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SCHWARZ LEMMA FOR REAL HARMONIC FUNCTIONS ONTO SURFACES WITH NON-NEGATIVE GAUSSIAN CURVATURE

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Abstract Assume that f is a real ρ-harmonic function of the unit disk $\mathbb{D}$ onto the interval $(-1,1)$, where $\rho(u,v)=R(u)$ is a metric defined in the infinite strip $(-1,1)\times\mathbb{R}$. Then we prove that $|\nabla f(z)|(1-|z|^2)\le \frac{4}{\pi}(1-f(z)^2)$ for all $z\in\mathbb{D}$, provided that ρ has a non-negative Gaussian curvature. This extends several results in the field and answers to a conjecture proposed by the first author in 2014. Such an inequality is not true for negatively curved metrics.

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1. Introduction

1.1. Schwarz lemma

The standard Schwarz lemma states that if f is a holomorphic mapping of the unit disk $\mathbb{D}$ into itself such that $f(0)=0$, then $|f(z)|\le|z|$.

Its counterpart for harmonic mappings states the following ([8, Section 4.6]). Let f be a complex-valued function harmonic in the unit disk $\mathbb{D}$ into itself, with $f(0)=0$. Then

$$|f(z)|\le\frac{4}{\pi}\tan^{-1}|z|,$$

and this inequality is sharp for each point $z\in\mathbb{D}$. Furthermore, the bound is sharp everywhere (but is attained only at the origin) for univalent harmonic mappings f of $\mathbb{D}$ onto itself with $f(0)=0$.

The standard Schwarz lemma (also called Schwarz–Pick lemma) for holomorphic mappings states that every holomorphic mapping f of the unit disk onto itself satisfies the inequality

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}. \tag{1.1}$$

A very important version of the Schwarz lemma for holomorphic functions has been obtained by Ahlfors [1], who proved the following: Let f be a holomorphic map of the unit disk $\mathbb{D}$ into a Riemann surface S endowed with a Riemannian metric ρ with Gaussian curvature $\mathcal{K} \leq -1$. Then the hyperbolic length of any curve in $\mathbb{D}$ is no less than the length of its image. Equivalently,

$$d_\rho(f(z), f(w)) \leq d_h(z, w) \quad \text{for all } z, w \in \mathbb{D}$$

or $\|df(z)\| \leq 1$ everywhere, where the norms are taken with respect to the hyperbolic metric on $\mathbb{D}$ and the given metric on the image. For some other generalizations of the Schwarz lemma, we refer to the papers of Yau [23, 24], Osserman [20], Yang and Zheng [22], Royden [21], Ni [18, 19] and Broder [4]. The recent survey by Broder [3] also provides references to the Schwarz lemma in other contexts. Most of mentioned papers deal with Schwarz lemma for holomorphic functions, and the target space has a non-positive curvature.

We refer as well to some generalizations of Schwarz lemma for harmonic functions in the papers [5, 7, 9, 12–17].

In particular, the following result was proved in [13]. If $f\colon \mathbb{D} \to (-1, 1)$ is a real harmonic mapping, then it satisfies the inequality

$$|\nabla f(z)| \leq \frac{4}{\pi}\frac{1-f(z)^2}{1-|z|^2} \quad \text{for all } z \in \mathbb{D}.$$

Later, by using the same approach as that in [13], Chen [6] improved the latter inequality by showing that

$$|\nabla f(z)| \leq \frac{\cos(\frac{\pi}{2}f(z))}{1-|z|^2} \quad \text{for all } z \in \mathbb{D}.$$

In order to state our main result, let us introduce the class of ϱ-harmonic mappings.

1.2. ϱ-Harmonic mappings

Assume that Ω is a connected open set in the complex plane. Assume that ϱ is a positive continuous function in Ω. Then (by abusing the notation), it defines a conformal metric $\varrho(z) = \varrho(z)\,\mathrm{d}z \otimes \overline{\mathrm{d}z}$ in Ω. Then ϱ defines a Riemann surface (Ω, ϱ).

