On the number of conditions determining geometrical figures.

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§ 1. This paper is an attempt (I) to deduce from first principles the number of conditions required to determine a plane polygon of n sides; (II) thence to deduce the numbers for special cases; and (III) to discuss the effects of a redundancy and a deficiency in the number of conditions. An investigation of this kind should form an important as well as interesting accompaniment to the ordinary study of elementary geometry.

§ 2. Figures may agree in many respects, as in perimeter, size of angles,* &c.; but here they are to be considered only as they agree or differ in (*a*) Shape, (*b*) Size, (*c*) Position; or combinations of these.

When figures are to be determined in position, a plane is assumed containing axes of reference.

§ 3. A condition is the limitation to which an element or part of a figure is to be subjected. This limitation is always ultimately reducible to an assertion of equality between an element of the figure and a given magnitude of the same kind. The given magnitude may be (*a*) A ratio, (*b*) an angle, (*c*) a length; or any known function of these.

In enumerating conditions certain precautions must be taken that spurious conditions be not admitted. Conditions must be consistent, otherwise the problem is impossible; but they must in addition stand the two tests now to be discussed. One of these tests will be seen to relate to the subjects, the other to the predicates of conditions.

§ 4. *In the first place in each condition the element given must have a determinate and stated relation to the figure to be constructed.* In other words the number of conditions is in every case deduced on the supposition that the points, lines, angles or parts of figures mentioned are described in language free from ambiguity.

For example, it is not in general sufficient to say "Given two sides of a quadrilateral"; the sides should be further described as either "adjacent" or "opposite" as the case may be. So in general it is not sufficient to say "Given a diagonal"; the particular diagonal should be indicated. Whether a condition satisfies this requirement

* In crystallography, for example, angular relations are the all-important thing, not linear relations.

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may always be tested thus: Imagine the figure determined, and see if the condition in question be now determined; if so, and only if so, it is to be admitted as in this respect a proper condition.

In the second place the conditions must be independent. It should not be possible to deduce any one condition from the others.

Without actually trying to deduce a suspected condition from the others, the following test is often of easy application: A condition that is independent, and only such a condition, may be varied without in any way changing the others. As an example, apply this test to the triangle having given first the three sides (independent), second the three angles (dependent).

I.

§ 5. The point.

Of a point the position is all that has to be determined, and this is done by two conditions. This number may be deduced from the conditions for the straight line in much the same way as will be done for an infinite line; or, independently, from the fact that a point is determined by two cöordinates parallel to the axes.

It will be observed that the set of conditions for a point being determined, the set for a line can be deduced; and reciprocally. No matter then which of the two we begin with, a good reason can be given for beginning with the other. To those who think the point has an irresistible claim to priority it may be remarked that to determine a point by its cöordinates (a, b) is really to determine it by the intersecting of two lines whose equations are $x = a$, $y = b$.

§ 6. The straight line.

The number of conditions determining the shape of a straight line is zero. For the shape is fixed by the definition.

The magnitude of a straight line is determined by one condition. For the length only is required.

The position of a line of given magnitude is determined by three conditions. These may be the intercepts OA, OB, on the axes, and the distance of one end of the line from the origin; but there are many more sets of three equally suitable.

Thus a finite straight line is completely determined by four conditions. Hence an infinite straight line is determined by two conditions; for the characteristic "infinite" involves two conditions, first the distance of one end from the origin (infinite), and second the length of the line (infinite). Compare this with the $y = mx + c$ of cöordinate geometry, where the determining conditions are m and c.

7. An important principle, to be used later on, has to be stated here. If a figure be already determined in shape, its size will be etermined if one line of the figure be given in length, and then its position if one line be given in position. Hence in any figure,

if	N	= number of conditions for shape,		
then	N + 1	= ,,	,,	shape + size,
and	N + 1 + 3 =	,,	,,	shape + size + position.

8. The triangle.

The shape of a triangle is determined by two conditions.

If there be two triangles ABC and PQR, and if $PQ = k \cdot AB$, $QR = k \cdot BC$ and $RP = k \cdot CA$; then by giving k in succession all possible values we get the series of triangles having what we recognise as similarity of shape to ABC. By eliminating k we get the two independent conditions $PQ : QR = AB : BC$, and $QR : RP = BC : CA$. This and other methods of deducing the number of conditions are of course to be found in the Sixth Book of Euclid. It need hardly be pointed out that among the conditions for similarity of shape a length can never occur.

The shape and size of a triangle are determined by three conditions.

The shape, size, and position of a triangle are determined by six conditions.

These results might be got from § 7 by putting $N = 2$; but they can be got independently in a great variety of ways. For example, the magnitude and position of the side AB of a triangle are given by four conditions (§ 6), and other two conditions, $\angle B$ and side BC, will determine the triangle completely.*

§ 9. The polygon.

The shape of an n*-gon is determined by* 2n – 4 *conditions.*

This result may be got by dividing the n-gon into $n - 2$ triangles by lines drawn from a vertex, and applying the first proposition in § 8.

