A NOTE ON ORTHOGONAL POLYNOMIALS

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1. Introduction and preliminaries

During an investigation into the existence of Gauss-type quadrature formulae for the numerical solution of Fredholm integral equations with weakly singular kernels an intermediate result was found which is of independent interest.

The quality and position of the zeros of the orthogonal polynomials which satisfy

$$\int_a^b m(t)t^{i-1}\Pi_n(t)dt = 0, \quad i = 1, 2, \ldots, n \text{ for } n = 1, 2, \ldots$$

where $m(t)$ is of constant sign in $]a, b[$ are well known. The results of this paper show that these properties also belong to the polynomials which satisfy a generalized orthogonality relation of the form

$$\int_a^b m_i(t)\Pi_n(t)dt = 0, \quad i = 1, 2, \ldots, n \text{ for } n = 1, 2, \ldots$$

The proof of the results shows that the conditions under which the classical result is valid can be weakened slightly.

Definition. The sequence $m_i(t)$, $i = 1, 2, \ldots$ of real valued functions will be said to form an *integrable Markov system* for $a<t<b$ if

(*a*) $m_i(t)$ is defined at each point of $a<t<b$, and is integrable-R (a, b) for $i = 1, 2, \ldots$;

(*b*) for $n = 1, 2, \ldots$ the linear combination

$$\sum_{i=1}^{n} a_i m_i(t)$$

has not more than $n-1$ zeros in $a<t<b$, where $a_1, a_2, \ldots$ are arbitrary scalars, not all of which are zero.

It follows from the definition that if $a<t_i<b$, $i = 1, 2, \ldots, n$ then the determinant

$$M_n(t_1, t_2, \ldots, t_n) = \begin{vmatrix} m_1(t_1) & m_2(t_1) & \ldots & m_n(t_1) \\ \ldots & \ldots & \ldots & \ldots \\ m_1(t_n) & m_2(t_n) & \ldots & m_n(t_n) \end{vmatrix}$$

vanishes if and only if $t_i = t_j$ for some i, j.

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Further ((1), p. 25), if $t_1, t_2, \ldots, t_{n-1}$ are distinct points of $]a, b[$ then

$$M_n(t, t_1, t_2, \ldots, t_{n-1}),$$

considered as a function of t, changes sign only as t passes through $t_1, t_2, \ldots, t_{n-1}$. It follows that $M_n(t, t_1, t_2, \ldots, t_{n-1})$ behaves like a polynomial of degree $n-1$ with simple zeros at $t_1, t_2, \ldots, t_{n-1}$. This property of M_n will be restated as:

Property (P). If $\Pi_{n-1}(t)$ is a polynomial of degree $n-1$, with real coefficients, having precisely r zeros $t_1, t_2, \ldots, t_r$ at which $\Pi_{n-1}(t)$ changes sign then

$$\Pi_{n-1}(t)M_{r+1}(t, t_1, \ldots, t_r)$$

does not change sign for $a<t<b$. (Complex zeros, which will occur in conjugate pairs, or zeros which lie outside $]a, b[$ will not effect the sign changes of $\Pi_{n-1}(t)$).

In the following $m_i(t)$, $i = 1, 2, \ldots$ will always denote an integrable Markov system.

2. Preliminary lemmas

Lemma 1. *If $q_n(t)$ and $q_{n+1}(t)$ are polynomials of precise† degrees n and $n+1$ respectively such that*

$$(a) \int_a^b m_i(t)q_n(t)dt = \int_a^b m_i(t)q_{n+1}(t)dt = 0, \quad i = 1, 2, \ldots, n;$$

(b) *$q_n(t)$ has only simple zeros in $]a, b[$,*

then $q_n(t)$ and $q_{n+1}(t)$ do not have a common zero in $]a, b[$.