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Moreover, assume that ϱ is a smooth function in Ω with Gaussian curvature $\mathcal{K}_\varrho$, where

$$\mathcal{K}_\varrho(z) := -\frac{\Delta \log \varrho(z)}{\varrho^2(z)}. \tag{1.2}$$

Here Δ denote the usual Laplacian:

$$\Delta g(z) := g_{xx} + g_{yy}, \quad z = x + \mathrm{i}y.$$

We assume $\sup_{z\in\Omega} |\mathcal{K}_\varrho(z)| < \infty$ and ϱ has a finite area defined by

$$\mathcal{A}(\varrho) = \int_\Omega \varrho^2(u + \mathrm{i}v)\, \mathrm{d}u\, \mathrm{d}v.$$

Let $f : (D, \delta) \to (\Omega, \varrho)$ be a $\mathcal{C}^2$ map of two Riemann surfaces, where δ is the (pullback to D via the inclusion of the) Euclidean metric. We say that f is harmonic if

$$f_{z\bar{z}} + ((\log \varrho^2)_w \circ f) \cdot f_z\, f_{\bar{z}} = 0, \tag{1.3}$$

where z and w are holomorphic coordinates on D and Ω, respectively. Recall that a Euclidean harmonic function f is a solution of the Laplace equation $\Delta f = 0$, and in this case $\varrho \equiv 1$. Also, f satisfies Equation (1.3) if and only if its Hopf differential

$$\mathrm{Hopf}(f) := (\varrho^2 \circ f) f_z \overline{f_{\bar{z}}} \tag{1.4}$$

is a holomorphic quadratic differential on D.

Assume $f : \mathbb{D} \to (-1, 1)$ is a real ρ-harmonic function. Vuorinen and the first named author in [13] introduced the quantity

$$S(f) := |\nabla f(z)| \frac{1 - |z|^2}{1 - |f(z)|^2} \tag{1.5}$$

and showed that $S(f) \le \frac{4}{\pi}$ for Euclidean harmonic functions. In order to extend the results in [13], the first named author in [11] defined the class of admissible metrics. We say that a metric ϱ is admissible if $\varrho(z) = \varphi(|z|)$, where $\varphi : \mathbb{D} \to \mathbb{C}\setminus(-\infty, 0]$ is an analytic function defined in the unit disk satisfying the following properties:

(1) $\varphi(|z|) \le |\varphi(z)|$ and φ is nonincreasing in $[0, 1]$,
(2) $\varphi(-1, 1) \subset \mathbb{R}$ and $\int_0^1 (\sqrt{\varphi(x)} - \sqrt{\varphi(-x)})\, \mathrm{d}x = 0$.

Then inequality (1.5) was extended by the first named author [11] to ϱ-harmonic functions, where ϱ is an *admissible metric*. The following question was posed in [11].

Problem 1.1. *Let $f : \mathbb{D} \to (-1, 1)$ be a real ϱ-harmonic function. Suppose ϱ has non-negative Gaussian curvature. Does the bound $S(f) \le \frac{4}{\pi}$ hold?*

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Remark 1.2. The assumption that the target domain has a non-negative Gaussian curvature is crucial, and this is shown in Example 2.3. This problem is somehow complementary to the already mentioned Scharz lemma-type result of Ahlfors for holomorphic functions of the unit disk onto a surface with a non-positive Gaussian curvature.

We will see in Example 3.1 that the answer to the question posed in Problem 1.1 is no. However, it will be shown in this paper that a real ϱ-harmonic function is also harmonic with respect to the *modified* metric $\rho(u,v)=\varrho(u,0)$, and the positiveness of the Gaussian curvature of ρ will be crucial.

Indeed, we shall prove the following theorem, which is the main content of this paper.

Theorem 1.3. *Assume that f is a real ϱ-harmonic function of the unit disk onto the interval $(-1,1)$. If $\rho(u,v)=\varrho(u,0)$, then f is ρ-harmonic. Assume further that the Gaussian curvature of ρ is non-negative. Then we have the sharp inequality*

$$S(f)=|\nabla f(z)|\frac{1-|z|^2}{1-|f(z)|^2}\leqslant\frac{4}{\pi}\ \text{for all } z\in\mathbb{D}. \tag{1.6}$$

Corollary 1.4. *Assume that Ω is a hyperbolic domain in the complex plane, and let $\lambda=\lambda_\Omega$ be its hyperbolic metric of constant Gaussian curvature equal to -4. Let $f:\Omega\to(-1,1)$ be a ρ-harmonic function, where $\rho(u,v)=R(u)$ has a non-negative Gaussian curvature. Then we have the following sharp inequality:*

$$d_h(f(z),f(w))\leq\frac{4}{\pi}d_\lambda(z,w)\quad \text{for all } z,w\in\Omega. \tag{1.7}$$

