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INEQUALITIES AND REPRESENTATION FORMULAS FOR FUNCTIONS OF EXPONENTIAL TYPE

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Abstract

We generalise the classical Bernstein's inequality: $|f'(t)| \le \tau \sup_{-\infty<u<\infty} |f(u)|$, $-\infty < t < +\infty$. Moreover we obtain a new representation formula for entire functions of exponential type.

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1. Introduction

In this paper *the variable t is in* $(-\infty, +\infty)$. Let B_τ be the class of entire functions of exponential type $\le \tau$, bounded on the real axis. For $f \in B_\tau$ the classical Bernstein's inequality [2, p. 211] may be stated as:

$$(1) \qquad |f'(t)| = \left| -\frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+t) \, \frac{d}{dx}\left(\frac{\sin \tau x}{x}\right) dx \right| \le \tau \sup_{-\infty<u<\infty} |f(u)| \, .$$

In this paper we give a generalisation of (1). In order to prove it we need the following result:

THEOREM A. [3, Theorem 1] *Let* $f \in B_\tau$ *be such that* $f(x) = O(|x|^{-\epsilon})$, $\epsilon > 0$, $x \to \pm\infty$. *For all reals* γ *and* $0 \le \alpha \le 1$ *we have*

$$-4\pi\tau \sum_{k=-\infty}^{+\infty} e^{-\alpha i (k\pi+\gamma)} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{\tau}\right)$$

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$$(2)\qquad = \int_{-\infty}^{+\infty} f(x) \frac{e^{-i\tau x} - \alpha i \tau x e^{-i\tau x} - 2e^{-\alpha i \tau x} + e^{(1-\alpha)i\tau x}}{x^2}\, dx + e^{-2i\gamma} \int_{-\infty}^{+\infty} f(x) \frac{e^{(1-\alpha)i\tau x} - e^{i\tau x} + \alpha i \tau x e^{i\tau x}}{x^2}\, dx.$$

Furthermore, we generalise the following representation formulas for functions in B_τ:

THEOREM B. [3, Theorems 2 and 3] *Let $f \in B_\tau$ be such that $f(x) = O(|x|^{-\epsilon})$, $\epsilon > 0$, $x \to \pm\infty$. For all reals γ we have*

$$(3)\qquad \begin{aligned} &-4\pi\tau e^{2i\gamma} \sum_{k=-\infty}^{+\infty} e^{-\alpha i(k\pi+\gamma)} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\alpha \frac{k\pi+\gamma}{\tau}\right) \\ &= \int_{-\infty}^{+\infty} f(\alpha x) \frac{e^{\alpha i\tau x} + e^{(1-\alpha)i\tau x} - 2e^{(2-\alpha)i\tau x} - (2\alpha-3)i\tau x e^{\alpha i \tau x}}{x^2}\, dx \\ &\quad + e^{2i\gamma} \int_{-\infty}^{+\infty} f(\alpha x) \frac{e^{(1-\alpha)i\tau x} - e^{-\alpha i\tau x} - i\tau x e^{-\alpha i\tau x}}{x^2}\, dx \end{aligned}$$

for $1 \le \alpha \le 3/2$, *and*

$$(3')\qquad \begin{aligned} &-4\pi\tau e^{4i\gamma} \sum_{k=-\infty}^{+\infty} e^{-\alpha i(k\pi+\gamma)} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\alpha \frac{k\pi+\gamma}{\tau}\right) \\ &= \int_{-\infty}^{+\infty} f(\alpha x) \frac{(2\alpha-3)i\tau x e^{\alpha i\tau x} + e^{(3-\alpha)i\tau x} - e^{\alpha i\tau x}}{x^2}\, dx \\ &\quad + e^{2i\gamma} \int_{-\infty}^{+\infty} f(\alpha x) \frac{e^{(1-\alpha)i\tau x} - 2e^{(2-\alpha)i\tau x} + e^{(3-\alpha)i\tau x}}{x^2}\, dx \\ &\quad + e^{4i\gamma} \int_{-\infty}^{+\infty} f(\alpha x) \frac{e^{(1-\alpha)i\tau x} - e^{-\alpha i\tau x} - i\tau x e^{-\alpha i\tau x}}{x^2}\, dx \end{aligned}$$

for $3/2 \le \alpha \le 2$.

