim\sim\sim\sim p_1 \supset \sim\sim p_1$, (2) $\sim\sim p_1 \supset \sim\sim\sim\sim p_1$ and (3) $\sim^{2m}W_n$ for each (m, n) in S. $P_S(W_n)$ is specified by adding the well-formed formula $\sim\sim W_n \supset L$ to the axioms of P_S. Then if $\{P\}_D$ is taken to consist of all calculi of the form of $P_S(W_n)$ we can prove that the problem to determine of an arbitrary member P of $\{P\}_D$ whether or not P is finitely axiomatizable is of degree D. (*Received February 15, 1965.*)

DANIEL E. ANDERSON and RICHARD B. ANGELL. *Venn diagrams for n classes.*

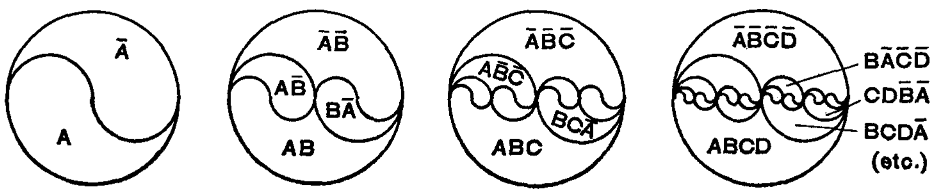
This paper presents two methods for the construction of Venn-type diagrams representing all possible relations between any given set of n classes. Venn diagrams for four or fewer classes present little difficulty. Previous methods for extending Venn diagrams to more than four classes, however, have either 1) required discontinuous regions, or 2) been incapable of generalization for any n, or, 3) were difficult to grasp visually, or, 4) did not lend themselves to insertion of shading or stars to denote emptiness or non-emptiness of the classes. The two methods below avoid these shortcomings.

The first method, which was treated in detail in this JOURNAL, vol. 30, pp. 113–118 by D. E. Anderson and F. L. Cleaver, represents classes by regions resembling "pants-legs" or "comb-teeth". A topological proof is available to show that such diagrams are constructible for any n terms.

The second method, suggested by Angell, is a special case of the general theorem referred to above. In this case a circle with diameter, d, represents the universe of discourse. The first class is established by bisecting the circle with a wave of wave-length $1d$ and amplitude $1/4d$. This and each succeeding nth class, is determined by a wave with 2^{n-1} crests, having a wave-length $d/2^{n-1}$, and amplitude $d/4(2^{n-1})$. The formula for the line defining the nth class is

$$y = (-1)^{k-1}\sqrt{\frac{1}{2(n+1)} - \left(x - \frac{2k-1}{2^{n+1}}\right)^2}, \text{ where } k = 1, 2, \ldots, 2^n \text{ and } \frac{k-1}{2^n} \leqq x \leqq \frac{k}{2^n},$$

and the diagrams, for the first four classes, will look like this:



This method will provide a diagram for any n classes, with each class visibly distinguishable, constructed by a rule which remains the same from the introduction of the first class on. Some additional properties of these diagrams will be discussed. (*Received February 3, 1965.*)

DAVID KAPLAN. *Rescher's plurality-quantification.*

In [1], Rescher introduces the plurality-quantifier 'M', where 'MxFx' is read 'most objects x are such that Fx', and comments on the validity of a number of interesting schemata expressed in the first-order predicate calculus with the addition of the plurality quantifier. Semantical notions for languages containing plurality quantification are based on the following clause in the definition of 'satisfaction',

f satisfies $M\alpha\Phi$ in $\langle DR\rangle$ if and only if $K(\underset{x}{E}[x \in D$ and f^{α}_{x} satisfies Φ in $\langle DR\rangle]) >$ $K(\underset{x}{E}[x \in D$ and f^{α}_{x} satisfies $\neg\Phi$ in $\langle DR\rangle])$;

here $\langle DR\rangle$ is a model consisting of a non-empty set D and an assignment R of denotations to non-logical constants, f is an assignment of values to variables, $f^{\alpha}_{x} = (f \sim \{\langle \alpha f(\alpha)\rangle\}) \cup \{\langle \alpha x\rangle\}$, and 'K' is read 'the cardinal of'.