Here d_h is the hyperbolic metric in the unit disk defined by

$$d_h(z,w)=\tanh^{-1}\frac{|z-w|}{|1-z\bar{w}|}.$$

The proof of the first part of Theorem 1.3 is an easy matter, and it is presented in § 1.3, while the second part is the content of Theorem 2.1. Corollary 1.4 is a straightforward application of the definition of the hyperbolic metric. We only need to notice the following. If $g\colon\mathbb{D}\to\Omega$ is a covering map, then $h(z)=f(g(z))$ is a real ρ-harmonic mapping of the unit disk onto $(-1,1)$. Moreover,

$$\lambda_\Omega(g(z))=\lambda_{\mathbb{D}}(z)|g'(z)|.$$

So, for $w=g(z)$, in view of Equation (1.6), we have

$$\frac{|\nabla f(w)|}{\lambda_\Omega(w)}=|\nabla h(z)|(1-|z|^2)\leq\frac{4}{\pi}(1-h(z)^2)=\frac{4}{\pi}(1-|f(w)|^2).$$

Thus,

$$\frac{|\nabla f(w)|}{(1-|f(w)|^2)}\leq\frac{4}{\pi}\lambda_\Omega(w).$$

By integrating the previous inequality throughout the family of paths joining z_1 and z_2 (as at the end of the proof of Theorem 2.1), we get

$$d_h(f(z_1), f(z_2)) \leq \frac{4}{\pi} d_\lambda(z_1, z_2) \quad \text{for all } z_1, z_2 \in \Omega. \tag{1.8}$$

1.3. Real ϱ-harmonic mappings and our setting (real R-harmonic mappings)

If f is real, then Equation (1.3) can be re-stated as follows:

$$\Delta f + \frac{\varrho_u(f(z), 0) - \mathrm{i}\varrho_v(f(z), 0)}{\varrho(f(z), 0)}(f_x^2 + f_y^2) = 0. \tag{1.9}$$

In particular, we see that $\varrho_v(u, 0) \equiv 0$ or f is a constant function.

Let $R(u) = \varrho(u, 0)$. If f is a real harmonic function of the unit disk onto the interval (α, β), then

$$\Delta f + \frac{R'(f)}{R(f)}(|\nabla f|^2) = 0, \tag{1.10}$$

where R is a metric defined in the interval (α, β). Observe that R can be extended to the infinite strip-domain $S(\alpha, \beta) := \{x + \mathrm{i}y, x \in (\alpha, \beta), y \in \mathbb{R}\}$ by setting $\rho(u, v) = R(u) = \varrho(u, 0)$.

Moreover, we have this important fact: *f is real ϱ-harmonic if and only if f is real ρ-harmonic.* This is why we will consider the Gaussian curvature of ρ instead of ϱ. We will refer to such real harmonic mappings as *real R-harmonic mappings.*

The Gaussian curvature of ρ is given by

$$\mathcal{K}_\rho(u, v) = -\frac{1}{R(u)^2}\left(\frac{R'(u)}{R(u)}\right)'. \tag{1.11}$$

In fact, Equation (1.10) is equivalent to the Laplace equation

$$\Delta g = 0,$$

where

$$g := \frac{H(f)}{H(1)} : \mathbb{D} \to (-1, 1), \tag{1.12}$$

while

$$H(u) := -\frac{1}{2}\left(\int_0^1 R(u)\,\mathrm{d}u + \int_0^{-1} R(u)\,\mathrm{d}u\right) + \int_0^u R(u)\,\mathrm{d}u, \tag{1.13}$$

and

$$H(1) = \frac{1}{2}\int_{-1}^1 R(u)\,\mathrm{d}u < \infty,$$

provided that R belongs to the Lebesgue space $\mathcal{L}^1(-1,1)$. We will, however, prove that this condition $R \in \mathcal{L}^1(-1,1)$ is a priori satisfied for metrics of non-negative Gaussian curvature, with which we deal in our main result.

The following theorem contains some results for metrics, which are not necessarily positively curved.

Theorem 1.5. *Assume that f is a real R-harmonic mapping of the unit disk into the interval $(-1,1)$, and assume that R is an increasing function in $(-1,0)$ and decreasing in $(0,1)$. Then we have the following sharp inequality*

$$|\nabla f(z)| \le 2\frac{1-|f(z)|}{1-|z|^2} \quad \text{for all } z \in \mathbb{D}. \tag{1.14}$$

If $f(0)=0$ and $\int_{-1}^{0} R(t)\,\mathrm{d}t = \int_0^1 R(t)\,\mathrm{d}t$, then we have the sharp inequality

$$|f(z)| \le \frac{4}{\pi}\tan^{-1}|z|, \quad z \in \mathbb{D}. \tag{1.15}$$

The proof of Theorem 1.5 is presented in § 2.1. We also have the following straightforward corollary of Theorem 1.5.