Proof. Let $t-x_1$ be a factor common to $q_n(t)$ and $q_{n+1}(t)$ and set

$$q_n(t) = (t-x_1)r_{n-1}(t), \quad q_{n+1}(t) = (t-x_1)r_n(t).$$

Then

$$\int_a^b m_i(t)(t-x_1)[r_{n-1}(x_1)r_n(t)-r_{n-1}(t)r_n(x_1)]dt = 0,$$

for $i = 1, 2, \ldots, n$. But as $r_{n-1}(x_1)r_n(t)-r_{n-1}(t)r_n(x_1)$ is a polynomial of degree n with x_1 as a zero it can be written $(t-x_1)s_{n-1}(t)$.

Hence $$\int_a^b m_i(t)(t-x_1)^2 s_{n-1}(t)dt = 0, \quad i = 1, 2, \ldots, n.$$

Using property (P), there exists $M_{r+1}(t, t_1, \ldots, t_r)$, where $t_1, \ldots, t_r$ are the zeros of $s_{n-1}(t)$ at which it changes sign and which lie in $]a, b[$, such that

$$M_{r+1}(t, t_1, \ldots, t_r)s_{n-1}(t)$$

does not change sign in $]a, b[$. But

$$\int_a^b M_{r+1}(t, t_1, \ldots, t_r)s_{n-1}(t)(t-x_1)^2 dt = 0.$$

† A polynomial is of precise degree n if it is of degree n and the coefficient t^n does not vanish.

The last equation shows that $s_{n-1}(t) \equiv 0$, and so as $r_{n-1}(x_1) \neq 0$, the coefficient of t^n in $r_n(t)$ is zero and $q_{n+1}(t)$ is not a polynomial of degree $n+1$. The contradiction proves the lemma. (It may be noted that the simplicity of the zeros of $q_n(t)$ can be deduced rather than assumed but it is more convenient to have the statement of the lemma in this form.)

Lemma 2. *If the n-fold integral exists then*

$$\overbrace{\int_a^b \cdots \int_a^b}^{n} F(t_1, t_2, \ldots, t_n)dt_1\ldots dt_n = \int_a^b \int_a^{t_n} \cdots \int_a^{t_2} \sum F(t_{i_1}, t_{i_2}, \ldots, t_{i_n})dt_1\ldots dt_n$$

where the summation extends over all the $n!$ *permutations of* $(1, 2, \ldots, n)$.

Proof. The integral over the hypercube

$$a \leqq t_i \leqq b, \quad i = 1, 2, \ldots, n$$

is the sum of the integrals over all the $n!$ hypertriangles

$$a \leqq t_{i_1} \leqq t_{i_2} \leqq \ldots \leqq t_{i_n} \leqq b$$

into which it can be decomposed, where $(i_1, i_2, \ldots, i_n)$ is any permutation of $(1, 2, \ldots, n)$. The result follows.

For convenience in the following an integral over the hypercube $a \leqq t_i \leqq b$, $i = 1, 2, \ldots, n$, will be written

$$\int_{C_n} (.)d\tau$$

and an integral over the hypertriangle $a \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_n \leqq b$ will be written

$$\int_{T_n} (.)d\tau.$$

$V_{n+1}(t, t_1, \ldots, t_n)$ denotes the $(n+1)\times(n+1)$ determinant

$$\begin{vmatrix} t^n & t^{n-1} & \ldots & t & 1 \\ t_1^n & t_1^{n-1} & \ldots & t_1 & 1 \\ . & . & . & . & . \\ t_n^n & t_n^{n-1} & \ldots & t_n & 1 \end{vmatrix}$$

and an analogous notation is used later for an $n \times n$ determinant.

A corollary of lemma 2 can now be stated as

Corollary.

$$\int_{C_n} m_1(t_1)m_2(t_2)\ldots m_n(t_n)V_{n+1}(t, t_1, \ldots, t_n)d\tau$$
$$= \int_{T_n} M_n(t_1, t_2, \ldots, t_n)V_{n+1}(t, t_1, \ldots, t_n)d\tau.$$

For the integral over the hypercube is, by the lemma,

$$\int_{T_n} \Sigma \pm m_1(t_{i_1})m_2(t_{i_2})\dots m_n(t_{i_n})V_{n+1}(t, t_1, \dots, t_n)d\tau$$

where the sign of $m_1(t_{i_1})\dots m_n(t_{i_n})$ is determined by the parity of the permutation of $(1, 2, \dots, n)$. The sign attached to $m_1(t_1)\dots m_n(t_n)$ is *plus*, and as an even permutation results in *plus*, and an odd permutation in *minus* the result follows from the definition of a determinant.