Let $\mathscr{L}_1$ be the set of all formulas of the first-order monadic predicate calculus without identity; $\mathscr{L}_1^M$ is obtained from $\mathscr{L}_1$ by admitting plurality quantification.

Lemma 1. If $\Phi \in \mathscr{L}_1^M$, then we can effectively find an equivalent $\Psi \in \mathscr{L}_1^M$ which contains no quantifier within the scope of another quantifier.

Using this lemma we obtain:

Lemma 2. If $\Phi \in \mathscr{L}_1^M$, then we can effectively find a sentence Φ^a expressed in first-order arithmetic with '+' as its only non-logical sign, such that Φ is valid in all finite models if and only if Φ^{a} is true.

Lemma 3. If $\Phi \varepsilon \mathscr{L}_1^M$, then we can effectively find a finite number of denumerable models $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, such that Φ is valid in all infinite domains if and only if Φ is true in $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.

By lemma 2 and Presberger's theorem that the arithmetical theory mentioned is decidable, the finitely valid formulas of M are decidable. Hence by lemma 3:

Theorem 1. The valid formulas of $\mathscr{L}_1^M$ are decidable.

Lemma 4. There is a sentence $\Phi \varepsilon \mathscr{L}_1^M$ and a finite number b such that

(i) Φ is false in all infinite models.

(ii) Φ is satisfiable in all finite domain with more than b elements.

(iii) Φ is false in all models with b or less elements.

From lemma 4 and the well-known fact that no such sentence can be formed in first-order languages, we have:

Theorem 2. (stated in [1]) The plurality quantifier is not definable in the (full) first order predicate calculus with identity.

The plurality quantifier is obviously definable if we allow quantification on a binary predicate variable.

Let $\mathscr{L}_2$ be obtained from $\mathscr{L}_1$ by admitting binary predicates; $\mathscr{L}_2^M$ is obtained from $\mathscr{L}_2$ by admitting plurality quantification. Using lemma 4 and the fact that corresponding to every $\Psi \varepsilon \mathscr{L}_2$ and finite number b, we can find a formula $X \varepsilon \mathscr{L}_1$ which is valid just in case Ψ is valid in all domains with b or less elements, we have the result:

Lemma 5. If $\Psi \varepsilon \mathscr{L}_2$, we can effectively find Φ, $X \varepsilon \mathscr{L}_1^M$, such that $(\Phi \rightarrow \Psi) \wedge X$ is (universally) valid if and only if Ψ is valid in all finite domains.

But from the theorem of Trahtenbrot and Craig we know that the set of $\Psi \varepsilon \mathscr{L}_2$ which are valid in all finite domains is not axiomatizable. Therefore

Theorem 3. The valid formulas of $\mathscr{L}_2^M$ are not axiomatizable.

Finally, in contrast to lemma 3, and Löwenheim's theorem for the full predicate calculus with equality, we have:

Theorem 4. There is a sentence $\Phi \varepsilon \mathscr{L}_2^M$ which is valid in all and only countable domains.

[1] Nicholas Rescher, *Plurality-quantification*, ***The Journal of Symbolic Logic,*** vol. 27, pp. 373–374. (*Received March 10, 1965.*)

David Kaplan. *Generalized plurality-quantification.*

For each rational number m/n, $0 \leqq m/n < 1$, we provide an interpretation of the plurality quantifier (see the preceding abstract for notation and reference) according to which 'MxFx' is read 'more than m/n of the objects x are such that Fx'. Semantical notions for each such interpretation are based on the following clause:

$$\text{f m/n-satisfies } M\alpha\Phi \text{ in } \langle DR\rangle \text{ if and only if}$$

$$n \cdot K(\underset{x}{E}[x \varepsilon D \wedge f_x^{\alpha}\text{m/n-satisfies } \Phi \text{ in } \langle DR\rangle]) > m \cdot K(D)$$

It is readily seen that the familiar existential quantifier and Rescher's plurality quantifier are generalized plurality quantifiers for the ratios 0 and 1/2 respectively.