Corollary 1.6. *If R is even in $(-1,1)$ and decreasing in $[0,1)$, then*

$$|\nabla f\,(z)| \leqslant 2\frac{1-f\,(z)^2}{1-|z|^2} \text{ for all } z \in \mathbb{D},$$

so that

$$d_h\,(f\,(z)\,, f\,(w)) \leqslant 2d_h\,(z,w) \text{ for all } z,w \in \mathbb{D}.$$

Further, if $f(0)=0$, then

$$|f\,(z)| \leqslant \frac{4}{\pi}\tan^{-1}|z| \text{ for all } z \in \mathbb{D}.$$

2. Proof of main results

Theorem 2.1 is the main part of Theorem 1.3, and it solves Problem 1.1 for the modified metrics.

Theorem 2.1. *Assume that R is a metric of non-negative Gaussian curvature in $(-1,1)$. If f is an R-harmonic function of the unit disk into $(-1,1)$, then it satisfies the sharp inequalities*

$$|\nabla f(z)| \le \frac{4}{\pi}\frac{1-f(z)^2}{1-|z|^2} \tag{2.1}$$

and

$$d_h(f(z), f(w)) \le \frac{4}{\pi} d_h(z,w) \tag{2.2}$$

for $z, w \in \mathbb{D}$, where d_h is the hyperbolic metric.

To prove Theorem 2.1, we need the following lemma, which is of interest in its own right.

Lemma 2.2. *Assume that f is an increasing $\mathcal{C}^1$ diffeomorphism of $[-1,1]$ onto itself such that f' is log-concave. Then for all $x \in [-1,1]$, we have the inequality*

$$1 - f(x)^2 \le f'(x)(1 - x^2). \tag{2.3}$$

Proof of Lemma 2.2. Let $h := \log(f')$. Take any $x \in (-1,1)$. Since h is concave, for some real k and all $t \in (-1,1)$, we have

$$h(t) \le h_x(t) := h(x) + k(t - x); \tag{2.4}$$

by approximation, without loss of generality $k \neq 0$. Also, the condition that f is an increasing diffeomorphism of $[-1,1]$ onto itself implies that $f(-1) = -1$ and $f(1) = 1$. So,

$$\begin{aligned} f(x) &= 1 - \int_x^1 f'(t)\,\mathrm{d}t \\ &= 1 - \int_x^1 \mathrm{e}^{h(t)}\,\mathrm{d}t \\ &\geqslant 1 - \int_x^1 \mathrm{e}^{h_x(t)}\,\mathrm{d}t \\ &= g^+(U,x) := 1 - U\frac{1 - \mathrm{e}^{k(1-x)}}{-k}, \end{aligned}$$

where

$$U := \mathrm{e}^{h(x)} = f'(x) > 0. \tag{2.5}$$

Similarly,

$$\begin{aligned} f(x) &= -1 + \int_{-1}^x \mathrm{e}^{h(t)}\,\mathrm{d}t \\ &\le -1 + \int_{-1}^x \mathrm{e}^{h_x(t)}\,\mathrm{d}t \\ &= -g^-(U,x) := -1 + U\frac{\mathrm{e}^{-k(1+x)} - 1}{-k}. \end{aligned}$$

So,

$$f(x)^2 \geqslant g_2(U,k,x) := \max[g^+(U,x)_+^2, g^-(U,x)_+^2], \tag{2.6}$$

where $z_+ := \max(0,z)$.

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We also have

$$2 = \int_{-1}^{1} f' = \int_{-1}^{1} \mathrm{e}^{h} \leq \int_{-1}^{1} \mathrm{e}^{h_x(t)}\,\mathrm{d}t = U\,\mathrm{e}^{-kx}\,\frac{2\,\sinh k}{k},$$

so that

$$U \geqslant U_{k,x} := \mathrm{e}^{kx}\frac{k}{\sinh k}. \tag{2.7}$$

Thus, it is enough to show that

$$\rho(U,k,x) := \frac{1 - g_2(U,k,x)}{U(1-x^2)} \leq 1 \tag{2.8}$$

for $U \geqslant U_{k,x}$. Note that $\rho(U,k,x)(1-x^2)$ is a continuous piecewise-rational function of U such that $\mathbb{R}$ can be partitioned into several intervals such that on each of the intervals of the partition, the expression $\rho(U,k,x)(1-x^2)$ coincides with one of the following three expressions:

$$\rho_+(U) := \frac{1 - g^+(U,x)^2}{U}, \quad \rho_-(U) := \frac{1 - g^-(U,x)^2}{U}, \quad \rho_0(U) := \frac{1-0}{U}. \tag{2.9}$$

We have

$$\rho_+'(U) = -\frac{\left(\mathrm{e}^{k-kx} - 1\right)^2}{k^2} \leq 0, \quad \rho_-'(U) = -\frac{e^{-2k(x+1)}\left(\mathrm{e}^{k(x+1)} - 1\right)^2}{k^2} \leq 0 \tag{2.10}$$

and $\rho_0'(U) < 0$.