3. Proof of Theorem

Theorem. *If $\{m_i(t)\}$ is an integrable Markov system in (a, b) then*

(i) *there exists a polynomial $p_n(t)$, of degree n, which is unique up to an arbitrary non-zero scalar multiplier, such that*

$$\int_a^b m_i(t)p_n(t)dt = 0, \quad i = 1, 2, \dots, n; \tag{1}$$

(ii) *the zeros of $p_n(t)$ are real, distinct and lie in $]a, b[$;*

(iii) *if $p_{n+1}(t)$ is a polynomial of degree $n+1$ which satisfies*

$$\int_a^b m_i(t)p_{n+1}(t)dt = 0, \quad i = 1, 2, \dots, n, \tag{2}$$

and has real, distinct zeros then between any pair of adjacent zeros of $p_{n+1}(t)$ lies a zero of $p_n(t)$, all lying in $]a, b[$.

Proof. (i) The polynomial defined by the determinant

$$\begin{vmatrix} t^n & t^{n-1} & \dots & t & 1 \\ \int_a^b t^n m_1(t)dt & \int_a^b t^{n-1}m_1(t)dt & \dots & \int_a^b tm_1(t)dt & \int_a^b m_1(t)dt \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \int_a^b t^n m_n(t)dt & \int_a^b t^{n-1}m_n(t)dt & \dots & \int_a^b tm_n(t)dt & \int_a^b m_n(t)dt \end{vmatrix} \tag{3}$$

satisfies (1). In order to show that it is indeed a polynomial of degree n it has to be verified that the cofactor of t^n does not vanish. Now if it did vanish then there would exist scalars $b_0, b_1, \dots, b_{n-1}$, not all zero, such that

$$\int_a^b m_i(t) \sum_{j=0}^{n-1} b_j t^j dt = 0, \quad i = 1, 2, \dots, n.$$

From property (P) we can now deduce a contradiction, thus showing that the determinant is a polynomial of degree n in t.

The uniqueness follows by taking $q_n(t)$ to satisfy (1) and such that $p_n(t) - q_n(t)$

is a polynomial of degree $n-1$. An application of property (P) to the equations

$$\int_a^b m_i(t)[p_n(t)-q_n(t)]dt = 0, \quad i = 1, 2, \ldots, n$$

shows that $p_n(t)-q_n(t) \equiv 0$. For later use another more convenient form of (3) will be found. Now (3) can be written:

$$\begin{vmatrix} t^n & t^{n-1} & \ldots & t & 1 \\ \int_a^b t_1^n m_1(t_1)dt_1 & \int_a^b t_1^{n-1} m_1(t_1)dt_1 & \ldots & \int_a^b t_1 m_1(t_1)dt_1 & \int_a^b m_1(t_1)dt_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \int_a^b t_n^n m_n(t_n)dt_n & \int_a^b t_n^{n-1} m_n(t_n)dt_n & \ldots & \int_a^b t_n m_n(t_n)dt_n & \int_a^b m_n(t_n)dt_n \end{vmatrix}$$

$$= \int_{C_n} m_1(t_1)m_2(t_2)\ldots m_n(t_n)V_{n+1}(t, t_1, \ldots, t_n)d\tau.$$

For brevity write

$$V_n(t_1, t_2, \ldots, t_n) = V_n,$$

this being an $n \times n$ determinant, and similarly

$$M_n(t_1, t_2, \ldots, t_n) = M_n.$$

Also let

$$P_n(t) = (t-t_1)(t-t_2)\ldots(t-t_n).$$

Then the polynomial $p_n(t)$ defined by (3) is

$$\int_{T_n} M_n V_n P_n(t)d\tau. \tag{4}$$

(ii) Let x_1 be a real zero of $p_n(t)$ which lies outside $]a, b[$, and define $q_{n-1}(t)$, a polynomial of degree $n-1$, by