2. Statement of results

We adopt the *convention* $\sum A_\nu = 0$ if ν varies on an empty set of integers.
For each $\alpha \in \mathbb{R}$ let us denote by L_α the linear functional

$$(4)\qquad f \longmapsto \frac{i}{\pi} \int_{-\infty}^{+\infty} f(x+t) \frac{\alpha i \tau x \cos(\tau x) - \cos(\tau x) + e^{-\alpha i \tau x}}{x^2}\, dx.$$

THEOREM 1. *Let $f \in B_\tau$ be such that $f(x) = O(|x|^{-\epsilon})$, $\epsilon > 0$. For all $-1 \le \alpha \le 1$ we have*

$$(5)\qquad |L_\alpha f(t)| \le \tau \sup_{-\infty<u<\infty} |f(u)|.$$

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In the same way, for each $\alpha \in \mathbb{R}$ let us denote by I_α the linear functional

$$(6)\qquad \begin{aligned} f \longmapsto \frac{i}{\pi}\int_{-\infty}^{+\infty} \left(f'(x+t) - \alpha i\tau f(x+t)\right) \frac{e^{-\alpha i\tau x} - 1}{x}\,dx \\ = \frac{i}{\pi}\int_{-\infty}^{+\infty} f(x+t)e^{-\alpha i\tau x}\frac{d}{dx}\left(\frac{e^{-\alpha i\tau x}-1}{x}\right)dx. \end{aligned}$$

Let $h_f(\theta) := \overline{\lim}_{r\to\infty}\,(1/r)\ln\left|f\left(r\,e^{i\theta}\right)\right|$ be the Phragmén-Lindelöf indicator function.

THEOREM 1′. *Let $f \in B_\tau$ be such that $f(x) = O(|x|^{-\epsilon})$, $\epsilon > 0$ and $h_f(\pi/2) \le 0$. For all $-1 \le \alpha \le 1$ we have*

$$(7)\qquad \left|(1+\alpha)i\tau f(t) - f'(t) + I_\alpha f(t)\right| \le \tau \sup_{-\infty<u<\infty} |f(u)|.$$

REMARK 1. Using the forthcoming Lemma 1 we see that (1) is the particular case $\alpha = 1$ of (5).

REMARK 2. It follows from Lemma 4 that $I_\alpha f(t) \equiv 0$ if $\alpha \le 0$. The particular case $\alpha = 0$ of (7) gives, with the help of Lemma 2, the inequality [8]:

$$\left|i\tau f(t) - f'(t)\right| \le \tau \sup_{-\infty<u<\infty}|f(u)|, \qquad f \in B_\tau, \quad h_f(\pi/2) \le 0.$$

THEOREM 2. *Let $f \in B_\tau$ be such that $f(x) = O(|x|^{-\epsilon})$, $\epsilon > 0$. For all real numbers γ and $\alpha > 0$ we have*

$$(8)\qquad \begin{aligned} &\tau\sum_{k=-\infty}^{+\infty}(-1)^k e^{-\alpha i(k\pi+\gamma)}\frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\alpha\frac{k\pi+\gamma}{\tau}+t\right) \\ &= \sum_{0\le\nu\le\alpha-1}\frac{e^{-(2\nu+1)i\gamma}}{\pi}\int_{-\infty}^{+\infty} f(\alpha x+t)e^{(2\nu+1-\alpha)i\tau x}\frac{\sin^2(\tau x/2)}{x^2}\,dx \\ &+\frac{e^{-(2[\alpha]+1)i\gamma}}{4\pi}\int_{-\infty}^{+\infty} f(\alpha x+t)e^{\alpha i\tau x}\frac{2e^{(1-2\{\alpha\})i\tau x} - e^{-2\{\alpha\}i\tau x} - 2(1-\{\alpha\})i\tau x - 1}{x^2}\,dx, \end{aligned}$$

where $\{\alpha\} := \alpha - [\alpha]$ is the fractional part of α.

If in addition $h_f(\pi/2) \le 0$ then, in the righthand member of (8), the summation is over the integers ν such that $0 \le \nu < \alpha/2$, $\alpha \ge 2$; moreover the last integral is zero for $\alpha \ge 1$.