* It may be of interest to mention that Bonnycastle's *Elements of Geometry*, sixth edition (1818), pp. 416-431, contains a list of one hundred and seventy-one cases of triangles determined by sets of three conditions, and this list might now be indefinitely increased.

It may also be got by noticing that the shape is determined if $n-1$ ratios of consecutive sides be given, and also $n-3$ consecutive angles. And $(n-1)+(n-3)=2n-4$.

The shape and size of an n*-gon are determined by* 2n – 3 *conditions.*

The shape, size, and position of an n*-gon are determined by* 2n *conditions.*

These results may be obtained as follows:

a. From § 7 by putting $N=2n-4$.

b. By dividing the n-gon into $n-2$ triangles and applying the results of § 8.

c. By giving all the sides (n) and all the angles save three ($n-3$), and so determining the n-gon in shape and size.

d. By supposing given the n infinite lines ($2n$ conditions), the intersections of which determine the n-gon completely.

e. By supposing the n vertices given ($2n$ conditions), which also determine the n-gon completely.*

The following problem is discussed in order to show a real and a fictitious set of $2n$ conditions: To construct an n-gon, having given the n middle points of the sides ($2n$ conditions). It is known that this is a determinate problem only if n is an odd number. The explanation of this is got on applying the second test of § 4, when it is found that n being even, the n points are not all independent. To prove this let us take the eight-sided polygon ABCDEFGH, the mid points of AB, BC, &c., being A′B′C′D′E′F′G′H′, and try to deduce H′ from the others. AC, CE, EG being parallel to and double A′B′, C′D′, E′F′, are known in direction and magnitude. Hence GA and therefore G′H′ are known in direction and magnitude; and the point G′ is known; and therefore H′ is known. And this will evidently hold whatever the number of sides be so long as it is even. Hence in putting down the n points, if the last one happen to be put down in the right place, the number of independent conditions is deficient, and the problem is in-

*Throughout this paper the assumption is made that if a set of m conditions determine a figure, any other set of m conditions will determine it. The whole investigation goes to support this assumption; but pending the production of a rigid proof, some considerations in its favour may be offered. Every plane figure is determined by its n angular points, and each of these has two degrees of freedom. Hence a plane figure has $2n$ degrees of freedom. And every condition reduces that number by one, and by one only. Thus for any figure the number of determining conditions is constant.

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determinate (as we shall see in § 19); and if the last point be put down in the wrong place the conditions are inconsistent, and the problem is impossible (§ 3). If now a similar method be taken with a figure of an odd number of sides, a side, and not the distance between two mid points, is determined in magnitude and direction; and the construction of the figure follows at once.

II.

§ 10. The numbers we have found hold for polygons, of which the only thing stated in their definition is the number of sides. When figures are more particularly characterized, as parallelograms, regular figures, &c., a number of the conditions are involved in the definition of the figure; and by that number has the general expression to be diminished.

§ 11. Denoting the sides of a figure by a, b, c..., and the angles by A, B, C..., let us take a few examples. In the case of triangles, isosceles triangles have involved in their definition the condition $a = b$, and there remain two conditions required to determine shape and size; right-angled triangles have $\angle$ A a right angle, again leaving two conditions; while equilateral triangles have $a = b$, $b = c$, and there is left only one condition. In the case of quadrilaterals, determined as to shape and size by five conditions $(2n - 3)$, there are the special cases trapezium $(a \parallel c)$, symmetric trapezium $(a \parallel c,\ b = d)$, parallelogram $(a \parallel c,\ b \parallel d)$, rectangle $(a \parallel c,\ b \parallel d$, A a right angle), rhombus $(a = b,\ b = c,\ c = d)$, and square $(a = b,\ b = c,\ c = d$, A a right angle), which, after the numbers of defining conditions are subtracted, are found to be determined respectively by four, three, three, two, two and one conditions. Examples in verification of these results are to be found everywhere in elementary geometry.

A regular n*-gon is determined in shape and size by one condition.* For the equality of the sides involves $n - 1$ conditions, and of the angles $n - 3$, that is $2n - 4$ in all. If this number be deducted from the general expression $2n - 3$, the remainder is one as was asserted. Another way of stating this is to say that the shape is determined by the definition, and one condition will determine the size.

§ 12. So far, mention has been made only of rectilineal figures; but a very curious problem is presented in this subject by the circle. Looked upon as the limiting case of the regular polygon, it might be supposed to be determined in size, like a regular polygon, by one condition. And it is sometimes so determined; for to give the radius determines the circle. But we find in other cases two conditions given, as a chord and the angle it subtends at the centre, or a chord and its distance from the centre; and even three conditions, as two chords meeting in the circumference and the angle between them.

At first sight this state of matters seems to imply a breach of continuity between a regular polygon and its limit; but there is no such breach, as will presently be seen.

§ 13. *The apparent anomalies in the circle arise from the failure to comply with the first requirement of* § 4; a failure that can easily be shown to produce the same results even in regular rectilineal polygons.