$$p_n(t) = (t-x_1)q_{n-1}(t).$$

From property (P) of the Markov system, a linear combination, $M_{r+1}(t, t_1, \ldots, t_r)$, of $m_1(t)$, $m_2(t)$, ..., $m_n(t)$ can be constructed so that

$$q_{n-1}(t)M_{r+1}(t, t_1, t_2, \ldots, t_r)$$

does not change sign in $]a, b[$. (The points $t_1, t_2, \ldots, t_r$ are those in $]a, b[$ at which $q_{n-1}(t)$ changes sign.) Consider now

$$\int_a^b M_{r+1}(t, t_1, \ldots, t_r)(t-x_1)q_{n-1}(t)dt.$$

As M_{r+1} is a linear combination of the members of the Markov system it follows from (1) that this last integral is zero. But by construction the integrand does not vanish in $]a, b[$. Hence no real zero of $p_n(t)$ lies outside $]a, b[$.

To show that the zeros are real and simple it is noted that as the coefficients of $p_n(t)$ are real then any complex zeros must occur in complex conjugate pairs. Let $(t-\alpha)^2+\beta^2$ be a factor of $p_n(t)$, where β may be zero.

Then $p_n(t) = [(t-\alpha)^2+\beta^2]r_{n-2}(t)$, and property (P) together with

$$\int_a^b m_i(t)[(t-\alpha)^2+\beta^2]r_{n-2}(t)dt = 0, \quad i = 1, 2, \ldots, n$$

will give a contradiction as before.

(iii) Let $p_{n+1}(t)$ be any polynomial of precise degree $n+1$ which has real distinct zeros, and which satisfies

$$\int_a^b m_i(t)p_{n+1}(t)dt = 0, \quad i = 1, 2, \ldots, n.$$

(At least one such polynomial exists for the imposition of the further condition

$$\int_a^b m_{n+1}(t)p_{n+1}(t)dt = 0$$

will define such a polynomial by the first part of the theorem.)

Name the zeros of $p_{n+1}(t)$ as $x_1, x_2, \ldots, x_n, x_{n+1}$, and let

$$p_{n+1}(t) = (t-x_1)q_n(t)$$

so that $q_n(t)$ has the real distinct zeros $x_2, x_3, \ldots, x_{n+1}$. Then

$$\int_a^b m_i(t)(t-x_1)q_n(t)dt = 0, \quad i = 1, 2, \ldots, n. \tag{5}$$

and so, as before, the polynomial defined by the determinant

$$\begin{vmatrix} t^n & t^{n-1} & \cdots & t & 1 \\ \int_a^b m_1(t)(t-x_1)t^n dt & \int_a^b m_1(t)(t-x_1)t^{n-1}dt & \cdots & \int_a^b m_1(t)(t-x_1)t\,dt & \int_a^b m_1(t)(t-x_1)dt \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \int_a^b m_n(t)(t-x_1)t^n dt & \int_a^b m_n(t)(t-x_1)t^{n-1}dt & \cdots & \int_a^b m_n(t)(t-x_1)t\,dt & \int_a^b m_n(t)(t-x_1)dt \end{vmatrix} \tag{6}$$

will satisfy (5). If the cofactor of t^n is non-zero then the determinant is, apart from a non-zero scalar multiplier, the polynomial $q_n(t)$. But if the cofactor is zero then there will exist scalars $b_0, b_1, \ldots, b_{n-1}$ such that

$$\int_a^b m_i(t)(t-x_1)\sum_{k=0}^{n-1} b_k t^k dt = 0, \quad i = 1, 2, \ldots, n. \tag{7}$$

Now from (i)

$$\int_a^b m_i(t)p_n(t)dt = 0, \quad i = 1, 2, \ldots, n$$

defines uniquely the zeros of $p_n(t)$. It follows that

$$p_n(t) = (t - x_1) \sum_{k=0}^{n-1} b_k t^k,$$

and so $p_n(t)$ and $p_{n+1}(t)$ have a common zero at $t = x_1$. But by Lemma 1 $p_n(t)$ and $p_{n+1}(t)$ cannot have a common zero. Hence the cofactor of t^n in (6) does not vanish and (6) defines $q_n(t)$.