From Theorem 2 we will deduce

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THEOREM 2′. *Let $f \in B_\tau$ be such that $f(x) = O(|x|^{-\epsilon})$, $\epsilon > 0$. For all real numbers γ and $\alpha \geq 0$ we have*

$$\tau \sum_{k=-\infty}^{+\infty} e^{-\alpha i(k\pi+\gamma)} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\alpha \frac{k\pi+\gamma}{\tau}+t\right)$$
$$= \sum_{0\leq\nu\leq\alpha} \frac{e^{-2\nu i\gamma}}{\pi} \int_{-\infty}^{+\infty} f(\alpha x+t) e^{(2\nu-\alpha)i\tau x} \frac{\sin^2(\tau x/2)}{x^2}\,dx$$
$$+\frac{e^{-(2[\alpha]+1)i\gamma}}{4\pi} \int_{-\infty}^{+\infty} f(\alpha x+t) e^{(\alpha+1)i\tau x} \frac{2e^{(1-2\{\alpha\})i\tau x} - e^{-2\{\alpha\}i\tau x} - 2(1-\{\alpha\})i\tau x - 1}{x^2}\,dx. \tag{9}$$

If in addition $h_f(\pi/2) \leq 0$ then, in the righthand member of (9), *the summation is over the integers ν such that $0 \leq \nu < (\alpha+1)/2$; moreover the last integral is zero.*

3. Lemmas

We use the following integral representation of functions in B_τ.

LEMMA 1. [7, p. 143] *Let $f \in B_\tau$ be such that $f(x) = O(|x|^{-\epsilon})$, $\epsilon > 0$. We have*

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+t) \frac{\sin \tau x}{x}\,dx. \tag{10}$$

In order to prove Theorem 1′ we need the following result.

LEMMA 2. *Let $f \in B_\tau$ be such that $h_f(\pi/2) \leq 0$. We have*

$$\tau f(t) + i f'(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(2x+t) \left(\frac{\sin \tau x}{x}\right)^2 dx. \tag{11}$$

PROOF. Compare the following formulas (see [4, Theorems 2 and 3]):

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} f(2x/\tau+t) \left(\frac{\sin x}{x}\right)^2 dx - \frac{5}{6} f(t) - \frac{1}{\tau^2} f''(t) = \frac{1}{2\pi^2} \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{+\infty} \frac{f(2\pi\nu/\tau+t)}{\nu^2}, \qquad f \in B_\tau,$$

and

$$-2\left(\frac{\tau}{2\pi}\right)^2 \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{+\infty} \frac{f(2\pi\nu/\tau+t)}{\nu^2} = f''(t) - i\tau f'(t) - \frac{\tau^2}{6} f(t), \qquad f \in B_\tau, \quad h_f(\pi/2) \leq 0.$$

The function $e^{-i\epsilon z}$, $\epsilon > 0$, shows that Lemma 2 is not true without the restriction $h_f(\pi/2) \leq 0$.

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In the proof of Theorem 2 an essential tool will be

LEMMA 3. *Let* $t(\theta) := \sum_{j=-n}^{n} c_j e^{ij\theta}$ *be a trigonometric polynomial of degree* $\le n$. *For all real numbers* θ *and* γ *we have*

$$\frac{1}{2(n-m)} \sum_{k=1}^{2(n-m)} e^{-i(k\pi+\gamma)m/(n-m)} \frac{\sin^2((k\pi+\gamma)/2)}{\sin^2((k\pi+\gamma)/2(n-m))} t\left(\theta + \frac{k\pi+\gamma}{(n-m)}\right)$$
$$= \sum_{l=0}^{2} (-1)^l \binom{2}{l} \sum_{0\le s\le (2n-l(n-m))/(2(n-m))} e^{-2si\gamma} \sum_{j=-n}^{n-(2s+l)(n-m)} (n-(2s+l)(n-m)-j)\, c_j e^{ij\theta}, \tag{12}$$

where $m < n$ *is an integer.*

PROOF. Let us consider the integral

$$J_\rho(\theta) := \frac{1}{2\pi i} \oint_{|\zeta|=\rho} \frac{t(-i\ln\zeta)(\zeta^{n-m} - e^{i(n-m)\theta})^2}{\zeta^n(\zeta - e^{i\theta})^2 (\zeta^{2(n-m)} - e^{2i(\gamma+(n-m)\theta)})} d\zeta.$$