So, $\rho(U,k,x)$ is nonincreasing in U. It remains to show that

$$r(k,x) := \rho(U_{k,x},k,x) \leq 1. \tag{2.11}$$

Note that $g^+(U_{k,x},x) = -g^-(U_{k,x},x) = \left(\mathrm{e}^{kx} - \mathrm{e}^{k}\right)\operatorname{csch} k + 1$. So,

$$r(k,x) = \frac{1 - g^+(U_{k,x},x)^2}{U_{k,x}(1-x^2)} = \frac{2(\cosh k - \cosh kx)\operatorname{csch} k}{k\,(1-x^2)}. \tag{2.12}$$

Inequality (2.11) can be rewritten as

$$\operatorname{dif}(x) := (1-x^2)r(k,x) - (1-x^2) \leq 0 \tag{2.13}$$

for real $k > 0$ and $x \in [0,1]$ because $\operatorname{dif}(-k,x) = \operatorname{dif}(k,x) = \operatorname{dif}(k,-x)$.

We have $\operatorname{dif}'''(x) = -2k^2 \operatorname{csch} k \sinh kx \leq 0$. So, dif'' is non-increasing. Hence, there is some $c \in [0,1]$ such that dif is convex on $[0,c]$ and concave on $[c,1]$. Also, $\operatorname{dif}'(0) = \operatorname{dif}'(1) = \operatorname{dif}(1) = 0$. Thus, (2.13) follows. □

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Proof of Theorem 2.1. Let us show that $R \in \mathcal{L}^1(-1,1)$. In view of Equation (1.11), $\log R$ is concave. Therefore,

$$\log R(t) \le \log R(0) + \frac{R'(0)}{R(0)} t \quad \text{for all } t \in (-1,1),$$

and thus

$$R(t) \le R(0) \mathrm{e}^{\frac{R'(0)}{R(0)} t}.$$

Hence,

$$\int_{-1}^{1} R(t) \,\mathrm{d}t < \infty.$$

Now we put

$$r := H(1) = \frac{1}{2} \int_{-1}^{1} R(u) \,\mathrm{d}u.$$

Recall the Euclidean harmonic function g defined in Equation (1.12). It comes down to estimating the gradient of the derivative of the function g, which is equal to

$$|\nabla g| = R(f) |\nabla f| / H(1).$$

For the real Euclidean harmonic function $g : \mathbb{D} \to (-1,1)$, we have [6, 13]

$$|\nabla g(z)| = \frac{R(f(z)) |\nabla f(z)|}{r} \le \frac{4}{\pi} \frac{\cos \frac{\pi}{2} g(z)}{1 - |z|^2}, \tag{2.14}$$

where

$$g(z) := \frac{1}{r} \left(H(0) + \int_0^{f(z)} R(u) \,\mathrm{d}u \right).$$

Let

$$\mathcal{R} := \frac{4}{\pi} \frac{\cos \left(\frac{\pi}{2r} \left(H(0) + \int_0^{f(z)} R(u) \,\mathrm{d}u \right) \right)}{1 - |z|^2}. \tag{2.15}$$

Note that

$$\cos \frac{\pi}{2} b \le 1 - b^2$$

for $b \in [0,1]$.

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Let

$$\psi(v) := \frac{\pi}{2r}\left(H(0) + \int_0^v R(u)\,\mathrm{d}u\right)$$

and apply Lemma 2.2. We get

$$\mathcal{R} \le \frac{4}{\pi}\,\frac{1-\psi(v)^2}{\psi'(v)} \le \frac{4}{\pi}\,(1-v^2). \tag{2.16}$$

Combining Equations (2.15), (2.14) and (2.16), we obtain Equation (2.1). Concerning Equation (2.2), notice that the proof of [13, Theorem 1.2] can be applied in this case because the ρ-harmonicity is invariant under precomposition by Möbius transformations. □

The following example shows that one cannot omit the condition of positive Gaussian curvature. In fact, we cannot prove a weaker estimate with a constant larger than $4/\pi$.

