Bull. Aust. Math. Soc. **84** (2011), 53–66
doi:10.1017/S0004972710002042

THE QUASILINEARITY OF SOME FUNCTIONALS ASSOCIATED WITH THE RIEMANN–STIELTJES INTEGRAL

S. S. DRAGOMIR ✉ and I. FEDOTOV

(Received 6 October 2010)

Abstract

The superadditivity and subadditivity of some functionals associated with the Riemann–Stieltjes integral are established. Applications in connection to Ostrowski's and the generalized trapezoidal inequalities and for special means are provided.

2010 *Mathematics subject classification*: primary 26D15; secondary 26A42.

Keywords and phrases: Riemann–Stieltjes integral, superadditivity, subadditivity, Ostrowski's inequality, special means.

1. Introduction

In the theory of the *Riemann–Stieltjes integral* for scalar functions, it is well known that if $f : [a, b] \to \mathbb{R}(\mathbb{C})$ is continuous and $u : [a, b] \to \mathbb{R}(\mathbb{C})$ is of bounded variation, then the Riemann–Stieltjes integral $\int_a^b f(t)\, du(t)$ exists and the following sharp inequality holds:

$$\left|\int_a^b f(t)\, du(t)\right| \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u), \tag{1.1}$$

where $\bigvee_a^b(u)$ denotes the *total variation* of u on $[a, b]$. We recall that

$$\bigvee_a^b (u) = \sup\left\{\sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)|,\ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\right\}.$$

Inequality (1.1) plays an important role in obtaining various sharp bounds for the approximation error of the Riemann–Stieltjes integral by simpler quantities such as:

$$f(x)[u(b) - u(a)] \quad \text{(see [2, 5, 6])},$$
$$f(b)[u(b) - u(x)] + f(a)[u(x) - u(a)] \quad \text{(see [2, 7])}$$

© 2011 Australian Mathematical Publishing Association Inc. 0004-9727/2011 $16.00

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

and

$$\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt \quad \text{(see [8, 9])},$$

where $x \in [a, b]$.

Following the recent paper [1], for a continuous function $f : [a, b] \to \mathbb{R}$ and a function of bounded variation $u : [a, b] \to \mathbb{R}$ we define the functional

$$\Psi(f, u; [a, b]) := \max_{t\in[a,b]} |f(t)| \cdot \bigvee_a^b(u) - \left|\int_a^b f(t)\,du(t)\right|. \tag{1.2}$$

Due to the properties of the Riemann–Stieltjes integral, the functional Ψ is well defined and nonnegative. The following properties of this functional as a function of an interval hold [1].

THEOREM 1.1. *Assume that $f : [a, b] \to \mathbb{R}$ is continuous and $u : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Then for any $c \in (a, b)$,*

$$(0\le) \quad \Psi(f, u; [a, c]) + \Psi(f, u; [c, b]) \le \Psi(f, u; [a, b]), \tag{1.3}$$

that is, $\Psi(f, u; \cdot)$ is superadditive as a function of an interval.

If $[c, d] \subseteq [a, b]$, then

$$(0\le) \quad \Psi(f, u; [c, d]) \le \Psi(f, u; [a, b]), \tag{1.4}$$

that is, $\Psi(f, u; \cdot)$ is monotonic nondecreasing as a function of an interval.

In the same paper the following functional is also considered:

$$\Phi(f, u; [a, b]) := \left[\max_{t\in[a,b]} |f(t)| \frac{1}{b-a} \bigvee_a^b(u) - \left|\frac{1}{b-a}\int_a^b f(t)\,du(t)\right|\right]^{(b-a)},$$

which is well defined for continuous functions $f : [a, b] \to \mathbb{R}$ and functions of bounded variation $u : [a, b] \to \mathbb{R}$. The following result concerning the properties of this functional holds [1].

THEOREM 1.2. *Assume that $f : [a, b] \to \mathbb{R}$ is continuous and $u : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Then for any $c \in (a, b)$,*

$$\Phi(f, u; [a, b]) \ge \Phi(f, u; [a, c]) \cdot \Phi(f, u; [c, b]), \tag{1.5}$$

that is, $\Phi(f, u; \cdot)$ is supermultiplicative as a function of an interval.

For applications of the above results for the *Trapezoidal and Ostrowski error functionals* as well as applications for special means, see [1].

In this paper we consider other composite functionals that can be naturally associated with the functional $\Psi(f, u; [a, b])$ and study their quasilinearity properties. Some applications in connection with Ostrowski's and the generalized trapezoidal inequalities and for special means are provided as well [3, 4, 10, 11].