We have $\lim_{\rho\to\infty} J_\rho(\theta) = 0$. On the other hand, using the residue theorem (with $\rho > 1$),

$$J_\rho(\theta) = \mathrm{Res}\left(\zeta = e^{i\theta}\right) + \sum_{k=1}^{2(n-m)} \mathrm{Res}\left(\zeta_k = e^{i(\theta+(k\pi+\gamma)/(n-m))}\right) + \mathrm{Res}\,(\zeta = 0)\,.$$

Now it suffices to calculate the residues as in [4, Lemma 2].

On several occasions we shall use the following variant of Paley-Wiener theorem (see [4, Lemma 1] for references).

LEMMA 4. *If* $F \in B_\tau$ *is integrable then for every real number* δ *such that* $|\delta| \ge \tau$ *we have*

$$\int_{-\infty}^{+\infty} F(x) e^{i\delta x}\, dx = 0. \tag{13}$$

If, moreover, $h_F(\pi/2) \le 0$ *then we have*

$$\int_{-\infty}^{+\infty} F(x)\, dx = 0. \tag{14}$$

4. Proofs of the theorems

One way to prove Bernstein's inequality (1) is to use the case $\gamma = \pi/2$ in the interpolation formula [1, p. 143]:

$$\sin\gamma f'(t) - \cos\gamma \tilde{f}'(t) = 2\tau \sum_{k=-\infty}^{+\infty} (-1)^k \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{\tau} + t\right). \tag{15}$$

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PROOF OF THEOREM 1. Apply (2) to the function $x \mapsto f(x+t)$, multiply both members of the resulting formula by $e^{i\gamma}$ and use $e^{\pm i\gamma} = \cos\gamma \pm i\sin\gamma$. This leads us to the formula

$$\begin{aligned}
&\sin\gamma\frac{i}{\pi}\int_{-\infty}^{+\infty} f(x+t)\frac{\alpha i\tau x\cos(\tau x) - \cos(\tau x) + e^{-\alpha i\tau x}}{x^2}\,dx \\
(16)\quad &-\cos\gamma\frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+t)\frac{e^{(1-\alpha)i\tau x} - e^{-\alpha i\tau x} - i\sin(\tau x) - \alpha\tau x\sin(\tau x)}{x^2}\,dx \\
&\qquad = 2e^{(1-\alpha)i\gamma}\tau\sum_{k=-\infty}^{+\infty} e^{-\alpha ik\pi}\frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{\tau}+t\right),
\end{aligned}$$

from which inequality (5) follows for $0 \le \alpha \le 1$. Applying the result to $z \mapsto f(-z)$ shows that (5) is valid for $-1 \le \alpha \le 0$.

REMARK 3. Formula (15) is the case $\alpha = 1$ of (16).

PROOF OF THEOREM 1′. Recalling the definition (see (6)) of $I_\alpha f(t)$ we write $L_\alpha f(t)$ in the form

$$L_\alpha f(t) = \frac{i}{\pi}\int_{-\infty}^{+\infty} f(x+t)\frac{\alpha i\tau x\cos(\tau x) - \cos(\tau x) + 1 - \alpha i\tau x}{x^2}\,dx + I_\alpha f(t). \tag{17}$$

Denote the first integral in the righthand member by J. We have

$$J = i\frac{\tau}{\pi}\int_{-\infty}^{+\infty} f(2x+t)(1-2\alpha i\tau x)\left(\frac{\sin\tau x}{x}\right)^2 dx. \tag{18}$$

Now let us suppose for the time being that $f(x) = O(|x|^{-1-\epsilon})$, $\epsilon > 0$, and apply Lemma 2 to the function $z \mapsto f(z)(1 - \alpha i\tau(z-t))$ to conclude that

$$J = (1+\alpha)i\tau f(t) - f'(t). \tag{19}$$

For this subclass of B_τ the inequality (7) follows from (5), (17) and (19). Considering functions of the form

$$e^{2i\delta z}\left(\frac{\sin\delta z}{\delta z}\right)^2 f(z), \qquad \delta > 0,$$

the conclusion of Theorem 1′ follows by letting $\delta \to 0$.