Example 2.3. Assume that $\varrho(w) := \frac{1}{1-|w|^2}$ and let $\rho(w) := \varrho(u,0) = \frac{1}{1-u^2}$. Then $f : \mathbb{D} \to (-1,1)$ is ρ-harmonic (and ϱ-harmonic) if and only if

$$f(z) = \tanh g(z)$$

for a Euclidean harmonic mapping g of the unit disk in the real line $\mathbb{R}$. In particular, the functions

$$f(z) = \tanh(nx), \qquad z = x + \mathrm{i}y, \quad n \in \mathbb{N},$$

are ρ-harmonic. Then $|\nabla f(z)| = n\,\mathrm{sech}^{\,2}(nx)$, and so,

$$\frac{|\nabla f(z)|}{1-|f(z)|^2}(1-|z|^2)\bigg|_{z=0} = n,$$

so that in Theorem 2.1, we cannot omit the condition of the positiveness of Gaussian curvature, nor can we even prove a weaker statement with a larger constant factor instead of $4/\pi$. Observe that in this case

$$\mathcal{K}_\rho(z) = -2(1+x^2) < 0.$$

Of course, the curvature of the hyperbolic (Poincarè) metric is $\mathcal{K}_\varrho(z) = -4$, and it is not equal to the curvature of ρ, even though both curvatures are negative.

In the following example, a result for a metric R of zero Gaussian curvature is given.

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Example 2.4. Assume that the Gaussian curvature of R is zero. Then $R(x) = \mathrm{e}^{cx}$. Moreover, by Equation (2.14),

$$|\nabla f(z)| \le A := \frac{4\mathrm{e}^{-cf(z)} \sin\left[\frac{\pi}{2\sinh(c)}\left(\mathrm{e}^{c} - \mathrm{e}^{cf(z)}\right)\right]\sinh(c)}{c\pi\left(1-|z|^2\right)}.$$

Further, by the proof of Theorem 2.1,

$$|\nabla f(z)| \le A \le \frac{4}{\pi}\frac{1-f(z)^2}{1-|z|^2}.$$

2.1. Proof of Theorem 1.5

We need the following lemma.

Lemma 2.5. *If $R\colon (-1,1) \to (0,+\infty)$ is positive, increasing in $(-1,0)$ and decreasing in $(0,1)$, if $v \in (-1,1)$, and if*

$$r = \frac{1}{2}\int_{-1}^{1} R(u)\,\mathrm{d}u,$$

then we have the sharp inequality

$$\sin\left[\frac{\pi\int_v^1 R(u)\,\mathrm{d}u}{2r}\right] \le \frac{\pi}{2r}(1-|v|)\,R(v), \tag{2.17}$$

and in particular

$$\sin\left[\frac{\pi\int_v^1 R(u)\,\mathrm{d}u}{2r}\right] \le \frac{\pi}{2r}\left(1-v^2\right)R(v).$$

The constant $\pi/2$ is sharp even if we restrict the consideration to $\mathcal{C}^2$ diffeomorphisms $R\colon (-1,1) \to (0,\infty)$.

Proof of Lemma 2.5. The proof of inequality (2.17) is easy. We use here the fact that R is decreasing in $[0,1)$ and the elementary inequality $\sin x \le x$ for $x \in [0,\pi/2]$. Then for $v \in [0,1]$, we have

$$\begin{aligned}\sin\left[\frac{\pi\int_v^1 R(u)\,\mathrm{d}u}{2r}\right] &\le \frac{\pi\int_v^1 R(u)\,\mathrm{d}u}{2r}\\ &\le \pi\frac{(1-v)\,R(v)}{2r}.\end{aligned}$$

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If $v<0$, then we use the fact that R is increasing in $(-1,0)$. We come to the desired inequality as follows:

$$\sin\left(\frac{\pi}{2r}\int_v^1 R(u)\,\mathrm{d}u\right)=\sin\left(\frac{\pi}{2r}\int_{-1}^v R(u)\,\mathrm{d}u\right)\le\pi\frac{(1+v)\,R(v)}{2r}.$$

To prove the sharpness part, observe that inequality (2.17) is equivalent to

$$\cos\phi(v)\le(1-v^2)\phi'(v), \tag{2.18}$$

where

$$\phi(v)=\frac{\pi}{2}-\frac{\pi\int_v^1 R(u)\,\mathrm{d}u}{2\int_0^1 R(u)\,\mathrm{d}u}=\frac{\pi\int_0^v R(u)\,\mathrm{d}u}{2\int_0^1 R(u)\,\mathrm{d}u}.$$

For $s, as^2\in(0,1)$, we define the concave diffeomorphism $\psi\colon[0,1]\to[0,1]$ by the formula

$$\psi(x):=\begin{cases}\left(1+2as-as^2\right)x-ax^2 & \text{if } x>0\wedge x<s,\\ 1+\left(1-as^2\right)(-1+x) & \text{if } x\ge s\wedge x\le 1.\end{cases} \tag{2.19}$$

Now we define $\phi(x)=\frac{\pi}{2}\psi(x)$.