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

2. Some general results

Consider a continuous function $f : [a, b] \to \mathbb{R}$ and a function of bounded variation $u : [a, b] \to \mathbb{R}$ for which

$$\Psi(f, u; [x, y]) := \max_{t\in[x,y]} |f(t)| \cdot \bigvee_x^y(u) - \left| \int_x^y f(t)\, du(t) \right| \neq 0 \tag{2.1}$$

for any proper subinterval $[x, y]$ of the given interval $[a, b]$. Define the new functional

$$\begin{aligned}\Upsilon(f, u; [a, b]) &:= \frac{b-a}{\max_{t\in[a,b]} |f(t)| \frac{1}{b-a} \cdot \bigvee_a^b(u) - \left|\frac{1}{b-a} \int_a^b f(t)\, du(t)\right|} \\ &= \frac{(b-a)^2}{\Psi(f, u; [a, b])}.\end{aligned} \tag{2.2}$$

The following result holds.

THEOREM 2.1. *Assume that $f : [a, b] \to \mathbb{R}$ is continuous and $u : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ such that condition (2.1) is valid. Then for any $c \in (a, b)$,*

$$\Upsilon(f, u; [a, b]) \le \Upsilon(f, u; [a, c]) + \Upsilon(f, u; [c, b]), \tag{2.3}$$

that is, $\Upsilon(f, u; \cdot)$ is subadditive as a function of an interval.

PROOF. Since, by Theorem 1.1, the functional $\Psi(f, u; \cdot)$ is superadditive as a function of an interval, we have for any $c \in (a, b)$ that

$$\begin{aligned}\frac{\Psi(f, u; [a, b])}{b-a} &\ge \frac{\Psi(f, u; [a, c]) + \Psi(f, u; [c, b])}{b-a} \\ &= \frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a} + (b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}.\end{aligned} \tag{2.4}$$

Utilizing the elementary inequality between the *weighted arithmetic mean* and the *weighted harmonic mean*, that is,

$$\frac{\alpha a + \beta b}{\alpha + \beta} \ge \frac{\alpha + \beta}{\frac{\alpha}{a} + \frac{\beta}{b}}, \quad \alpha, \beta, a, b > 0,$$

for the choices

$$a = \frac{\Psi(f, u; [a, c])}{c-a}, \quad b = \frac{\Psi(f, u; [c, b])}{b-c}, \quad \alpha = c-a, \quad \beta = b-c,$$

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

we have

$$\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}$$
$$\geq \frac{(c-a)+(b-c)}{\frac{c-a}{\frac{\Psi(f,u;[a,c])}{c-a}}+\frac{b-c}{\frac{\Psi(f,u;[c,b])}{b-c}}}=\frac{(c-a)+(b-c)}{\frac{(c-a)^2}{\Psi(f,u;[a,c])}+\frac{(b-c)^2}{\Psi(f,u;[c,b])}} \tag{2.5}$$
$$=\frac{b-a}{\frac{(c-a)^2}{\Psi(f,u;[a,c])}+\frac{(b-c)^2}{\Psi(f,u;[c,b])}}.$$

Combining (2.4) with (2.5), we get

$$\frac{\Psi(f,u;[a,b])}{b-a}\geq \frac{b-a}{\frac{(c-a)^2}{\Psi(f,u;[a,c])}+\frac{(b-c)^2}{\Psi(f,u;[c,b])}},$$

which shows that the functional $\Upsilon(f,u;\cdot)$ is subadditive as a function of an interval. □

Further, for $q\in(0,1)$, we consider the following family of functionals:

$$\Omega_q(f,u;[a,b]):=(b-a)^{1-q}[\Psi(f,u;[a,b])]^q$$
$$=(b-a)^{1-q}\left[\max_{t\in[a,b]}|f(t)|\bigvee_a^b(u)-\left|\int_a^b f(t)\,du(t)\right|\right]^q. \tag{2.6}$$

The following result concerning the quasilinearity of the functional $\Omega_q(f,u;\cdot)$ may be stated.

Theorem 2.2. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous and $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then for any $q\in(0,1)$,*

$$(0\leq)\quad \Omega_q(f,u;[a,c])+\Omega_q(f,u;[c,b])\leq\Omega_q(f,u;[a,b]), \tag{2.7}$$

for any $c\in(a,b)$, that is, the functional $\Omega_q(f,u;\cdot)$ is superadditive as a function of an interval.

If $[c,d]\subseteq[a,b]$, then

$$(0\leq)\quad \Omega_q(f,u;[c,d])\leq\Omega_q(f,u;[a,b]), \tag{2.8}$$

that is, $\Omega_q(f,u;\cdot)$ is monotonic nondecreasing as a function of an interval.