PROOF OF THEOREM 2. The proof of Theorem 2 uses the method of approximation of Hörmander-Lewitan ([5, 6]).

Let

$$\varphi(x) := \left(\frac{\sin\pi x}{\pi x}\right)^2 \qquad \text{and} \qquad N := 1 + \left[\frac{\tau}{2\pi h}\right], \quad h > 0.$$

Given $f \in B_\tau$ the trigonometric polynomials

$$f_h(x) := \sum_{k=-\infty}^{+\infty} \varphi(hx+k) f\left(x + \frac{k}{h}\right) = \sum_{j=-N}^{N} c_j(h) e^{2\pi i j h x} \tag{20}$$

have the coefficients

$$c_j(h) = h \int_{-\infty}^{+\infty} \varphi(hx) f(x) e^{-2\pi i j h x}\, dx \tag{21}$$

and converge towards f uniformly on every bounded set of the complex plane.

We apply (12) to the trigonometric polynomial $f_h(x/2\pi h)$. We take $\theta = 0$ (the general case in (8) is obtained after an obvious translation), $n = N$ and $m = (p/q)N$ where p and q are integers such that $p/q < 1$ and where h can be chosen in the form $h = \tau/2\pi(S-1)$ where $S \equiv 0 \pmod{q}$ is a positive integer; thus m is an integer since $N \equiv 0 \pmod{q}$. We put

$$U(h) := \frac{1}{2(n-m)} \sum_{k=1}^{2(n-m)} e^{-i(k\pi+\gamma)m/(n-m)} \frac{\sin^2((k\pi+\gamma)/2)}{\sin^2((k\pi+\gamma)/2(n-m))} f_h\left(\frac{k\pi+\gamma}{2\pi h(n-m)}\right) \tag{22}$$

and

$$\begin{aligned} U_{s,l}(h) &:= \sum_{j=-n}^{n-(2s+l)(n-m)} (n-(2s+l)(n-m)-j)\, c_j(h) \\ &= \sum_{j=0}^{2n-(2s+l)(n-m)} j c_{n-(2s+l)(n-m)-j}(h). \end{aligned} \tag{23}$$

Now we can write

$$U(h) = \sum_{l=0}^{2} (-1)^l \binom{2}{l} \sum_{0 \le s \le (2n-l(n-m))/(2(n-m))} e^{-2si\gamma} U_{s,l}(h). \tag{24}$$

We get

$$\lim_{h\to 0} hU(h) = \left(1 - \frac{p}{q}\right) \frac{\tau}{\pi} \sum_{k=-\infty}^{+\infty} e^{-i(k\pi+\gamma)p/(q-p)} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{(1-p/q)\tau}\right). \tag{25}$$

We may assume from now on that f is integrable. Using (21) in (23) we obtain

$$hU_{s,l}(h) = h^2 \int_{-\infty}^{+\infty} \varphi(hx) f(x) e^{-2\pi h N i x (r-1)} K_{s,h}(x)\, dx, \tag{26}$$

where

$$K_{s,h}(x) := \frac{rN e^{2\pi i h x(rN+2)} + e^{2\pi i h x} - (rN+1) e^{2\pi i h x (rN+1)}}{(e^{2\pi i h x} - 1)^2} \tag{27}$$

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with $r := 2 - (2s + l)(1 - p/q)$. Then the dominated convergence theorem gives

$$\lim_{h\to 0} hU_{s,l}(h) = \int_{-\infty}^{+\infty} f(x)e^{-i\tau x(r-1)} K_s(x)\,dx, \tag{28}$$

where

$$K_s(x) := \frac{i\tau x r e^{i\tau x r} + 1 - e^{i\tau x r}}{(2\pi i x)^2}. \tag{29}$$