Theorem 1. If $0 < m/n < 1$, then all lemmas and theorems of the preceding abstract hold for m/n-quantification.

In his abstract, Rescher points out that all his remarks remain unaffected if 'M' were interpreted — for finite domains — as 'more than 80 per cent [instead of 50 per cent] of the objects,' etc. The following theorems illustrate the fact that validity is not, in general, preserved by such a reinterpretation.

Theorem 2. If $0 < m/n < 1$, then we can construct a sentence $\Phi \varepsilon \mathscr{L}_1^M$ such that for all rational j/k $0 < j/k < 1$, Φ is j/k-valid if and only if $j/k \geqq m/n$.

Theorem 3. If $0 < m/n < 1$, then we can construct a sentence $\Phi \varepsilon \mathscr{L}_1^M$ such that for all rational j/k $0 < j/k < 1$, Φ is j/k-valid in all finite domains if and only if $m/n \geqq j/k$.

Corollary. If $0 < m/n < 1$, then we can construct a sentence $\Phi \varepsilon \mathscr{L}_1^M$ such that for all rational j/k, $0 < j/k < 1$, Φ is j/k-valid in all finite domains if and only if $j/k = m/n$. *(Received March 10, 1965.)*

C. E. M. YATES. *On the degrees of index sets* (II).

We have already announced (Abstract, this JOURNAL, vol. 28 (1964), p. 161) an exact classification (within the arithmetical-hierarchy) of the set $G(\mathbf{a}) = \{e | R_e$ is of degree $\mathbf{a}\}$ for any r.e. degree $\mathbf{a}$. (Our notation is that of the previous abstract). A slight modification of the proof yields Sacks' theorem that the r.e. degrees are dense as follows. Let $\mathbf{a}$, $\mathbf{b}$ be r.e. degrees such that $\mathbf{a} < \mathbf{b}$. The first simple step is to show that $G(\mathbf{a}) \epsilon \Sigma_3(\mathbf{b})$. The second (slightly modified) step in the classification is to define a r.e. sequence of r.e. sets $C_0, C_1, \ldots$, such that if $\mathbf{c}_e$ is the degree of C_e then $\mathbf{a} \leqq \mathbf{c}_e \leqq \mathbf{b}$ and $R_e \epsilon G(\mathbf{a}) \equiv C_e \epsilon G(\mathbf{b})$ for all e. By the fixed-point theorem there is a number k such that $C_k = R_k$. Clearly, $\mathbf{c}_k \neq \mathbf{a}$ and $\mathbf{c}_k \neq \mathbf{b}$, so that $\mathbf{a} < \mathbf{c}_k < \mathbf{b}$. The only modification in the original classification procedure is to arrange that $\mathbf{a} \leqq \mathbf{c}_e$ for all e, but this introduces no major new difficulties.

Similar methods can be used to prove: (i) if $\mathbf{a} < \mathbf{0}^{(1)}$ and $\mathbf{a}_0 < \mathbf{a}_1 < \ldots$ is an infinite ascending sequence of uniformly r.e. degrees each $< \mathbf{a}$, then there is a r.e. degree $\mathbf{c}$ such that $\mathbf{a}_0 < \mathbf{a}_1 < \ldots < \mathbf{c}$ and $\mathbf{a}|\mathbf{c}$. (ii) if $\mathbf{0} < \mathbf{a} < \mathbf{0}^{(1)}$ and $\mathbf{b}$ is a degree which is $\geqq \mathbf{0}^{(1)}$ and r.e. in $\mathbf{0}^{(1)}$ then there is a r.e. degree $\mathbf{c}$ such that $\mathbf{c}^{(1)} = \mathbf{b}$ and $\mathbf{a}|\mathbf{c}$. (iii) if $\mathbf{0} < \mathbf{a} < \mathbf{0}^{(1)}$ then there is a degree $\mathbf{c}$ such that $\mathbf{c}|\mathbf{a}$ and $\mathbf{c}$ contains a maximal set. *(Received February 5, 1965.)*