Proof. We know from the proof of Theorem 2.2 that

$$\frac{\Psi(f,u;[a,b])}{b-a}\geq\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)} \tag{2.9}$$

for any $c\in(a,b)$. Taking the power $q\in(0,1)$ in (2.9), we get

$$\left[\frac{\Psi(f,u;[a,b])}{b-a}\right]^q\geq\left[\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}\right]^q \tag{2.10}$$

for any $c\in(a,b)$.

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

By the concavity of the function $g(t) = t^q$, $q \in (0, 1)$ we also have

$$\begin{aligned}
&\left[\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a} + (b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}\right]^q \\
&\quad \geq \frac{(c-a)\left[\frac{\Psi(f,u;[a,c])}{c-a}\right]^q + (b-c)\left[\frac{\Psi(f,u;[c,b])}{b-c}\right]^q}{(c-a)+(b-c)} \\
&\quad = \frac{(c-a)^{1-q}[\Psi(f,u;[a,c])]^q + (b-c)^{1-q}[\Psi(f,u;[c,b])]^q}{(c-a)+(b-c)} \\
&\quad = \frac{(c-a)^{1-q}[\Psi(f,u;[a,c])]^q + (b-c)^{1-q}[\Psi(f,u;[c,b])]^q}{b-a}
\end{aligned} \tag{2.11}$$

for any $c \in (a, b)$.

Combining (2.10) with (2.11), we deduce that

$$\frac{[\Psi(f,u;[a,b])]^q}{(b-a)^q} \geq \frac{(c-a)^{1-q}[\Psi(f,u;[a,c])]^q + (b-c)^{1-q}[\Psi(f,u;[c,b])]^q}{b-a}$$

for any $c \in (a, b)$, which shows that the functional $\Omega_q(f, u; \cdot)$ is superadditive as a function of an interval.

Now let $a < c < d < b$. Then by the superadditivity of $\Omega_q(f, u; \cdot)$,

$$\Omega_q(f,u;[a,b]) - \Omega_q(f,u;[c,d]) \geq \Omega_q(f,u;[a,c]) + \Omega_q(f,u;[d,b]) \geq 0,$$

which proves the monotonicity property. □

If $p \geq q \geq 0$, $p \geq 1$ we can also consider the mapping depending on two parameters:

$$\begin{aligned}
&\Lambda_{p,q}(f,u;[a,b]) \\
&\quad := (b-a)^{(p-q)/p}\Psi^q(f,u;[a,b]) = (b-a)^{(p-q+pq)/p} \\
&\qquad \times \left[\max_{t\in[a,b]}|f(t)| \cdot \frac{1}{b-a}\bigvee_a^b(u) - \left|\frac{1}{b-a}\int_a^b f(t)\,du(t)\right|\right]^q.
\end{aligned} \tag{2.12}$$

We have also the following general result.

THEOREM 2.3. *Assume that $f : [a, b] \to \mathbb{R}$ is continuous and $u : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Then for any $p \geq q \geq 0$, $p \geq 1$, we have that the functional $\Lambda_{p,q}(f, u; \cdot)$ defined by (2.13) is superadditive and monotonic nondecreasing as a function of an interval.*

PROOF. First of all, we observe that the following elementary inequality holds:

$$(\alpha + \beta)^p \geq (\leq)\alpha^p + \beta^p \tag{2.13}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

Indeed, if we consider the function $f_p : [0, \infty) \to \mathbb{R}$, $f_p(t) = (t+1)^p - t^p$, we have $f_p'(t) = p[(t+1)^{p-1} - t^{p-1}]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$, showing that f_p is strictly increasing on the interval $[0, \infty)$. Now for $t = \alpha/\beta$ ($\beta > 0$, $\alpha \geq 0$) we have $f_p(t) > f_p(0)$, giving that $(\alpha/\beta + 1)^p - (\alpha/\beta)^p > 1$, that is, the desired inequality (2.1).

For $p \in (0, 1)$ we have that f_p is strictly decreasing on $[0, \infty)$ which proves the second case in (2.13).