From (24), (25) and (28) we infer

$$\begin{aligned}\tau \sum_{k=-\infty}^{+\infty} (-1)^k e^{-\alpha i(k\pi+\gamma)} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\alpha\frac{k\pi+\gamma}{\tau}\right) \\ = \sum_{l=0}^{2} (-1)^{l+1}\binom{2}{l} \sum_{0\le s<\alpha-l/2} e^{-(2s+1)i\gamma} I_{s,l}\end{aligned} \tag{30}$$

where

$$I_{s,l} := \frac{1}{4\pi}\int_{-\infty}^{+\infty} f(\alpha x)e^{(2s+l-\alpha)i\tau x} \frac{(2\alpha-2s-l)i\tau x\, e^{(2\alpha-2s-l)i\tau x} + 1 - e^{(2\alpha-2s-l)i\tau x}}{x^2}\,dx.$$

For that we need firstly to replace p/q (< 1) by $(1 - p/q)^{-1}$ (> 0) and secondly to perform a limiting process on p/q to extend the formula for real values.

For the various passages to the limit we give only some indications (see [3, 4] for more details).

In order to obtain (25) we observe that $f_h(x)$ is periodic, with period $1/h$, and write (22) in the form

$$U(h) = \frac{1}{2(n-m)} \sum_{k=-(n-m)}^{n-m-1} e^{-i(k\pi+\gamma)m/(n-m)} \frac{\sin^2((k\pi+\gamma)/2)}{\sin^2((k\pi+\gamma)/2(n-m))} f_h\left(\frac{k\pi+\gamma}{2\pi h(n-m)}\right),$$

with $\sin^2((k\pi+\gamma)/(2N)) \ge ((k\pi+\gamma)/(\pi N))^2$, for $-(n-m) \le k \le n-m-1$. We may assume that $0 \le \gamma \le \pi$.

To obtain (28) we write (26) in the form $hU_{s,l}(h) = \int_{-\infty}^{+\infty} F_h(x)\,dx$, with

$$\begin{aligned}|F_h(x)| &= h^2\varphi(hx)\left|f(x)\sum_{j=-n}^{n-(2s+l)(n-m)} (n-(2s+l)(n-m)-j)\,e^{-2\pi i j h x}\right| \\ &\le c(\tau)\,|f(x)|.\end{aligned}$$

The result follows since we may assume that f is integrable; we have only to consider an auxiliary function of the form

$$f_\delta(x) = \left(\frac{\sin\delta x}{\delta x}\right)^2 f(x).$$

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The last passage to the limit is obtained similarly.

Regrouping the terms we express the righthand member of (30) as

$$-\sum_{0\le s\le \alpha-1}\frac{e^{-(2s+1)i\gamma}}{4\pi}\int_{-\infty}^{+\infty} f(\alpha x)e^{(2s-\alpha)i\tau x}\left(\frac{1-e^{i\tau x}}{x}\right)^2 dx$$
$$+\sum_{\alpha-1<s<\alpha-(1/2)}\frac{e^{-(2s+1)i\gamma}}{4\pi}$$
$$\times\int_{-\infty}^{+\infty} f(\alpha x)e^{\alpha i\tau x}\frac{2(\alpha-s-1)i\tau x-e^{-2(\alpha-s)i\tau x}+2e^{-(2\alpha-2s-1)i\tau x}-1}{x^2}\,dx$$
$$+\sum_{\alpha-(1/2)\le s<\alpha}\frac{e^{-(2s+1)i\gamma}}{4\pi}\int_{-\infty}^{+\infty} f(\alpha x)e^{\alpha i\tau x}\frac{1-e^{-2(\alpha-s)i\tau x}-2(\alpha-s)i\tau x}{x^2}\,dx.$$

Denote the second summation by J_1 and the third by J_2. We have $J_1=0$ if $\{\alpha\}\le 1/2$, $J_2=0$ if $\{\alpha\}>1/2$. If $\{\alpha\}>1/2$ then $s=[\alpha]$ in J_1 which leads us to formula (8). If $\{\alpha\}\le 1/2$ then $s=[\alpha]$ in J_2, so that

$$J_2=\frac{e^{-(2[\alpha]+1)i\gamma}}{4\pi}\int_{-\infty}^{+\infty} f(\alpha x)e^{\alpha i\tau x}\frac{1-e^{-2\{\alpha\}i\tau x}-2\{\alpha\}i\tau x}{x^2}\,dx.$$

But by (14) this is equal to the last integral appearing in formula (8).

The last observation of Theorem 2 follows also from (14).