By writing (6) as

$$\begin{vmatrix} t^n & t^{n-1} & \dots & 1 \\ \int_a^b m_1(t_1)(t_1-x_1)t_1^n dt_1 & \int_a^b m_1(t_1)(t_1-x_1)t_1^{n-1} dt_1 & \dots & \int_a^b m_1(t_1)(t_1-x_1) dt_1 \\ \cdot & \cdot & \cdot & \cdot \\ \int_a^b m_n(t_n)(t_n-x_1)t_n^n dt_n & \int_a^b m_n(t_n)(t_n-x_1)t_n^{n-1} dt_n & \dots & \int_a^b m_n(t_n)(t_n-x_1) dt_n \end{vmatrix} \tag{8}$$

it is seen that it can also be expressed as

$$\int_{C_n} m_1(t_1)m_2(t_2)\dots m_n(t_n)(t_1-x_1)(t_2-x_1)\dots(t_n-x_1)V_{n+1}(t, t_1, t_2, \dots, t_n)d\tau$$
$$= \int_{T_n} M_n V_{n+1}(t, t_1, \dots, t_n)(t_1-x_1)(t_2-x_1)\dots(t_n-x_1)d\tau. \tag{9}$$

Finally, as $V_{n+1}(t, t_1, \dots, t_n) = (t-t_1)(t-t_2)\dots(t-t_n)V_n(t_1, t_2, \dots, t_n)$, (9) is

$$(-1)^n \int_{T_n} M_n V_n (x_1-t_1)(x_1-t_2)\dots(x_1-t_n)(t-t_1)\dots(t-t_n)d\tau$$
$$= (-1)^n \int_{T_n} M_n V_n P_n(x_1) P_n(t) d\tau \tag{10}$$

using the abridged notation introduced earlier. This determinant defines the polynomial $q_n(t)$ which is assumed to vanish for $x_2, x_3, \dots, x_{n+1}$.

Now

$$P_n(t) = \sum_{r=0}^{n} \frac{(t-x_1)^r}{r!} P_n^{(r)}(x_1);$$

substituting this for $P_n(t)$ in (10) and putting $t = x_2, x_3, \dots, x_{n+1}$ in turn gives the equations

$$q_n(x_j) = \sum_{r=0}^{n} \frac{(x_j-x_1)^r}{r!} \int_{T_n} M_n V_n P_n^{(r)}(x_1) P_n(x_1) d\tau = 0, \quad j = 2, 3, \dots, n+1. \tag{11}$$

Eliminate now $\int_{T_n} M_n V_n P_n^{(r)}(x_1) P_n(x_1) dt$ $r = 1, 2, \dots, n-1$. This gives, noting that $P_n^{(n)}(x_1) = n!$,

$$\begin{vmatrix} (x_2-x_1)^n \int_{T_n} M_nV_nP_n(x_1)d\tau + \int_{T_n} M_nV_n[P_n(x_1)]^2 d\tau, & (x_2-x_1), \ldots, (x_2-x_1)^{n-1} \\ (x_3-x_1)^n \int_{T_n} M_nV_nP_n(x_1)d\tau + \int_{T_n} M_nV_n[P_n(x_1)]^2 d\tau, & (x_3-x_1), \ldots, (x_3-x_1)^{n-1} \\ \cdots & \cdots \\ (x_{n+1}-x_1)^n \int_{T_n} M_nV_nP_n(x_1)d\tau + \int_{T_n} M_nV_n[P_n(x_1)]^2 d\tau, & (x_{n+1}-x_1), \ldots, (x_{n+1}-x_1)^{n-1} \end{vmatrix} = 0.$$