Now let $c \in (a, b)$. Since $\Psi(f, u; \cdot)$ is superadditive as a function of an interval, we have by (2.13), for any $p \geq 1$, that

$$\begin{aligned} \Psi^p(f, u; [a, b]) &\geq [\Psi(f, u; [a, c]) + \Psi(f, u; [c, b])]^p \\ &\geq \Psi^p(f, u; [a, c]) + \Psi^p(f, u; [c, b]), \end{aligned} \tag{2.14}$$

which provides that

$$\begin{aligned} &\frac{\Psi(f, u; [a, b])}{b-a} \\ &\quad\geq \frac{[\Psi^p(f, u; [a, c]) + \Psi^p(f, u; [c, b])]^{1/p}}{(c-a)+(b-c)} \\ &\quad= \left(\frac{(c-a)\big[\frac{\Psi(f,u;[a,c])}{(c-a)^{1/p}}\big]^p + (b-c)\big[\frac{\Psi(f,u;[c,b])}{(b-c)^{1/p}}\big]^p}{(c-a)+(b-c)}\right)^{1/p} (b-a)^{1/p-1} \end{aligned} \tag{2.15}$$

for any $c \in (a, b)$.

Utilizing the monotonicity property of *power means*, that is,

$$\left(\frac{\alpha x^p + \beta y^p}{\alpha+\beta}\right)^{1/p} \geq \left(\frac{\alpha x^q + \beta y^q}{\alpha+\beta}\right)^{1/q}$$

where $p \geq q \geq 0$, and $\alpha, \beta, x, y \geq 0$ with $\alpha + \beta > 0$, we have

$$\begin{aligned} &\left(\frac{(c-a)\big[\frac{\Psi(f,u;[a,c])}{(c-a)^{1/p}}\big]^p + (b-c)\big[\frac{\Psi(f,u;[c,b])}{(b-c)^{1/p}}\big]^p}{(c-a)+(b-c)}\right)^{1/p} \\ &\quad\geq \left(\frac{(c-a)\big[\frac{\Psi(f,u;[a,c])}{(c-a)^{1/p}}\big]^q + (b-c)\big[\frac{\Psi(f,u;[c,b])}{(b-c)^{1/p}}\big]^q}{(c-a)+(b-c)}\right)^{1/q} \\ &\quad= \left(\frac{(c-a)^{1-q/p}\Psi^q(f, u; [a, c]) + (b-c)^{1-q/p}\Psi^q(f, u; [c, b])}{b-a}\right)^{1/q}. \end{aligned} \tag{2.16}$$

By making use of inequalities (2.15) and (2.16), we get

$$\begin{aligned} &\frac{\Psi(f, u; [a, b])}{b-a} \\ &\quad\geq \left(\frac{(c-a)^{1-q/p}\Psi^q(f, u; [a, c]) + (b-c)^{1-q/p}\Psi^q(f, u; [c, b])}{b-a}\right)^{1/q} \\ &\quad\quad\times (b-a)^{1/p-1} \end{aligned}$$

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

which is equivalent, by taking the power q, to

$$\begin{aligned}\frac{\Psi^q(f,u;[a,b])}{(b-a)^q} &\geq \left(\frac{(c-a)^{1-q/p}\Psi^q(f,u;[a,c])+(b-c)^{1-q/p}\Psi^q(f,u;[c,b])}{b-a}\right)\\ &\quad\times (b-a)^{q/p-q}\\ &= [(c-a)^{1-q/p}\Psi^q(f,u;[a,c])+(b-c)^{1-q/p}\Psi^q(f,u;[c,b])]\\ &\quad\times (b-a)^{q/p-q-1}.\end{aligned} \tag{2.17}$$

Moreover, if we multiply (2.17) by $(b-a)^{1+q-(q/p)}$, then we get

$$\begin{aligned}\Psi^q(f,u;[a,b])(b-a)^{1-q/p} \geq (c-a)^{1-q/p}\Psi^q(f,u;[a,c])\\ +(b-c)^{1-q/p}\Psi^q(f,u;[c,b])\end{aligned} \tag{2.18}$$

for any $c \in (a,b)$, which shows that $\Lambda_{p,q}(f,u;\cdot)$ is superadditive as a function of an interval.

The monotonicity follows as above and the proof is complete. □

3. Applications for Ostrowski's inequality

In [6], the author has proved the following inequality of *Ostrowski type* for the Riemann–Stieltjes integral:

$$\begin{aligned}\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\\ \leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(u),\end{aligned} \tag{3.1}$$

for all $x \in [a,b]$, where $f:[a,b]\to\mathbb{R}$ is of r-H-Hölder type, that is,

$$|f(x)-f(y)| \leq H|x-y|^r$$

for any $x, y \in [a,b]$, where $r \in (0,1]$ and $H > 0$ are given and u is of bounded variation on $[a,b]$.

We can now define the functional

$$\begin{aligned}\theta(f,u,x)(a,b) := H\max_{t\in[a,b]}|t-x|^r \bigvee_a^b(u)\\ -\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\end{aligned} \tag{3.2}$$

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

where f, u, x, a, b, r and H are as above. We observe that, when x is in the interior of $[a, b]$ then

$$\max_{t\in[a,b]} |t-x|^r = \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r$$

which provides a natural connection between inequality (3.1) and the functional (3.2).