PROOF OF THEOREM 2′. We change α to $\alpha+1$, $\alpha>0$ in formula (8). Since $1+1/\alpha>1$ we may also replace τ by $(1+1/\alpha)\tau$. An obvious change of variable in the formula so obtained gives us (9) for $\alpha>0$. The case $\alpha=0$ is a known formula. Here again the last observation of Theorem 2′ follows from (14).

5. Corollaries and remarks

5.1. The case of equality in Theorem 1 We observe that, for $-1<\alpha<1$, the equality in (5) is not possible for a function belonging to the class considered. However it is possible to find a sequence of functions $g_\delta\in B_\tau$, $g_\delta(x)=O(|x|^{-2})$, such that for any given $\epsilon>0$, $-1<\alpha<1$ and $-\infty<t_0<\infty$ the inequality

$$|L_\alpha g_\delta(t_0)|>(\tau-\epsilon)\sup_{-\infty<u<\infty}|g_\delta(u)|$$

holds for $\delta<\delta(\epsilon)$.

To prove this we examine the summation in the right-hand member of (16). We restrict ourselves to $\gamma=\pi/2$ (of course a more general discussion may be made here).

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Taking the absolute value on both sides we see that equality is possible, in (5), if and only if

$$f\left(\frac{2k+1}{2}\frac{\pi}{\tau}+t_0\right)=M\,e^{\alpha k\pi i}$$

for $k=0,\pm1,\pm2,\dots$ and $|M|=\sup_{-\infty<u<\infty}|f(u)|$. But then the function

$$g(z):=f\left(\frac{2z+1}{2}\frac{\pi}{\tau}+t_0\right)-M\,e^{\alpha\pi iz}$$

is an element of B_π and $g(k)=0$, $k=0,\pm1,\pm2,\dots$. Hence by a theorem of Valiron (see [2, p. 156] or [9]), $g(z)=A\sin(\pi z)$ where A is a constant, that is, $f(z)=K\,e^{\alpha i\tau z}-A\cos\tau(z-t_0)$ which does not satisfy the condition $f(x)=O(|x|^{-\epsilon})$, $\epsilon>0$.

On the other hand for the functions

$$g_\delta\in B_\tau,\qquad g_\delta(z):=e^{\alpha i\tau z}\left(\frac{\sin\delta z}{\delta z}\right)^2,\qquad 0<\delta\le(1-|\alpha|)\frac{\tau}{2},$$

we have $g_\delta(x)=O(|x|^{-2})$, $x\to\pm\infty$, so that in view of (16),

$$L_\alpha g_\delta(t_0)=i\frac{4\tau}{\pi^2}\sum_{k=-\infty}^{+\infty}e^{i\alpha\tau t_0}\left(\frac{\sin\delta\big((2k+1)(\pi/2\tau)+t_0\big)}{(2k+1)\delta\big((2k+1)(\pi/2\tau)+t_0\big)}\right)^2$$

with $|L_\alpha g_\delta(t_0)|\le\tau\sup_{-\infty<u<\infty}|g_\delta(u)|$. Finally, since

$$\lim_{\delta\to0}\frac{4\tau}{\pi^2}\sum_{k=-\infty}^{+\infty}\left(\frac{\sin\delta\big((2k+1)(\pi/2\tau)+t_0\big)}{(2k+1)\delta\big((2k+1)(\pi/2\tau)+t_0\big)}\right)^2=\tau,$$

it is clear that, for all sufficiently small δ,

$$|L_\alpha g_\delta(t_0)|>(\tau-\epsilon)\sup_{-\infty<u<\infty}|g_\delta(u)|.$$

5.2. Consequences of Theorem 2 When $\alpha=m$ is a positive integer Theorem 2 is reduced to

COROLLARY 1. *Let $f\in B_\tau$. For all real numbers γ and integers $m\ge1$ we have*

$$\begin{aligned}(31)\quad &\sum_{k=-\infty}^{+\infty}(-1)^{k(m-1)}\frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2}f\left(m\frac{k\pi+\gamma}{\tau}+t\right)\\ &=\frac{1}{\pi}\int_{-\infty}^{+\infty}f\left(m\frac{x+\gamma}{\tau}+t\right)e^{-i(m-1)x}\frac{\sin^2((x+\gamma)/2)}{(x+\gamma)^2}\frac{e^{2imx}-1}{e^{2ix}-1}\,dx.\end{aligned}$$

To see this perform the summation on the righthand side of (8) and note that, in view of the second part of Lemma 4, the last integral (appearing in that formula) vanishes.