P. H. G. ACZEL and J. N. CROSSLEY. *Constructive order types VI.*

Terminology and notation are as in the second author's previous abstracts in this JOURNAL.

We define a function E of coordinals following a construction of Parikh.

Theorem. The following are equivalent

(i) X is an infinite principal number for exponentiation

(ii) $X = W$ or $W^X = X$

(iii) $X = W$ or $X = E$ (A) for some coordinal A.

Corollary. The collection of all principal numbers for exponentiation is strictly ε_ω-unique.

A coordinal A is said to be full if $\Gamma < |A| \rightarrow (\exists C)\ (|C| = \Gamma \,\&\, C < A)$.

A coordinal A is said to be closed under f if $A_1, \ldots, A_n < A$ implies $f(A_1, \ldots, A_n) < A$.

Theorem. For any epsilon number ε_Γ there is a coordinal A such that

(i) $|A| = \Gamma$, $|E(A)| = \varepsilon_\Gamma$

(ii) E(A) is full and closed under addition, multiplication and exponentiation.

Notation. $||A|| =$ order of type $\{B : B < A\}$

Theorem. If A is closed under addition or multiplication or exponentiation then $||\ ||$ is an isomorphism from $\{B : B < A\}$ with respect to addition or multiplication or exponentiation, respectively, *onto* an initial segment of the ordinals. *(Received February 2, 1965.)*

W. W. TAIT. *Intensional interpretations of functionals of finite type I.*

0 is a *finite type* (f.t.), and (σ, τ) is a f.t. whenever σ and τ are. Every constant and variable of type τ is a *τ-term*, and (st) is a τ-term if s is a (σ, τ)-term and t a σ-term. $t_1t_2\ldots t_n$ abbreviates the term $(\ldots(t_1t_2)\ldots t_n)$. Gödel's theory T (***Dialectica***, v. 12 (1958), pp. 280–87) can be formulated as follows: The constants are $\bar{0}$ (zero) of type 0, S (successor) of type (0, 0), and constants P_i, C and R of appropriate types.

$\bar{1} = S\bar{0}$, $\bar{2} = S\bar{1}$, etc. The formulae of T are formed from equations $s^\tau = t^\tau$ (for *arbitrary* f.t. τ) by means of connectives (but no quantifiers). The axioms and rules are: those of *classical* propositional logic (so that $s^\tau = t^\tau$ is not to be interpreted as extensional equality, but as a decidable relation of "definitional equality"), the usual axioms for $=$ (for each f.t. τ), $\bar{0}$ and S, the rule of mathematical induction, and the defining axioms: $P_i t_1 \ldots t_n = t_i$, $Crst_1 \ldots t_n = rt_1 \ldots t_n(st_1 \ldots t_n)$, $Rrs\bar{0} = r$ and $Rrs(St) = s(Rrst)t$. (E.g. for all $n > 0$, $0 < i \leqq n$ and f.t. $\tau_1, \ldots, \tau_n$, there is a P_i of type $(\tau_1, \ldots(\tau_2, \ldots(\tau_{n-1}, \tau_n)\ldots))$.) The P_i and C yield explicit definition, and the R yield primitive recursion.