§ 14. We shall take as an example to work with the regular dodecagon (*fig.* 10); but the reasoning will apply whatever be the number of sides. This polygon is made up of twelve equal isosceles triangles, and if one of these be known the polygon is determined. To attain this end one condition will suffice; since to determine an isosceles triangle two conditions are necessary, and here one, the vertical angle, is known already. To give then anything, as side, base, perpendicular from vertex on opposite side, having a definite and stated relation to one of the triangles such as AOB will determine the polygon.

§ 15. But suppose now the datum * were, "Given the line joining two vertices of the polygon." This line may be AB, AC, AD, AE, AF or AG—all other cases may be looked upon as repetitions of these—and each of these gives a different polygon. This in fact amounts to giving the base of an isosceles triangle AOB or AOC, &c., whose vertical angle is not definitely known. The vertical angle is some one of those included in $p \times \frac{1}{3} rt \angle$ where p may be 1, 2…6. If then a line be given and merely said to be a diagonal, there is always a number of solutions. This number increases with the number of sides of the polygon, and in the case of the circle, where the diagonal has become the chord, it is infinite. The solution becomes definite if the vertices between which the diagonal is drawn be stated, or, what is the same thing, if the angle the diagonal subtends at the centre of the poly-

* "A datum is any quantity, condition, or other mathematical premiss which is given in a particular problem."—De Morgan, in *The Penny Cyclopædia*, vol. viii, p. 313.

gon be given. In an n-gon this angle is restricted in magnitude to one of the series $[1, 2, 3, \ldots, n-1] 4rt \angle s \div n$; a series that approximates as n increases to the case of the circle where any angle up to four right angles may be taken.

§ 16. To take yet another case, let it be required to construct a dodecagon from the two data, "Given two diagonals drawn from the same vertex." These two lines belong to some such quadrilateral as ALOD (*fig.* 10), which we are required to determine without five, the proper number of conditions. We know AL, AD, and that OL = OA, OA = OD; but the fifth condition is incomplete, for we know only that $\angle$ LOD is one of some half dozen possible angles. This angle has still to be got either directly, or by the naming of the vertices joined, or by the giving of the angle between the diagonals. Thus we may have as data for a regular polygon, two diagonals drawn from a common vertex, and the angle between them. For every regular polygon these three conditions must be taken from a finite number of possible sets of lines with their included angle. But the greater the number of sides of the polygon the greater is the number of these sets, till in the case of the circle the number of sets is infinite. In other words, any two lines may be given inclined at any angle, and a circle can be described through their extremities; or to put it in yet another way, a circle can be described through any three points.

In much the same way, the sides produced of a regular polygon being considered to correspond to the tangents to a circle, the case of the circle determined by its touching three given straight lines can be deduced and explained.

Thus the apparently redundant conditions are needed to specify completely the element of the one condition really given.

§ 17. One more comparison between a polygon and a circle may be worth mentioning. A regular polygon is determined in size and position if the coördinates of its centre, the length of a line from the centre to a vertex, and the angle this line makes with one of the axes be given—in all, four conditions. The last condition becomes, if we may so speak, of less importance as the sides of the polygon become more numerous, till in the circle it is dispensed with altogether.

III.

§ 18. *Redundancy of conditions means impossibility of construction.* For a figure being determined by a certain number of conditions, to give more than that number is to impose conditions we are in general unable to carry out.

There is sometimes an apparent redundancy which does not lead to impossibility of construction. When this happens, all the conditions being properly specified, it is a sure indication that the conditions are not all independent. In fact, if the determining number of conditions for the particular case be taken, then the remaining conditions will be found to be deducible from these. Examples of this may be got in Euclid's treatment of similar triangles, where sometimes (VI. 4, 5) it is made to appear as if three conditions were necessary to determine the shape of a triangle.

§ 19. *Deficiency of conditions results in an indeterminate figure.* In other words, an infinite number of figures can be got to satisfy these conditions. It is from this source we derive Locus problems. If a point or a line of a figure be given in position, and a number of conditions be given less, by one usually, than the determining number, an infinite number of figures will be got varying continuously. Then if a particular point be followed up in all these figures, the curve thus got is the locus of the point.

To speak of 'the curve' is hardly correct, for the locus often consists of two or more curves. We may bring the paper to a conclusion by pointing out how this plurality of curves arises.

§ 20. At the outset the number of conditions determining a line (on which the subsequent results were founded) was deduced on the supposition that a line is symmetrical. But the moment a line is made the side of a figure, lateral symmetry ceases ; for now a line has a side towards and a side away from the figure—has an inside and an outside. Consequently when a figure is fixed in position by the fixing of a line the figure will in general have two positions according to the side of the line that is placed towards one of the axes. A similar remark may be made about the end symmetry of a line, and a consequent pair of possible positions.

The four cases occur when we construct on the same base four congruent triangles. And now it will be seen why a locus may consist of more than one curve. For fix the base and the area, and the locus of the vertex is a pair of straight lines. Fix the base and a base angle, and the locus of the vertex is four straight lines. Other examples will suggest themselves.