Hence

$$\int_{T_n} M_nV_nP_n(x_1)d\tau \begin{vmatrix} (x_2-x_1)^n & (x_2-x_1) & \ldots & (x_2-x_1)^{n-1} \\ (x_3-x_1)^n & (x_3-x_1) & \ldots & (x_3-x_1)^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ (x_{n+1}-x_1)^n & (x_{n+1}-x_1) & \ldots & (x_{n+1}-x_1)^{n-1} \end{vmatrix} +$$

$$+ \int_{T_n} M_nV_n[P_n(x_1)]^2 d\tau \begin{vmatrix} 1 & x_2-x_1 & \ldots & (x_2-x_1)^{n-1} \\ 1 & x_3-x_1 & \ldots & (x_3-x_1)^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_{n+1}-x_1 & \ldots & (x_{n+1}-x_1)^{n-1} \end{vmatrix} = 0.$$

After removing the non-zero Vandermonde determinant factor, the result is that

$$(x_1-x_2)(x_1-x_3)\ldots(x_1-x_{n+1}) \int_{T_n} M_nV_nP_n(x_1)d\tau = \int_{T_n} M_nV_n[P_n(x_1)]^2 d\tau \qquad (12)$$

Writing $\Pi_{n+1}(t) = (t-x_1)(t-x_2)\ldots(t-x_{n+1})$ it follows that

$$\Pi'_{n+1}(x_1) \int_{T_n} M_nV_nP_n(x_1)d\tau = \int_{T_n} M_nV_n[P_n(x_1)]^2 d\tau, \quad i = 1, 2, \ldots, n+1, \qquad (13)$$

where the prime denotes differentiation. As M_nV_n is of constant sign in T_n

$$\int_{T_n} M_nV_nP_n(x_i)d\tau$$

has the same sign as $\Pi'_{n+1}(x_i)$. But, from (4), the polynomial defined by

$$\int_{T_n} M_nV_nP_n(x)d\tau$$

is, apart from a scalar multiplier, that polynomial $p_n(t)$ which satisfies

$$\int_a^b m_i(t)p_n(t)dt = 0, \quad i = 1, 2, \ldots, n.$$

Hence, at the points $x_1, x_2, \ldots, x_{n+1}$, $p_n(t)$ has the sign of

$$\Pi'_{n+1}(x_1), \ldots, \Pi'_{n+1}(x_{n+1})$$

and so between each pair of consecutive zeros of $p_{n+1}(t)$ lies a zero of $p_n(t)$.

4. Remark

It will have been noticed that $p_{n+1}(t)$ was not required to satisfy the additional condition

$$\int_a^b m_{n+1}(t)p_{n+1}(t)dt = 0,$$

but merely that its zeros should be real and distinct. In the classical situation, when $m_i(t) = m(t)t^{i-1}$, the separation of zeros of $p_{n+1}(t)$ by $p_n(t)$ then follows under the weaker assumption that $p_{n+1}(t)$ should satisfy

$$\int_a^b m(t)t^{i-1}p_{n+1}(t) = 0 \quad i = 1, 2, \ldots, n$$

together with the requirement that its zeros should be real and distinct.

5. Gaussian quadrature

The results of the theorem have an application to the existence of Gauss-type quadrature formula for the simultaneous calculation of integrals of the form

$$\int_a^b m_i(t)f(t)dt,$$

where it is required to approximate the integrals by weighted sums of $f(t)$ at certain points. For, using Lagrange's interpolation formula,

$$\int_a^b m_i(t)f(t)dt = \sum_{j=1}^{n} f(x_j)H_{ij} + \frac{1}{n!}\int_a^b m_i(t)\Pi_n(t)f^{(n)}(\xi)dt$$

for $i = 1, 2, \ldots, n$, where $\Pi_n(t) = (t-x_1)(t-x_2)\ldots(t-x_n)$ and

$$H_{ij} = \frac{1}{\Pi'_n(x_j)}\int_a^b m_i(t)\frac{\Pi_n(t)dt}{t-x_j}. \tag{14}$$

If the remainders

$$\int_a^b m_i(t)\Pi_n(t)f^{(n)}(\xi)dt, \quad i = 1, 2, \ldots, n, \quad a < \xi < b,$$

are to vanish when $f(t)$ is a polynomial of degree n, then

$$\int_a^b m_i(t)\Pi_n(t)dt = 0, \quad i = 1, 2, \ldots, n,$$

thus giving, with the appropriate conditions on $m_i(t)$, a set of quadrature points.