LEMMA 3.1. *Assume that $f : [a, b] \to \mathbb{R}$ is of r-H-Hölder type and u is of bounded variation on $[a, b]$. If $c \in (a, b)$, then*

$$\theta(f, u, x)(a, b) \geq \theta(f, u, x)(a, c) + \theta(f, u, x)(c, b) \tag{3.3}$$

for any $x \in [a, b]$, that is, $\theta(f, u, x)$ is superadditive as a function of an interval.

PROOF. Observe that, for any $c \in (a, b)$, we have successively

$$\begin{aligned}
&\theta(f, u, x)(a, b) \\
&\quad= H\Big(\max_{t\in[a,b]} |t-x|^r\Big) \bigvee_a^b(u) - \left|\int_a^b [f(x) - f(t)]\, du(t)\right| \\
&\quad= H \max\Big\{\max_{t\in[a,c]} |t-x|^r, \max_{t\in[c,b]} |t-x|^r\Big\}\Big(\bigvee_a^c(u) + \bigvee_c^b(u)\Big) \\
&\qquad - \left|\int_a^c [f(x) - f(t)]\, du(t) + \int_c^b [f(x) - f(t)]\, du(t)\right|.
\end{aligned} \tag{3.4}$$

Now, since

$$\begin{aligned}
&H \max\Big\{\max_{t\in[a,c]} |t-x|^r, \max_{t\in[c,b]} |t-x|^r\Big\}\Big(\bigvee_a^c(u) + \bigvee_c^b(u)\Big) \\
&\qquad \geq H\Big[\max_{t\in[a,c]} |t-x|^r \bigvee_a^c(u) + \max_{t\in[c,b]} |t-x|^r \bigvee_c^b(u)\Big]
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\left|\int_a^c [f(x) - f(t)]\, du(t) + \int_c^b [f(x) - f(t)]\, du(t)\right| \\
&\qquad \leq \left|\int_a^c [f(x) - f(t)]\, du(t)\right| + \left|\int_c^b [f(x) - f(t)]\, du(t)\right|,
\end{aligned} \tag{3.6}$$

then, by (3.4),

$$\begin{aligned}
\theta(f, u, x)(a, b) &\geq H \max_{t\in[a,c]} |t-x|^r \bigvee_a^c(u) + H \max_{t\in[c,b]} |t-x|^r \bigvee_c^b(u) \\
&\qquad - \left|\int_a^c [f(x) - f(t)]\, du(t)\right| - \left|\int_c^b [f(x) - f(t)]\, du(t)\right| \\
&= \theta(f, u, x)(a, c) + \theta(f, u, x)(c, b)
\end{aligned}$$

and the statement is proved. □

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

COROLLARY 3.2. *Assume that f and u are as above. If $x \in [c, d] \subset [a, b]$, then*

$$\theta(f, u, x)(a, b) \geq \theta(f, u, x)(c, d)$$

or, equivalently,

$$\begin{aligned} H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(u) - \left|f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t)\right| \\ \geq H\left[\frac{1}{2}(d-c) + \left|x - \frac{c+d}{2}\right|\right]^r \bigvee_c^d(u) \\ - \left|f(x)[u(d)-u(c)] - \int_c^d f(t)\,du(t)\right|. \end{aligned} \tag{3.7}$$

As in the general case presented above, we consider the following composite functionals that can be attached to $\theta(f, u, x)$:

$$\Phi(f, u, x)(a, b) := \left[\frac{\theta(f, u, x)(a, b)}{b-a}\right]^{(b-a)},$$

$$\Upsilon(f, u, x)(a, b) := \frac{(b-a)^2}{\theta(f, u, x)(a, b)},$$

provided the denominator is not zero, and the families of functionals

$$\Omega_q(f, u, x)(a, b) := (b-a)^{1-q}[\theta(f, u, x)(a, b)]^q, \quad q \in (0, 1),$$

and

$$\Lambda_{p,q}(f, u, x)(a, b) := (b-a)^{(p-q)/p}[\theta(f, u, x)(a, b)]^q, \quad p \geq q \geq 0,\ p \geq 1.$$

PROPOSITION 3.3. *Assume that $f : [a, b] \to \mathbb{R}$ is of r-H-Hölder type and u is of bounded variation on $[a, b]$. For $x \in (a, b)$, the functional $\Phi(f, u, x)$ is supermultiplicative, $\Upsilon(f, u, x)$ is subadditive and $\Omega_q(f, u, x)$ and $\Lambda_{p,q}(f, u, x)$ are superadditive as functions of an interval.*

The proof follows from Lemma 3.1 and the results from the preceding sections. The details are omitted.