Then for $u=as^2$, we have

$$\frac{\cos\phi(s)}{(1-s^2)\phi'(s)}=\frac{2\,\sin\left[\frac{1}{2}\pi(1-s)(1-u)\right]}{\pi\,(1-s^2)\,(1-u)}.$$

The supremum of the latter expression is equal to 1. It is 'attained in the limit', for instance, if $s=1/n\to 0$ and $u=(n-1)^2/n^2=as^2\to 1$, with $n\to\infty$.

To prove the last statement, extend ψ in $[-1,1]$ by $\psi(x)=-\psi(-x)$ and define $R(x)=\psi'(x)$, for $x\in[-1,1]$. Then R is not smooth, but it is continuous on $[-1,1]$.

We introduce appropriate mollifiers: Fix a smooth even function $\sigma:\mathbb{R}\to[0,1]$, which is compactly supported in the interval $(-1,1)$ and satisfies $\int_{\mathbb{R}}\sigma=1$. For $\varepsilon>0$, consider the mollifier

$$\sigma_\varepsilon(t):=\frac{1}{\varepsilon}\,\sigma\left(\frac{t}{\varepsilon}\right). \tag{2.20}$$

It is compactly supported in the interval $(-\varepsilon,\varepsilon)$ and satisfies $\int_{\mathbb{R}}\sigma_\varepsilon=1$. For $\varepsilon>0$, define

$$\varphi_\varepsilon(x):=\int_{\mathbb{R}}\psi(y)\frac{1}{\varepsilon}\sigma\left(\frac{x-y}{\varepsilon}\right)\,\mathrm{d}y=\int_{\mathbb{R}}\psi(x-\varepsilon z)\sigma(z)\,\mathrm{d}z.$$

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Because σ is even, we have

$$\begin{aligned}\varphi_\varepsilon(-x) &= \int_{\mathbb{R}} \psi(-x-\varepsilon z)\sigma(z)\,\mathrm{d}z \\ &= \int_{\mathbb{R}} \psi(x+\varepsilon z)\sigma(z)\,\mathrm{d}z \\ &= -\int_{\mathbb{R}} \psi(x-\varepsilon z)\sigma(z)\,\mathrm{d}z \\ &= -\varphi_\varepsilon(x),\end{aligned}$$

and

$$\varphi_\varepsilon'(x) = \int_{\mathbb{R}} \psi'(x-\varepsilon z)\sigma(z)\,\mathrm{d}z.$$

So φ_ε is an increasing and odd function. Further, we define $\psi_\varepsilon(x) := \frac{1}{\varphi_\varepsilon(1)}\varphi_\varepsilon(x)$. Then $\psi_\varepsilon(x) : [-1,1] \to [-1,1]$ is a $\mathcal{C}^\infty$ increasing odd diffeomorphism. Then $\psi_\varepsilon(x)$ converges uniformly to ψ and $\psi_\varepsilon'(x)$ converges uniformly to $\psi'(x)$ as $\varepsilon \to 0$. Thus, the function $R_\varepsilon(x) = \psi_\varepsilon'(x)$ is an even function, increasing in $[-1,0]$ and decreasing in $[0,1]$ that converges uniformly to R.

This implies that the constant $\pi/2$ is sharp even if we restrict the consideration to $\mathcal{C}^\infty$ diffeomorphisms. □

Proof of Theorem 1.5. Since R is positive, increasing in $(-1,0)$ and decreasing in $(0,1)$, it is clear that $R \in \mathcal{L}(-1,1)$. From Equation (2.14), we have

$$|\nabla g(z)| = \frac{R(f(z))|\nabla f(z)|}{r} \le \frac{4}{\pi}\frac{\cos\frac{\pi}{2}g(z)}{1-|z|^2}, \tag{2.21}$$

where

$$g(z) := \frac{H(0)}{2r} + \frac{1}{r}\left(\int_0^{f(z)} R(u)\,\mathrm{d}u\right),$$

with $H(0)$ defined in Equation (1.13).

Further,

$$\cos\frac{\pi}{2}g(z) = \sin\left[\frac{\pi\int_v^1 R(u)\,\mathrm{d}u}{2r}\right].$$

In view Lemma 2.5, the inequality (1.14) is proved. To prove Equation (1.15), in view of the assumption, we observe first that $H(0) = 0$. Since the function $\psi(u) = \int_0^u R(t)\frac{\mathrm{d}t}{r}$ is concave on $[0,1]$ with $\psi(0) = \psi(1) - 1 = 0$, it satisfies the inequality $\psi(u) \geqslant u$. Therefore,

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$$|f(z)| \leq \frac{1}{r}\left|\int_0^{f(z)} R(u)\,\mathrm{d}u\right| = |g(z)|.$$

Now we use the Schwarz lemma for Euclidean harmonic functions ([8, 10]), which implies that

$$|g(z)| \leq \frac{4}{\pi}\tan^{-1}|z|.$$

Inequality (1.14) is sharp because of Lemma 2.5. Inequality (1.15) is sharp, since it coincides with the corresponding inequality [2, p. 124] for Euclidean harmonic mappings (planar case), where the sharpness part is established. Observe that, if $R \equiv 1$, then R defines the Euclidean metric and satisfies the conditions of our theorem. This finishes the proof of the theorem. □

3. Concluding remarks

The answer to the general question posed in Problem 1.1 is negative. In the following example, it is shown that for metrics of zero Gaussian curvature, the quantity $S(f)$ defined in Equation (1.5) can be arbitrary big.