The need for simultaneous Gaussian quadrature arises in the solution of integral equations where the integral equation has the form

$$\lambda \sum_{i=1}^{n} \int_a^b m_i(s)k_i(s, t)f(t)dt + g(s) = f(s), \quad a \leqq s \leqq b,$$

the kernels $k_i(s, t)$ being continuous functions of t. It is essential here that if the integrals are replaced by quadrature formulae then the resulting equations have the form:

$$\lambda \sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij}k_i(s, x_j)\tilde{f}(x_j) + g(s) = \tilde{f}(s)$$

in order that putting $s = x_1, x_2, \ldots, x_n$ will lead to a determinate set of equations. Here $\tilde{f}(x_j)$ denotes the putative approximation of $f(x_j)$.

It is natural to ask what can be said about the weights H_{ij} in (14), for in the usual Gaussian quadrature the weights are known to have constant sign. No such result has been found for simultaneous Gaussian quadrature with a Markov set. In fact if $m_i(t) = t^{i-1}m(t)$, $i = 1, 2, \ldots, n-1$, $m(t) > 0$ say, then

$$\begin{aligned} H_{ij} &= \frac{1}{\Pi_n'(x_j)} \int_a^b m(t)t^{i-1} \frac{\Pi_n(t)}{t-x_j} dt \\ &= \frac{1}{\Pi_n'(x_j)} \int_a^b m(t) \frac{(t^{i-1} - x_j^{i-1})}{t-x_j} \Pi_n(t)dt + x_j^{i-1}H_{1j} \\ &= x_j^{i-1}H_{1j}. \end{aligned} \tag{15}$$

It is easy to prove that $H_{1j} > 0$, $j = 1, 2, \ldots, n$, for

$$H_{1j} = \frac{1}{\Pi_n'(x_j)} \int_a^b m(t) \frac{\Pi_n(t)}{t-x_j} dt,$$

and as

$$\frac{1}{\Pi_n'(x_j)} \int_a^b t^{i-1}m(t)(t-x_j) \frac{\Pi_n(t)}{t-x_j} dt = 0, \quad i = 1, 2, \ldots, n-1,$$

it now follows by induction that

$$x_j^{i-1}H_{1j} = \frac{1}{\Pi_n'(x_j)} \int_a^b m(t)t^{i-1} \frac{\Pi_n(t)}{t-x_j} dt, \quad i = 1, 2, \ldots, n.$$

Hence

$$\Pi_n'(x_j)H_{1j} = \frac{1}{\Pi_n'(x_j)} \int_a^b m(t) \left(\frac{\Pi_n(t)}{t-x_j}\right)^2 dt$$

by forming the polynomial $\Pi_n(t)/(t-x_j)$ from $1, t, \ldots, t^{n-1}$. Hence H_{1j} are all positive, but if $\{x_j\}$ changes sign in (a, b) then H_{2j}, $j = 1, \ldots, n$ will have positive and negative components.

6. Generalization

The conditions of the theorem can be weakened and still lead to the same conclusion. Let the sequence of functions $m_i(t)$, $i = 1, 2, \ldots$ have property (P), that is, given a polynomial $\Pi_{n-1}(t)$ of degree $n-1$ with real coefficients having r zeros at which $\Pi_{n-1}(t)$ changes sign, it is possible to find a linear combination M_{r+1} of $m_1, m_2, \ldots, m_{r+1}(t)$ such that

$$M_{r+1}(t)\Pi_{n-1}(t)$$

does not change sign in (a, b). Further, let $m_i(t)$ be such that the determinant

$$\begin{vmatrix} m_1(t_1) & m_2(t_1) & \ldots & m_n(t_1) \\ m_1(t_2) & m_2(t_2) & \ldots & m_n(t_2) \\ \cdot & \cdot & \cdot & \cdot \\ m_1(t_n) & m_2(t_n) & \ldots & m_n(t_n) \end{vmatrix}$$

does not change sign in $a \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_n \leqq b$.

With these conditions replacing condition (6) of the definition of an integrable Markov system the theorem still holds.

REFERENCE

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