4. Applications for the generalized trapezoidal formula

In [7], in order to approximate the Stieltjes integral $\int_a^b f(t)\,du(t)$ with the generalized trapezoidal rule $[u(b)-u(x)]f(b) + [u(x)-u(a)]f(a)$, where f is a function of bounded variation and u is continuous on $[a, b]$, the authors considered the *generalized trapezoidal error functional*

$$T(f, u; x, [a, b]) := [u(b)-u(x)]f(b) + [u(x)-u(a)]f(a) - \int_a^b f(t)\,du(t)$$

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

and showed that

$$|T(f, u; x, [a, b])| \leq \max_{t\in[a,b]} |u(t) - u(x)| \bigvee_a^b (f). \tag{4.1}$$

Now, if $f : [a, b] \to \mathbb{R}$ is of r-H-Hölder type, where $r \in (0, 1]$ and $H > 0$ are given and u is of bounded variation on $[a, b]$, then by (4.1) we have the *generalized trapezoid inequality*

$$|T(f, u; x, [a, b])| \leq H\left[\frac{1}{2}(b - a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (u) \tag{4.2}$$

for any $x \in [a, b]$.

We may define the functional

$$\eta(f, u, x)(a, b) := H \max_{t\in[a,b]} |t - x|^r \bigvee_a^b (u) - |T(f, u; x, [a, b])| \tag{4.3}$$

where f, u, x, a, b, r and H are as above.

LEMMA 4.1. *Assume that $f : [a, b] \to \mathbb{R}$ is of r-H-Hölder type and u is of bounded variation on $[a, b]$. If $c \in (a, b)$, then*

$$\eta(f, u, x)(a, b) \geq \eta(f, u, x)(a, c) + \eta(f, u, x)(c, b) \tag{4.4}$$

for any $x \in [a, b]$, that is, $\eta(f, u, x)$ is superadditive as a function of an interval.

The proof is similar to that of Lemma 3.1 by observing that

$$T(f, u; x, [a, b]) = T(f, u; x, [a, c]) + T(f, u; x, [c, b])$$

for $x, c \in [a, b]$, and we omit the details.

COROLLARY 4.2. *Assume that f and u are as above. If $x \in [c, d] \subset [a, b]$, then*

$$\eta(f, u, x)(a, b) \geq \eta(f, u, x)(c, d)$$

or, equivalently,

$$\begin{aligned} &H\left[\frac{1}{2}(b - a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (u) \\ &\quad - \left|[u(b) - u(x)]f(b) + [u(x) - u(a)]f(a) - \int_a^b f(t)\,du(t)\right| \\ &\quad \geq H\left[\frac{1}{2}(d - c) + \left|x - \frac{c+d}{2}\right|\right]^r \bigvee_c^d (u) \\ &\qquad - \left|[u(d) - u(x)]f(d) + [u(x) - u(c)]f(c) - \int_c^d f(t)\,du(t)\right|. \end{aligned} \tag{4.5}$$

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

As in the general case presented above, we consider the following composite functionals that can be attached to $\eta(f, u, x)$:

$$F(f, u, x)(a, b) := \left[\frac{\eta(f, u, x)(a, b)}{b-a}\right]^{(b-a)},$$

$$\Delta(f, u, x)(a, b) := \frac{(b-a)^2}{\eta(f, u, x)(a, b)},$$

provided the denominator is not zero, and the families of functionals

$$\Xi_q(f, u, x)(a, b) := (b-a)^{1-q}[\eta(f, u, x)(a, b)]^q, \quad q \in (0, 1),$$

and

$$\Pi_{p,q}(f, u, x)(a, b) := (b-a)^{(p-q)/p}[\eta(f, u, x)(a, b)]^q, \quad p \geq q \geq 0,\ p \geq 1.$$

PROPOSITION 4.3. *Assume that $f : [a, b] \to \mathbb{R}$ is of r-H-Hölder type and u is of bounded variation on $[a, b]$. For $x \in (a, b)$, the functional $F(f, u, x)$ is supermultiplicative, $\Delta(f, u, x)$ is subadditive and $\Xi_q(f, u, x)$ and $\Pi_{p,q}(f, u, x)$ are superadditive as functions of an interval.*