Example 3.1. For $z = x + \mathrm{i}y$, let $g(z) = \mathrm{i}ky$, where $k > 0$, and assume that ϕ is a conformal automorphism of $\mathcal{S} = \{x + \mathrm{i}y : x \in (-1, 1), y \in \mathbb{R}\}$, which maps y-axis onto $(-1, 1)$. Let $g_1 = \phi \circ g$. For instance, one may define a conformal automorphism ϕ as follows:

$$\phi(z) := -\frac{2\mathrm{i}\,\log\left[-\mathrm{i} + \frac{2}{-\mathrm{i}+\mathrm{e}^{\frac{\mathrm{i}\pi z}{2}}}\right]}{\pi}.$$

Next let $\varrho(w) = |\zeta'(w)|$, where we use notation $w = \phi(\zeta)$ and $w \mapsto \zeta(w)$ denotes the inverse function to ϕ. Then g_1 is ϱ-harmonic, $\lambda_0(iy) = \pi/2$ and $|\nabla g_1(iy)| = k|\phi'(iy)$. Also, $|\nabla g_1(0)| = 2k$. Here $\lambda_0(z)$ is the hyperbolic metric of the strip. Since the expression (1.5) is invariant with respect to conformal maps and hyperbolic metrics, by taking a conformal mapping a of the unit disk onto the strip $\mathcal{S}$ satisfying $a(0) = 0$ and defining $f(z) = g_1(a(z))$, we see that

$$S(f(z)) = \frac{|\nabla f(z)|(1 - |z|^2)}{1 - |f(z)|^2} = \frac{|\nabla g_1(a(z))|}{(1 - g_1(a(z))^2)\lambda_0(a(z))}$$

can be arbitrary big for $z = 0$, namely $S(f(0)) = 4k/\pi$. We remark that in this case

$$\varrho^2(u, v) = \frac{2}{\cos[\pi u] + \cosh[\pi v]},$$

so that $\mathcal{K}_\varrho = 0$, but $\mathcal{K}_\rho = -\pi^2/4$, where $\rho(u, v) = \varrho(u, 0)$.

The following example raises a similar question for positive harmonic functions.

Example 3.2. It is well known that a positive harmonic function defined in the half-plane is a contraction with respect to hyperbolic metric (see e.g. [14]). So, it is natural to ask whether such a result is true for positive R-harmonic functions defined in the half-plane, where R is a metric of non-negative Gaussian curvature. The following example shows that this is not true. Let $R(x) = 1 - \mathrm{e}^{-x}$ and define the positive R-harmonic function on $S(0,\infty) := \{x + \mathrm{i}y : x > 0, y \in \mathbb{R}\}$ by

$$f(x,y) := \log\left[\frac{\pi}{\frac{\pi}{2} - \tan^{-1}\left[\frac{y}{x}\right]}\right] = R\left(\Re\left[-\mathrm{i}/\pi\ \log(\mathrm{i}z)\right]\right).$$

Observe that $-\log(R(x))'' = \frac{1}{4}\ \mathrm{csch}\left[\frac{x}{2}\right]^2$. So, R has a non-negative curvature.

On the other hand,

$$x\frac{|\nabla f(x,y)|}{f(x,y)} = \frac{2x\sqrt{\frac{1}{\left(x^2+y^2\right)\left(\pi-2\tan^{-1}\left[\frac{y}{x}\right]\right)^2}}}{\log\left[\frac{2\pi}{\pi-2\tan^{-1}\left[\frac{y}{x}\right]}\right]} = \frac{2\sqrt{\frac{1}{\left(1+t^2\right)(\pi-2\tan^{-1}[t])^2}}}{\log\left[\frac{2\pi}{\pi-2\tan^{-1}[t]}\right]}$$

for $y = tx$. The last expression has its maximum at $t = -1.4771\ldots$, and it is equal to $1.0482\ldots$. This implies, in particular, that $f\colon S(0,\infty) \to (0,+\infty)$ is not a contraction with respect to corresponding hyperbolic metrics. It would be of interest to find the best Lipschitz constant in this context.

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