When $\alpha = 1/2$ the integral involved in (8) can be explicitly evaluated using Lemma 1 twice and one integration by parts. For all $f \in B_\tau$ we have

$$(32)\quad 4\tau \sum_{k=-\infty}^{+\infty} e^{i(k\pi+\gamma)/2} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{2\tau}+t\right) = \tau f(t) + i f'(t).$$

5.3. Consequences of Theorem 2′ Theorem 2′ is simplified when $\{\alpha\} \le 1/2$ if we observe that the last integral vanishes by Lemma 4.

COROLLARY 2. *Let $f \in B_\tau$. For all real numbers γ and $\alpha \ge 0$ such that $\{\alpha\} \le 1/2$ we have*

$$(33)\quad \begin{aligned} &\tau \sum_{k=-\infty}^{+\infty} e^{-\alpha(k\pi+\gamma)i} \frac{\sin^2((k\pi+\gamma)/2)}{(k\pi+\gamma)^2} f\left(\alpha\frac{k\pi+\gamma}{\tau}+t\right) \\ &\qquad = \sum_{0\le\nu\le\alpha} \frac{e^{-2\nu i\gamma}}{\pi} \int_{-\infty}^{+\infty} f(\alpha x+t)\, e^{(2\nu-\alpha)i\tau x} \frac{\sin^2(\tau x/2)}{x^2}\, dx. \end{aligned}$$

Note also that Theorem 2′ generalises Theorem B: apply Lemma 4 several times.

5.4. Another proof of Theorem 2′ The approximation method which was used to prove Theorem 2 could also be applied to prove Theorem 2′. Instead of Lemma 3 we would need the following interpolation formula which is more general than [3, Lemma 4] but can be proved in the same way.

AN INTERPOLATION FORMULA FOR TRIGONOMETRIC POLYNOMIALS. *Let $t(\theta) := \sum_{j=-n}^{n} c_j e^{ij\theta}$ be a trigonometric polynomial of degree $\le n$. Let us write $c_j := 0$ if $j < -n$. We have*

$$(34)\quad \begin{aligned} &\frac{1}{2(n-m)} \sum_{k=1}^{2(n-m)} (-1)^k e^{-i(k\pi+\gamma)m/(n-m)} A_{k,m}(R,\gamma)\, t\left(\theta+\frac{k\pi+\gamma}{n-m}\right) \\ &= \sum_{-m/(n-m)\le s\le 1} \sum_{j=0}^{n-m} \left(R^j-1\right) c_{j+2(n-m)(s-1)+m} e^{i\theta(j+2(n-m)(s-1)+m)+(2s-1)i\gamma} \\ &\quad + \sum_{-(1/2)-\frac{m}{n-m}<s<1} \sum_{j=0}^{n-m-1} \left(R^j-1\right) c_{2(n-m)s+m-j} e^{i\theta(2(n-m)s+m-j)+(2s-1)i\gamma}, \end{aligned}$$

where $m < n$ is an integer and

$$A_{k,m}(R,\gamma) := R^{n-m} - 1 + 2\sum_{\nu=1}^{n-m} \left(R^{n-m-\nu}-1\right) \cos\left(\nu\,\frac{k\pi+\gamma}{n-m}\right).$$

5.5. A representation of the conjugate function For the conjugate function $\tilde{f}$ defined and normalised as in [1, p. 138] the following representation analogous to (10) holds.

Let $f \in B_\tau$ be such that $f(x) = O\left(|x|^{-\epsilon}\right)$, $\epsilon > 0$. We have

$$\tilde{f}(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} (f(x+t) - f(x)) \frac{1 - \cos \tau x}{x} \, dx. \tag{35}$$

To obtain (35) compare formulas (15) with $\gamma = 0$ and (16) with $\alpha = 1$ and $\gamma = 0$, perform an integration by parts and then integrate from 0 to t.

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