5. Applications for means

If $g : [a, b] \to \mathbb{R}$ is a continuous function on $[a, b]$ then $u(t) = \int_a^t g(s)\, ds$ is differentiable on (a, b) and the functionals Ψ, Υ, Ω_q and $\Lambda_{p,q}$ become

$$\tilde{\Psi}(f, g; [a, b]) := \max_{t\in[a,b]} |f(t)| \cdot \int_a^b |g(t)|\, dt - \left|\int_a^b f(t)g(t)\, dt\right|, \tag{5.1}$$

$$\begin{aligned}\tilde{\Upsilon}&(f, g; [a, b]) \\ &:= \frac{b-a}{\max_{t\in[a,b]} |f(t)| \frac{1}{b-a} \cdot \int_a^b |g(t)|\, dt - \left|\frac{1}{b-a}\int_a^b f(t)g(t)\, dt\right|} \\ &= \frac{(b-a)^2}{\tilde{\Psi}(f, g; [a, b])},\end{aligned} \tag{5.2}$$

$$\begin{aligned}\tilde{\Omega}_q(f, g; [a, b]) &:= (b-a)^{1-q}[\tilde{\Psi}(f, g; [a, b])]^q \\ &= (b-a)\left[\max_{t\in[a,b]} |f(t)| \frac{1}{b-a} \cdot \int_a^b |g(t)|\, dt \right.\\ &\qquad \left. - \left|\frac{1}{b-a}\int_a^b f(t)g(t)\, dt\right|\right]^q\end{aligned} \tag{5.3}$$

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

and

$$\begin{aligned}\tilde{\Lambda}_{p,q}(f, g; [a, b]) &:= (b-a)^{(p-q)/p}\tilde{\Psi}^{q}(f, g; [a, b]) \\ &= (b-a)^{(p-q+pq)/p}\bigg[\max_{t\in[a,b]}|f(t)|\cdot\frac{1}{b-a}\int_a^b |g(t)|\,dt \\ &\quad - \bigg|\frac{1}{b-a}\int_a^b f(t)g(t)\,dt\bigg|\bigg]^{q}.\end{aligned} \tag{5.4}$$

Obviously $\tilde{\Psi}$ remains *superadditive and monotonic nondecreasing* as a function of an interval while $\tilde{\Upsilon}$ inherits the *subadditivity property* of Υ. Also, any member of the families of functionals $\tilde{\Omega}_q$, $q \in (0, 1)$, and $\tilde{\Lambda}_{p,q}$, $p \geq q \geq 0$, $p \geq 1$, is *superadditive and monotonic nondecreasing as a function of an interval.*

Let us recall the following means:

arithmetic mean: $A(a, b) = \dfrac{a+b}{2}$,

geometric mean: $G(a, b) = \sqrt{ab}$,

harmonic mean: $H(a, b) = \dfrac{2}{\frac{1}{a}+\frac{1}{b}}$,

logarithmic mean: $L(a, b) = \dfrac{b-a}{\ln b - \ln a}$, $\quad b \neq a$;

identric mean: $I(a, b) = \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$, $\quad b \neq a$;

p-logarithmic mean: $L_p(a, b) = \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}$,
$p \in \mathbb{R}\setminus\{-1, 0\}$, $b \neq a$;

with $a, b > 0$. It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities:

$$H(a, b) \leq G(a, b) \leq L(a, b) \leq I(a, b) \leq A(a, b). \tag{5.5}$$

If we consider $u(t) = t^p$, $p \in \mathbb{R}\setminus\{-1, 0\}$, $u : [a, b] \to \mathbb{R}$, $0 < a < b$, then obviously

$$\frac{1}{b-a}\int_a^b u(t)\,dt = L_p^p(a, b).$$

If $u(t) = 1/t$, $t \in [a, b]$, $0 < a < b$, then

$$\frac{1}{b-a}\int_a^b u(t)\,dt = \frac{1}{L(a, b)},$$

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

while for $u(t) = \ln t$, $t \in [a, b]$, $0 < a < b$,

$$\frac{1}{b-a}\int_a^b u(t)\,dt = \ln[I(a, b)].$$

If we choose above $g(t) = 1/t$, $f(t) = t^p$ with $p \in \mathbb{R} \setminus \{0, 1\}$ and observing that $0 < a < b$, we have

$$\max_{t\in[a,b]} |f(t)| = \max\{a^p, b^p\}, \qquad \int_a^b g(t)\,dt = \frac{b-a}{L(a, b)}$$

and

$$\int_a^b f(t)g(t)\,dt = L_{p-1}^{p-1}(a, b)(b-a).$$

Then we deduce that

$$v_p(a, b) := \left[\frac{\max\{a^p, b^p\}}{L(a, b)} - L_{p-1}^{p-1}(a, b)\right](b-a), \qquad p \in \mathbb{R} \setminus \{0, 1\}, \tag{5.6}$$

is *superadditive and monotonic nondecreasing* as a function of an interval, while

$$z_p(a, b) := \frac{b-a}{\frac{\max\{a^p, b^p\}}{L(a,b)} - L_{p-1}^{p-1}(a, b)}, \qquad p \in \mathbb{R} \setminus \{0, 1\}, \tag{5.7}$$

is *subadditive* as a function of an interval.