$r \dashv s$ means that s is obtained from r by a sequence of substitutions of right-hand sides of defining axioms by the corresponding left-hand sides. $T_\tau(s)$ means that s is a closed τ-term. Define $E(s, t) \equiv \bigvee r(s \dashv r \wedge t \dashv r)$ (s and t are *definitionally equal*), $C_0(s) \equiv \bigvee n(s \dashv \bar{n})$ and $C_{(\sigma,\tau)}(s) \equiv \bigwedge t(C_\sigma(t) \rightarrow C_\tau(st))$ (s is a *convertible term*). In intuitionistic arithmetic I we can derive $\bigwedge x(T_\tau(x) \rightarrow C_\tau(x)) \bigwedge xy(C_\tau(x) \wedge C_\tau(y) \rightarrow E(x, y) \vee \neg E(x, y))$. Let $v_0, v_1, v_2, \ldots$ be the variables of type 0 of T. If s is a term $\bar{s}(x)$ is the function whose value is the Gödel number of the result of replacing v_i in s by the numeral $(x)_i$ for all $i \geqq 0$. Thus, $\vdash_I \bigwedge x T_\tau(\bar{s}(x))$ if s is a τ-term. If A is a formula of T without variables of types $\neq 0$, then $A^*(x)$ is the result of replacing each part $s = t$ of A by $E(\bar{s}(x), \bar{t}(x))$.

Theorem. $\vdash_T A \Rightarrow \vdash_I \bigwedge x A^*(x)$. We can extend this result to get an interpretation in I of $T^+ = T + I^\omega$, where I^ω is I with quantification over all f.t. By Gödel's result (loc. cit.) then, T, T^+ and I all have the same quantifier free theorems (in their common notation). (*Received March 5, 1965.*)

W. W. Tait. *Intensional interpretations of functionals of finite type II.*

The notation follows the previous abstract (I). T_1 is like T except that we add a constant B_0 for bar recursion of type 0 [Spector, *Recursive function theory* (***Proc. of Symposia in Pure Math.***, vol. V) AMS, pp. 1–27] and a constant K of type (0, 0). (Bar recursion of type 0 is essentially recursion on the unsecured sequences of a variable functional of type ((0, 0), 0).) The defining axioms for B_0 are given by Spector (loc. cit.), and the axioms for K is given relative to a fixed free choice sequence γ of numbers: namely, $K\bar{n} = \overline{\gamma(n)}$. We define $r \dashv_\gamma s$ as in (I), but relative to the new axioms as well. If r does not contain K, then $r \dashv_\gamma s$ is written $r \dashv' s$. $T'_\tau(x)$, means x is a closed term of T_1 not containing K, $E(x, y) \equiv \bigvee z(x \dashv' z \wedge y \dashv' z)$ and $C_0(x) \equiv \bigvee n(x \dashv \bar{n})$, $C_{((0,0),0)}(x) \equiv \bigwedge \gamma \bigvee n(xK \dashv_\gamma \bar{n})$, and for $(\sigma, \tau) \neq ((0, 0)\ 0)$, $C_{(\sigma,\tau)}(x) \equiv \bigwedge y(C_\sigma(y) \rightarrow C_\tau(xy))$. Then by the same lemmas as in I:

Theorem. $\vdash_{T_1} A \Rightarrow \vdash_{I_1} \bigwedge x A^*(x)$, where I_1 is obtained by adding quantification over free choice sequences and the axiom of bar induction (see below) to I. Let T_2 have constants and defining equations for bar recursion of each f.t., together with those of T. Let I_2 be I_1 together with the axiom of *generalized bar induction*: $\bigwedge x(R(x) \vee \neg R(x)) \wedge \bigwedge \gamma(\bigwedge y D(\gamma(y)) \rightarrow \bigvee x R(\bar{\gamma}(x))) \wedge \bigwedge x(R(x) \rightarrow Q(x)) \wedge \bigwedge x(\bigwedge y Q(x^\frown y) \rightarrow Q(x)) \rightarrow Q(\langle\,\rangle)$, where $x \vee y$ is $\langle x_1, \ldots, x_n, y\rangle$ when $x = \langle x_1, \ldots, x_n\rangle$, and $\langle\,\rangle$ is the empty sequence number. This is equivalent to ordinary bar recursion when D is decidable. By the same methods:

Theorem. $\vdash_{T_2} A \Rightarrow \vdash_{I_2} \bigwedge x A^*(x)$. By Spector's theorem (loc. cit.) the Gödel interpretation of classical analysis holds in $T_2 (= \Sigma_4)$, and so we can interpret classical analysis in I_2. The converse of this is evident. Unlike I_1, however, no constructive foundation for I_2 is known. Ordinary bar induction is founded on Brouwer's theory of spreads, where the elements of the spreads are finite sequences of elements from some decidable species D. Generalized bar induction requires D to be undecidable. No constructive theory of spreads with undecidable D has been worked out. (*Received March 5, 1965.*)

RICHMOND H. THOMASON. *A decision procedure for Fitch's propositional calculus.* Consider the L-system (in the sense of Gentzen 4422) **LF**, having multiple constituents on the right and the usual structural rules and logical rules for classical disjunction, as well as the following rules:

$$\frac{\alpha, A \vdash B}{\alpha \vdash A \supset B} \qquad\qquad \frac{\alpha \vdash A, \beta \qquad \alpha, B \vdash \beta}{\alpha, A \supset B \vdash \beta}$$

$$\frac{\alpha \vdash A \vee B, \beta}{\alpha \vdash A, B, \beta} \qquad \frac{\alpha \vdash A, \beta}{\alpha, {\sim}A \vdash \beta} \qquad \frac{\alpha, A \vdash \beta \qquad \alpha, B \vdash \beta}{\alpha, A \vee B \vdash \beta}$$

$$\frac{\alpha \vdash {\sim}A, \beta \qquad \alpha \vdash {\sim}B, \beta}{\alpha \vdash {\sim}(A \vee B), \beta} \qquad\qquad \frac{\alpha, {\sim}A, {\sim}B \vdash \beta}{\alpha, {\sim}(A \vee B) \vdash \beta}$$

$$\frac{\alpha \vdash A, \beta}{\alpha \vdash {\sim}{\sim}A, \beta} \qquad\qquad \frac{\alpha, A \vdash \beta}{\alpha, {\sim}{\sim}A \vdash \beta}$$

This system is equivalent to the propositional calculus of Fitch (XXI 414), in the following sense: $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is provable in **LF** if and only if there is a proof in Fitch's system of $B_1 \vee \ldots \vee B_m$ on the hypotheses $A_1, \ldots, A_n$ (or, in case $m = 0$, a proof of $p_0 \wedge {\sim}p_0$ on hypotheses $A_1, \ldots, A_n$).

By means of arguments similar to those of Gentzen, an elimination theorem (or *Hauptsatz*) can be established for **LF**; accordingly, the methods commonly used to devise decision procedures for L-systems of propositional logic will yield such a procedure for **LF** (*Received February 1, 1965.*)

NOTICE

FELLOWSHIP AND RESEARCH OPPORTUNITIES IN MATHEMATICS

The Division of Mathematics, National Academy of Sciences—National Research Council, calls attention to a variety of fellowship and other support for basic research in mathematics at both the predoctoral and postdoctoral levels to be awarded during the year 1964–65. Copies of the complete announcement are available from the Division of Mathematics, National Academy of Sciences—National Research Council, 2101 Constitution Avenue, N.W., Washington, D.C. 20418.

NOTEBOOKS OF EMIL POST

The Fine Hall Library at Princeton University has a Xerox copy of the original notes of the late Emil Post consisting of 41 volumes of approximately 200 pages each distributed as follows:

Vols. 1–17	Theory of finite processes
18–35	Creative logic
36–39	Closed truth systems