Finally, if we consider the families of functionals

$$y_{p,q}(a, b) := (b-a)\left[\frac{\max\{a^p, b^p\}}{L(a, b)} - L_{p-1}^{p-1}(a, b)\right]^q, \tag{5.8}$$

where $p \in \mathbb{R} \setminus \{0, 1\}$, $q \in (0, 1)$ and

$$u_{p,r,q}(a, b) = (b-a)^{(r-q+rq)/r}\left[\frac{\max\{a^p, b^p\}}{L(a, b)} - L_{p-1}^{p-1}(a, b)\right]^q, \tag{5.9}$$

where $p \in \mathbb{R} \setminus \{0, 1\}$, $r \geq q \geq 0$ and $r \geq 1$, then we can conclude that each functional $y_{p,q}$ is *superadditive and monotonic nondecreasing as a function of an interval* for any $p \in \mathbb{R} \setminus \{0, 1\}$, $q \in (0, 1)$. Also, each functional $u_{p,r,q}$ is *superadditive and monotonic nondecreasing as a function of an interval* for any $p \in \mathbb{R} \setminus \{0, 1\}$ and $r \geq q \geq 0$ and $r \geq 1$.

Similar results may be stated for other choices of f and g. However, the details are omitted.

References

[1] P. Cerone, S. S. Dragomir and A. McAndrew, 'Superadditivity and supermultiplicity of two functionals associated with the Stieltjes integral', *RGMIA Res. Rep. Coll.* **12**(2) (2009), Article 8, Preprint, http://www.staff.vu.edu.au/RGMIA/v12n2.asp.

[2] W. S. Cheung and S. S. Dragomir, 'Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions', *Bull. Aust. Math. Soc.* **75**(2) (2007), 299–311.

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press

[3] Y. J. Cho, S. S. Dragomir, S. S. Kim and C. E. M. Pearce, 'Cauchy–Schwarz functionals', *Bull. Aust. Math. Soc.* **62**(3) (2000), 479–491.
[4] D. Comanescu, S. S. Dragomir and C. E. M. Pearce, 'Geometric means, index mappings and entropy', in: *Inequality Theory and Applications,* Vol. 3 (Nova Science Publications, Hauppauge, NY, 2003), pp. 85–96.
[5] S. S. Dragomir, 'On the Ostrowski's inequality for Riemann–Stieltjes integral', *Korean J. Appl. Math.* **7** (2000), 477–485.
[6] S. S. Dragomir, 'On the Ostrowski's inequality for Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications', *J. KSIAM* **5**(1) (2001), 35–45.
[7] S. S. Dragomir, C. Buşe, M. V. Boldea and L. Braescu, 'A generalisation of the trapezoidal rule for the Riemann–Stieltjes integral and applications', *Nonlinear Anal. Forum (Korea)* **6**(2) (2001), 337–351.
[8] S. S. Dragomir and I. Fedotov, 'An inequality of Grüss type for the Riemann–Stieltjes integral and applications for special means', *Tamkang J. Math.* **29**(4) (1998), 287–292.
[9] S. S. Dragomir and I. Fedotov, 'A Grüss type inequality for mappings of bounded variation and applications for numerical analysis', *Nonlinear Funct. Anal. Appl.* **6**(3) (2001), 425–433.
[10] S. S. Dragomir and C. E. M. Pearce, 'Quasilinearity & Hadamard's inequality. Inequalities, 2001 (Timişoara)', *Math. Inequal. Appl.* **5**(3) (2002), 463–471.
[11] L. Losonczi, 'Sub- and superadditive integral means', *J. Math. Anal. Appl.* **307**(2) (2005), 444–454.

S. S. DRAGOMIR, Mathematics, School of Engineering and Science,
Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia
and
School of Computational and Applied Mathematics, University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg, South Africa
e-mail: sever.dragomir@vu.edu.au

I. FEDOTOV, Department of Mathematics and Statistics,
Tshwane University of Technology, Private Bag X680, Pretoria 001, South Africa
e-mail: FedotovI@tut.ac.za

https://doi.org/10.1017/S0004972710002042 Published online by Cambridge University Press